Hypergraphs, Simplicial Complexes and Geometric Graphs
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Abstract

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1 Introduction

A *hypergraph* represents an arbitrary set of subsets of its vertex set, where each subset is called a *hyperedge*. We denote by K_n^r the complete r-uniform hypergraph with n vertices. In an *r-uniform hypergraph*, each hyperedge contains exactly r vertices from the vertex set of the hypergraph. An r-uniform hypergraph is *complete* if all possible distinct hyperedges consisting of exactly r vertices are present in the hypergraph.

K_n^r may be emdedded in $\mathbb{R}^m$ by first fixing the vertices and then drawing a *convex* set of points for each hyperedge such that the convex set contains exactly the vertices in the hyperedge. The convex set containing all the vertices of a hyperedge *represents* the corresponding hyperedge.

Hyperplanes and linear/affine independence

A *hyperplane* is a higher-dimensional analog of a (two-dimensional) plane in three or higher dimensional space. In one-dimensional space (in a line), every hyperplane is a point; each hyperplane divides a line into two rays. In the two-dimensional plane, every hyperplane is a line, each dividing the plane into two half-planes. In three-dimensional space, every hyperplane is an ordinary plane, each dividing the space into two half-spaces. This concept can also be applied to four-dimensional space and beyond, where the dividing object is simply referred to as a hyperplane. [Formally speaking, a hyperplane is an affine subspace of *codimension* 1.] The *dimension* of a hyperplane is one less than the dimension of the space of which it is a subspace of codimension 1. A set of points a_0, a_1, ..., a_p in $\mathbb{R}^m$ is said to be *linearly or affinely independent* if they are not contained in any hyperplane of dimension less than p. Consequently, every $(m+2)$ or more points in $\mathbb{R}^m$ are linearly dependent. The points a_i, $i = 0, 1, \ldots, p$ with coordinates $(x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(m)})$ are linearly independent in $\mathbb{R}^m$ if and only if, the matrix

$$\begin{pmatrix} x_0^{(1)} & \ldots & x_0^{(m)} & 1 \\ \ldots & \ldots & \ldots & 1 \\ x_p^{(1)} & \ldots & x_p^{(m)} & 1 \end{pmatrix}$$

has rank $p+1$.

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Convexity

In Euclidean space $\mathbb{R}^m$, an object is said to be *convex* if for every pair of points within the object, each point on the straight line segment joining the pair is also within the object. The *convex hull* of a set of points $M \subset \mathbb{R}^m$ is defined as the intersection of all the convex sets containing M. So, it is also the *smallest* convex set containing M.

Simplices and simplicial complexes

A *k-simplex* σ is the convex hull *conv* S of a collection S of $k+1$ linearly independent points. The *dimension* of σ is denoted by *dim* σ and is equal to k, if σ is a k-simplex. For example, a 0-simplex is a point, a 1-simplex is a straight line, a 2-simplex is a triangle, a 3-simplex is a tetrahedron and so on. The convex hull of any subset $T \subseteq S$ is again a simplex. It is a subset τ of the convex hull of S and is called a *face* of σ. If *dim* $\tau = l$ then τ is called an l-face.

A *simplicial complex* is the collection of faces of a finite number of simplices, any two of which are either disjoint or meet in a common face [6], which is also a simplex. The *dimension dim* K of a simplicial complex K is the largest dimension of any simplex belonging to K. If $k = dim\ K$, the K is a *k-complex*.

A *subcomplex* of a simplicial complex is a subset that is a simplicial complex itself.

A particular subcomplex of a simplicial complex K is the *i-skeleton* that consists of all simplices $\sigma \in K$ whose dimension is i or less [6].

Abstract simplicial complexes

By substituting the set of vertices for each simplex, we get a system of subsets of the vertex set. Formally, a finite system A of finite sets is an *abstract simplicial complex* if $\alpha \in A$ and $\beta \subseteq \alpha$ implies $\beta \in A$. The *vertex set* of A is $Vert\ A = \bigcup_{\alpha \in A} \alpha$. The dimension of an abstract simplicial complex is defined similarly as above.

Geometric realization of an abstract simplicial complex

A *geometric realization* of an abstract simplicial complex A is defined as a simplicial complex K together with a bijection $f : Vert\ A \rightarrow Vert\ K$ such that $\alpha \in A$ if and only if $convf(\alpha) \in K$ [6].

A set of points in $\mathbb{R}^m$ is said to be in *general position* if no $(m+1)$ of them lie in a $(m-1)$-dimensional hyperplane.

Geometric hypergraphs and their geometric realization

A *geometric hypergraph* H is a collection of i-simplices induced by some hyperedges made of $(i+1)$ vertices (from the vertex set of H) in *general position* in $\mathbb{R}^m$. Note that a geometric hypergraph may not be free from intersections between its hyperedges. [The formal definition of *intersection* is introduced in the next paragraph.] A geometric hypergraph H without any intersection between its hyperedges in $\mathbb{R}^m$ is a *geometric realization* of H.

Crossings and intersections

Now we will formally define the notions of *crossing* and *intersection* between two hyperedges in a convex embedding of a hypergraph. Before that, we need few more definitions. The *closure* $Cl(S)$ of a set S is the smallest closed set containing S. The *exterior* of an m-dimensional convex set C in $\mathbb{R}^m$ is $\mathbb{R}^m \setminus Cl(C)$. The *boundary* of C is defined as the collection of those points $p \in Cl(C)$ such that some points in the exterior of C are contained in an ϵ-neighbourhood of p, for any $\epsilon > 0$. The *interior* of C is the collection of all the points of C excluding those on the boundary. Two simplices are said to have a *nontrivial intersection*, if their relative interiors have a point in common. If, in addition, the two simplices are vertex disjoint, then they are said to *cross* [4]. Similarly, we say that two convex sets have an intersection if their relative

interiors have a point in common. Two vertex-disjoint hyperedges in a convex embedding of a hypergraph are said to *cross* if the convex sets representing these two hyperedges have an intersection. A convex embedding of K_n^r in $\mathbb{R}^m$ is called *intersection-free* if no two convex sets representing two of its hyperedges intersect.

Geometric Ramsey theorems

It is a simple exercise to show that 5 or more points admit a convex quadrilateral, provided no three points are collinear. Coupled with the general version of *Ramsey's* theorem, as given below, we can establish the Erdos-Szekeres theorem, which states that for every natural number k, there exists a number $N(k)$ such that any $N(k)$-point set $X \subset \mathbb{R}^2$ in general position contains a k-point convex independent subset.

The general form of Ramsey's theorem may be stated on the basis of three paramenters, p, r, n, where we consider coloring the p-subsets of an N-element set X with r colors, and look for an n-element subset $Y \subset X$ such that all p-element subsets of Y are assigned the same color. For sufficiently large N, there exists such a set Y.

In order to apply Ramsey's theorem, set $p = 4$, $r = 2$ and $n = k \geq 5$. We color all 4-tuples $T \subset X$ red if they form a convex quadrilateral, and blue, otherwise. If $N = N(k)$ is sufficiently large then there is a k-point subset $Y \subset X$ such that all 4-tuples from Y have the same color. For $k \geq 5$, this color is red because any five points determine one red 4-tuple. Since all 4 of its points are convex independent (form a convex quadrilateral), so the whole set Y is convex independent. [We need to check each point of Y only against all triples in Y for convex independence of that point (due to Caratheodory's theorem).] So, we get a k-point subset Y as desired.

Helly's theorem

We know that $n \geq 2$ (connected/convex) intervals on a line have a common intersection if and only if they have pairwise intersections. So, for n persons attending a party, where each visits the party for a single time duration, everyone would have been present simultaneously in the party if and only if each pair of persons met in the party. Not much difficult to show is the fact that a set of X- Y- oriented rectangles in the plane have a common intersection if and onlf of they have pairwise intersections. This is not true however for rectangles that are titled with respect to the axes. On the other hand, Helly's theorem for $d = 2$ dimensions may be stated as follows. Any collection of 4 or more convex sets in the plane have a common intersection if and only if each triple of convex sets from that collection have a common intersection.

We discuss one application of Helly's theorem here. Given any set S of n points in $\mathbb{R}^2$, there exists is a point $x \in \mathbb{R}^2$ such that each closed halfspace containing x also contains at least $\frac{n}{d+1}$ points of S. Such a point is called a *centerpoint*. We have to perform a non-trivial application of Helly's theorem in order to establish this result of existence of centerpoints. Sometimes the contrapositive of Helly's theorem is useful.

Radon's and Caratheodory's theorems

Radon's theorem is as follows. For any $d + 2$ points in $\mathbb{R}^2$, there exists two disjoint subsets whose convex hulls are not disjoint. Establish this result based on the convexity property that convex combinations have non-zero coefficients whereas affine combinations have signed coefficients summing to zero. Also, you may prove Caratheodory's theorem using this result.

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