Hidden Markov Models – II

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Outline

1. Review: Markov Models, HMM, Forward

2. Backward algorithm

3. Viterbi algorithm

4. Baum-Welch estimation algorithm

Markov Models

- **Observable states:**

$$1, 2, \ldots, N$$

- **Observed sequence:**

$$q_1, q_2, \ldots, q_t, \ldots, q_T$$

- **First order Markov assumption:**

$$P(q_t = j | q_{t-1} = i, q_{t-2} = k, \ldots) = P(q_t = j | q_{t-1} = i)$$

- **Stationarity:**

$$P(q_t = j | q_{t-1} = i) = P(q_{t+l} = j | q_{t+l-1} = i)$$

Markov Models

- **State transition matrix A :**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2N} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{iN} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{Nj} & \cdots & a_{NN} \end{bmatrix}$$

where

$$a_{ij} = P(q_t = j | q_{t-1} = i) \qquad 1 \leq i, j, \leq N$$

- **Constraints on a_{ij} :**

$$\begin{aligned} a_{ij} &\geq 0, \qquad \forall i, j \\ \sum_{j=1}^{N} a_{ij} &= 1, \qquad \forall i \end{aligned}$$

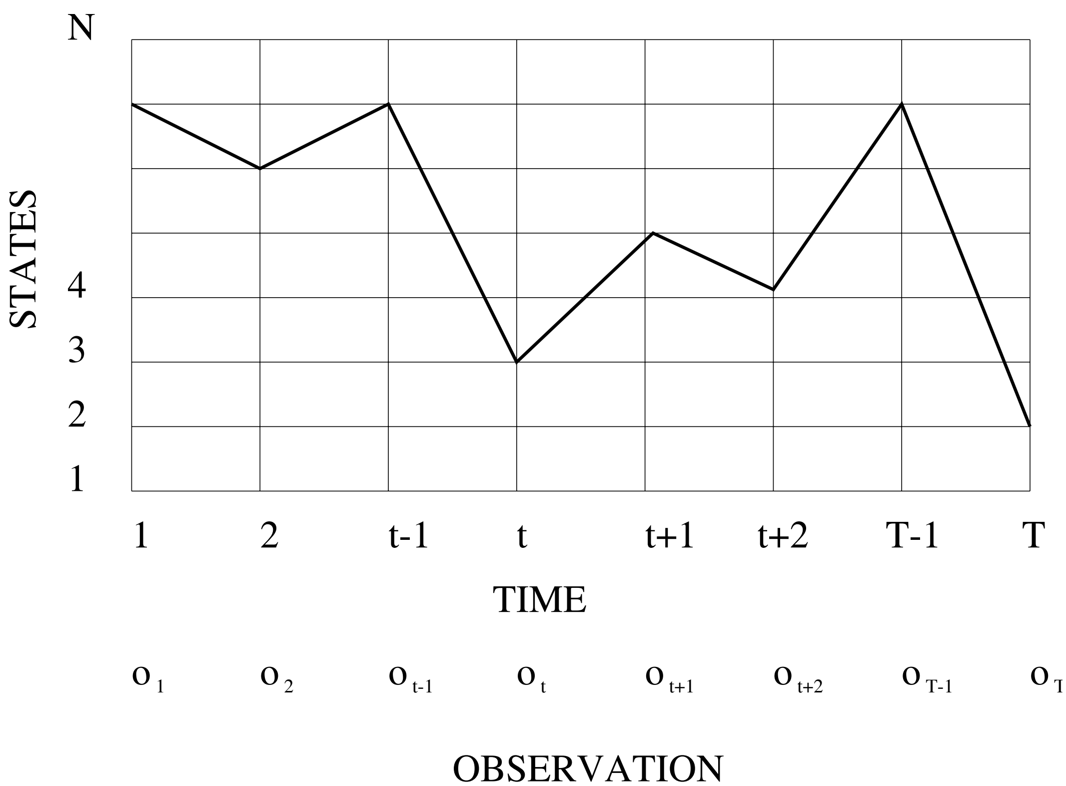
Hidden Markov Models

- **States are not observable**
- **Observations are probabilistic functions of state**
- **State transitions are still probabilistic**

Urn and Ball Model

- N urns containing colored balls
- M distinct colors of balls
- Each urn has a (possibly) different distribution of colors
- Sequence generation algorithm:
  1. Pick initial urn according to some random process.
  2. Randomly pick a ball from the urn and then replace it
  3. Select another urn according a random selection process associated with the urn
  4. Repeat steps 2 and 3

The Trellis


N
STATES
4
3
2
1
1
2
t-1
t
t+1
t+2
T-1
T
TIME
o 1
o 2
o t-1
o t
o t+1
o t+2
o T-1
o T
OBSERVATION

Elements of Hidden Markov Models

- N – the number of hidden states
- Q – set of states $Q = \{1, 2, \ldots, N\}$
- M – the number of symbols
- V – set of symbols $V = \{1, 2, \ldots, M\}$
- A – the state-transition probability matrix.

$$a_{ij} = P(q_{t+1} = j | q_t = i) \quad 1 \leq i, j, \leq N$$

- B – Observation probability distribution:

$$B_j(k) = P(o_t = k | q_t = j) \quad 1 \leq k \leq M$$

- π – the initial state distribution:

$$\pi_i = P(q_1 = i) \quad 1 \leq i \leq N$$

- λ – the entire model $\lambda = (A, B, \pi)$

Three Basic Problems

1. Given observation $O = (o_1, o_2, \ldots, o_T)$ and model $\lambda = (A, B, \pi)$, efficiently compute $P(O|\lambda)$.
   - Hidden states complicate the evaluation
   - Given two models λ_1 and λ_2, this can be used to choose the better one.
2. Given observation $O = (o_1, o_2, \ldots, o_T)$ and model λ find the optimal state sequence $q = (q_1, q_2, \ldots, q_T)$.
   - Optimality criterion has to be decided (e.g. maximum likelihood)
   - "Explanation" for the data.
3. Given $O = (o_1, o_2, \ldots, o_T)$, estimate model parameters $\lambda = (A, B, \pi)$ that maximize $P(O|\lambda)$.

Solution to Problem 1

- Problem: Compute $P(o_1, o_2, \ldots, o_T|\lambda)$
- Algorithm:
  - Let $q = (q_1, q_2, \ldots, q_T)$ be a state sequence.
  - Assume the observations are independent:

$$\begin{aligned} P(O|q,\lambda) &= \prod_{i=1}^{T} P(o_t|q_t,\lambda) \\ &= b_{q_1}(o_1)b_{q_2}(o_2)\cdots b_{q_T}(o_T) \end{aligned}$$

  - Probability of a particular state sequence is:

$$P(q|\lambda) = \pi_{q_1} a_{q_1 q_2} a_{q_2 q_3} \cdots a_{q_{T-1} q_T}$$

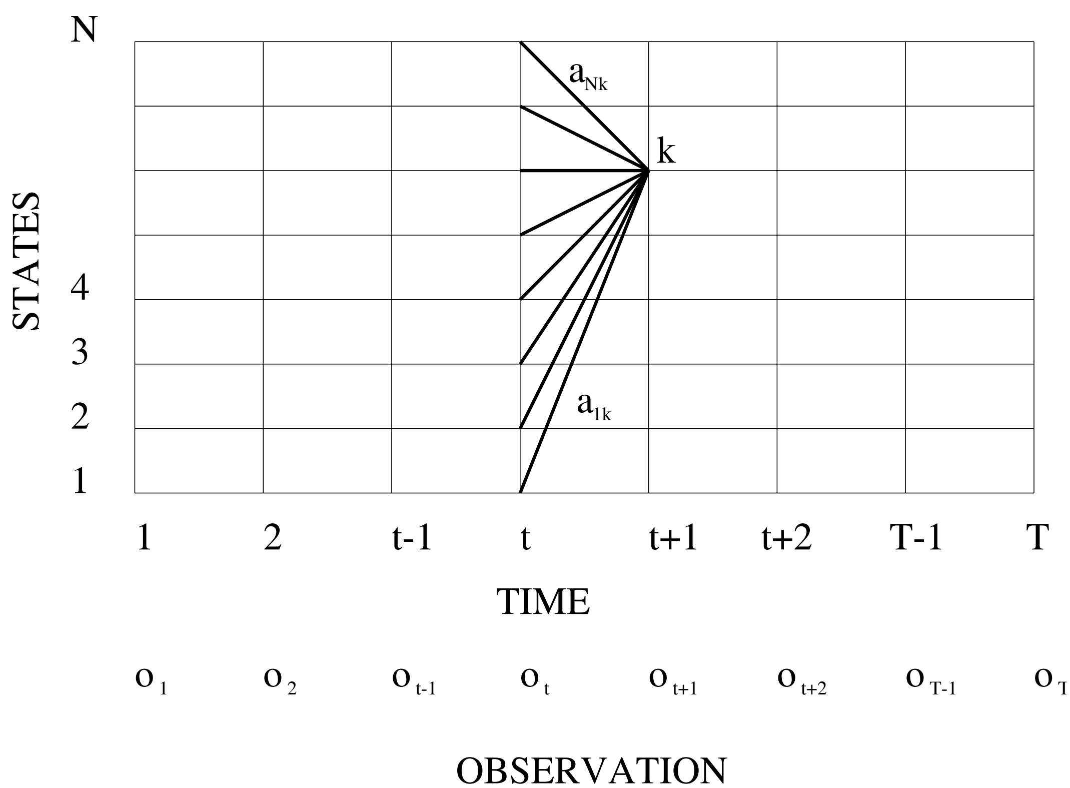
  - Also, $P(O,q|\lambda) = P(O|q,\lambda)P(q|\lambda)$
  - Enumerate paths and sum probabilities:

$$P(O|\lambda) = \sum_q P(O|q,\lambda)P(q|\lambda)$$

- N^T state sequences and $O(T)$ calculations.

  Complexity: $O(TN^T)$ calculations.

Forward Algorithm: Intuition



Forward Algorithm

- **Define forward variable $\alpha_t(i)$ as:**

$$\alpha_t(i) = P(o_1, o_2, \ldots, o_t, q_t = i|\lambda)$$

- **$\alpha_t(i)$ is the probability of observing the partial sequence $(o_1, o_2 \ldots, o_t)$ such that the state q_t is i.**

- **Induction:**

  1. **Initialization:** $\alpha_1(i) = \pi_i b_i(o_1)$
  2. **Induction:**

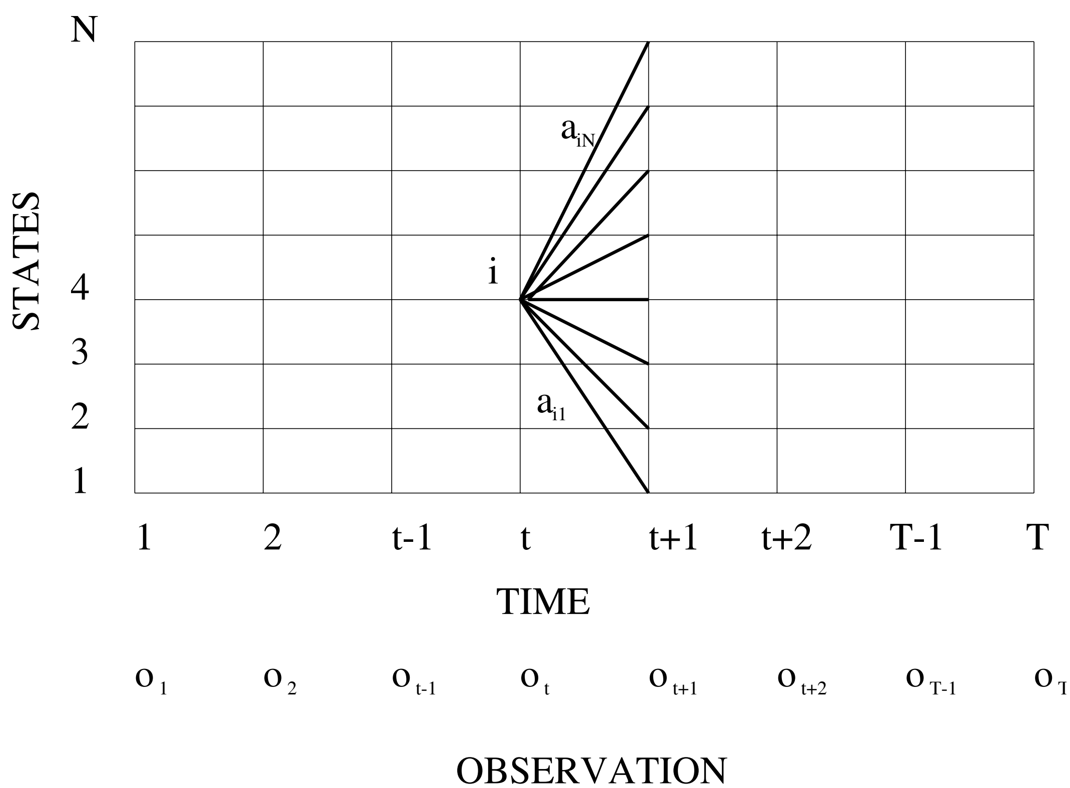
$$\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij}\right] b_j(o_{t+1})$$

  3. **Termination:**

$$P(O|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$$

- **Complexity:** $O(N^2T)$.

Backward Algorithm



Backward Algorithm

- Define backward variable $\beta_t(i)$ as:

$$\beta_t(i) = P(o_{t+1}, o_{t+2}, \ldots, o_T | q_t = i, \lambda)$$

- $\beta_t(i)$ is the probability of observing the partial sequence $(o_{t+1}, o_{t+2} \ldots, o_T)$ such that the state q_t is i.

- Induction:

  1. Initialization: $\beta_T(i) = 1$

  2. Induction:

$$\begin{aligned}\beta_t(i) \;&=\; \sum_{j=1}^{N} a_{ij} b_j(o_{t+1}) \beta_{t+1}(t), \\ &\qquad 1 \leq i \leq N, \\ &\qquad t = T-1, \ldots, 1\end{aligned}$$

Solution to Problem 2

- Choose the most likely path
- Find the path $(q_1, q_t, \ldots, q_T)$ that maximizes the likelihood:

$$P(q_1, q_2, \ldots, q_T | O, \lambda)$$

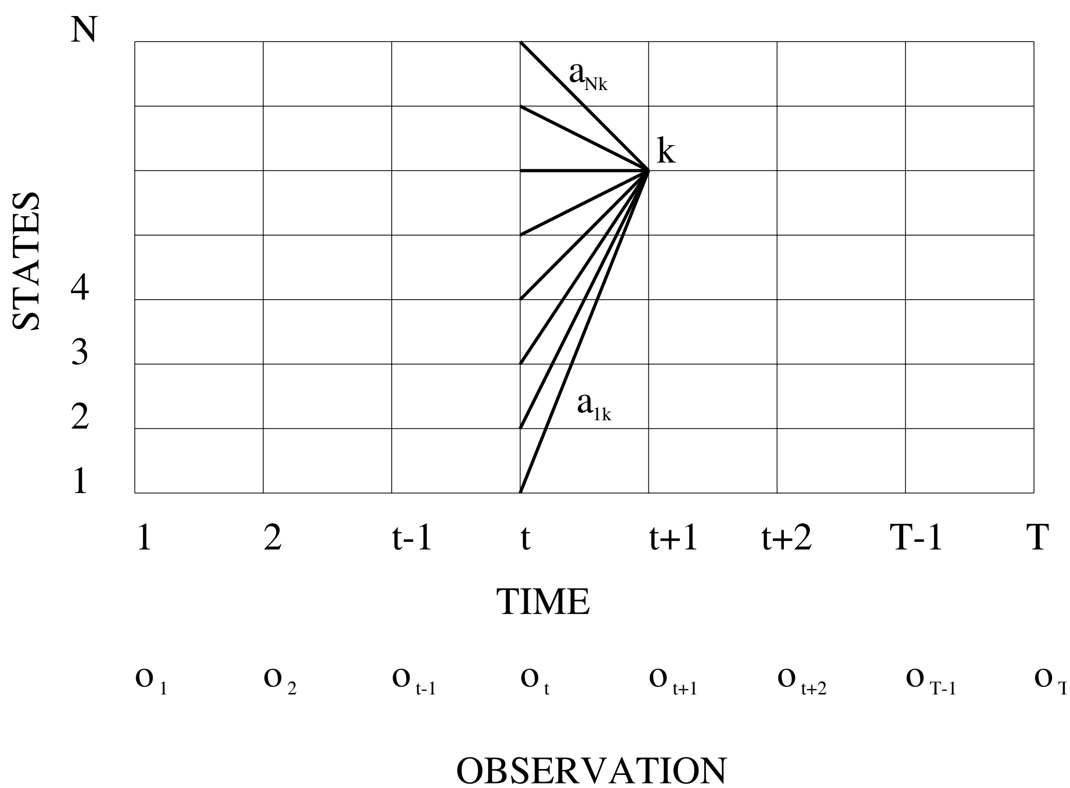
- Solution by Dynamic Programming
- Define:

$$\delta_t(i) = \max_{q_1, q_2, \ldots, q_{t-1}} P(q_1, q_2, \ldots, q_t = i, o_1, o_2, \ldots, o_t | \lambda)$$

- $\delta_t(i)$ is the highest prob. path ending in state i
- By induction we have:

$$\delta_{t+1}(j) = \max_i [\delta_t(i) a_{ij}] \cdot b_j(o_{t+1})$$

Viterbi Algorithm



Viterbi Algorithm

- **Initialization:**

$$\delta_1(i) = \pi_i b_i(o_1), \quad 1 \le i \le N$$
$$\psi_1(i) = 0$$

- **Recursion:**

$$\delta_t(j) = \max_{1 \le i \le N} [\delta_{t-1}(i) a_{ij}] b_j(o_t)$$
$$\psi_t(j) = \arg\max_{1 \le i \le N} [\delta_{t-1}(i) a_{ij}]$$
$$2 \le t \le T, 1 \le j \le N$$

- **Termination:**

$$P^* = \max_{1 \le i \le N} [\delta_T(i)]$$
$$q_T^* = \arg\max_{1 \le i \le N} [\delta_T(i)]$$

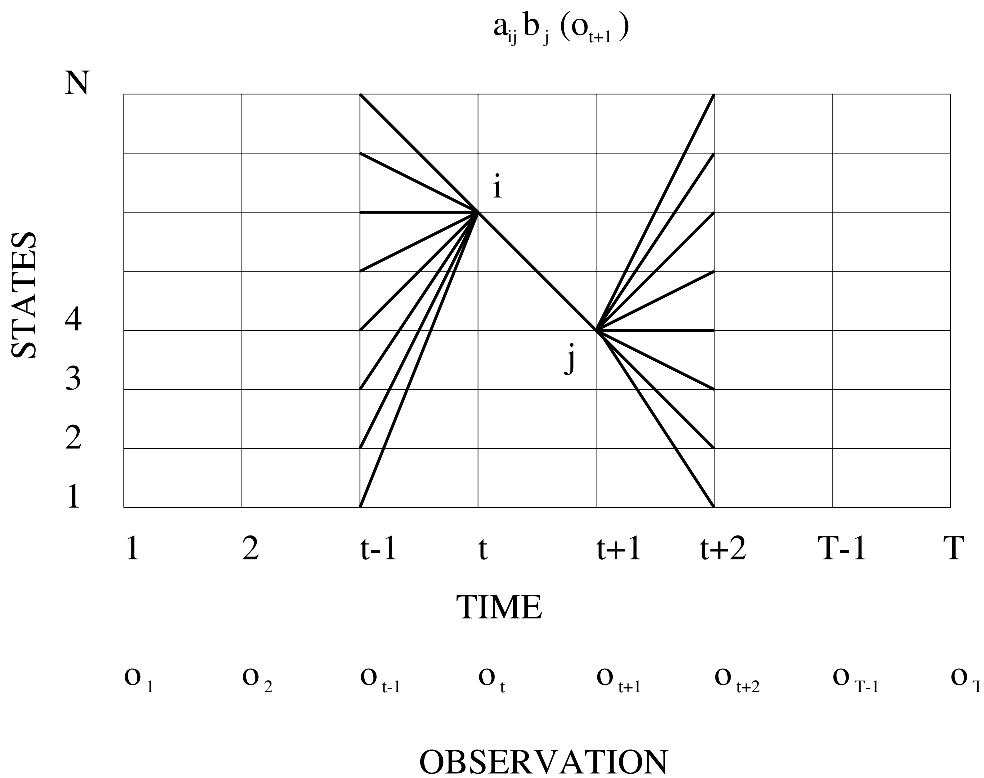
- **Path (state sequence) backtracking:**

$$q_t^* = \psi_{t+1}(q_{t+1}^*), \quad t = T-1, T-2, \ldots, 1$$

Solution to Problem 3

- Estimate $\lambda = (A, B, \pi)$ to maximize $P(O|\lambda)$
- No analytic method because of complexity – iterative solution.
- $\xi(i, j)$ is the probability of being in state i at time t and in state j at time $t + 1$.
- $$\begin{aligned} \xi(i, j) &= \frac{\alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}{P(O|\lambda)} \\ &= \frac{\alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_t(i) a_{ij} b_j(o_{t+1}) \beta_{t+1}(j)} \end{aligned}$$

Baum-Welch Algorithm



Baum-Welch Algorithm

- Define $\gamma_t(i)$ as prob. of being in state i at time t, given the observation sequence.

$$\gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)$$

- $\sum_{t=1}^{T} \gamma_t(i)$ is the expected number of times state i is visited.
- $\sum_{t=1}^{T-1} \xi_t(i,j)$ is the expected number of transitions from state i to state j.
- π_i = expected frequency in state i at time $(t = 1)$ $= \gamma_1(i)$.
- a_{ij} = (expected number of transition from state i to state j)/ (expected nubmer of transitions from state i):

$$a_{ij} = \frac{\sum \xi_t(i,j)}{\sum \gamma_t(i)}$$

- $b_j(k)$ = (expected number of times in state j and observing symbol k) / (expected number of times in state j:

$$b_j(k) = \frac{\sum_{t,o_t=k} \gamma_t(i)}{\sum_t \gamma(i)}$$

Properties

- **Covariance of the estimated parameters**
- **Convergence rates**

Types of HMM

- **Continuous density**
- **Ergodic**
- **State duration**

Implementation Issues

- **Scaling**
- **Initial parameters**
- **Multiple observation**

Comparison of HMMs

- **What is a natural distance function?**
- **If $\rho(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ is large, does it mean that the models are really different?**