Department of Mathematics, Indian Institute of Technology, Kharagpur
Assignment 4-5 on Probability and Stochastic Processes, April 2015.
Due:- 16 April, 2015

1. (a) If $\{N_1(t)\}$ and $\{N_2(t)\}$ are 2 independent Poisson processes with parameters λ_1 and λ_2. Show that $P[N_1(t) = k[\{N_1(t) + N_2(t) = n\}] = \binom{n}{k} p^k q^{n-k}$, where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ and $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$.

   (b) The traffic of cars on a road in the same direction is an elementary flow of events with intensity λ per 10 minutes. Mr. X desires a ride and tries to stop a car. The car drivers are decent persons in the this road and oblige such requests free of cost. State the probability distribution of time T for which Mr. X will have to wait. Given that Mr. X has been waiting for 10 minutes, what is the probability that he will have to wait another 10 minutes?

2. (a) Show that all states of an irreducible Markov Chain are of the same type.

   (b) Over a locality weather pattern in the month of July on given day is classified either as a rainy day (R) if it rains any time of the day or a sunny day (S), otherwise. In the year 2006, the outcome of this pattern during the month, July were as follows:

   RRRRSSR RSRRSRS SRRRSRR RSRRRRS RRR

   Suppose that the weather pattern is stable over this locality for at least 5 future years. Using Markov chain theory determine

   (i) the probability that it will be rainy day in a given third day,
   (ii) the probability that any three consecutive days will be rainy,
   (iii) the average number of consecutive sunny days after a rainy day,
   (iv) the expected number of sunny days,
   for the month of July 2010.

3. (a) Derive the steady state probabilities that there are n customers in the system for $M/M/1/K$ queueing system.

   (b) An organization has an interactive time sharing system with 10 active terminals. The average CPU time including swapping is 4 seconds, while mean think time is 24 seconds. With the help of machine repairman model study the system performance with regard to its number of terminals connected to the CPU. Do you think that the system has reached its saturation point? What is the probability that all the terminals are in failed state?

   (c) Customers arrive at WINE BAR (not permitted with license) at an average rate of five per hour. Service times at the counter is exponentially distributed random variable with a mean of 10 minutes per customer. There is one server on duty at the counter. If the BAR is raided, how many customers will be caught, on the average?

4. Suppose $\{N(t), t \geq 0\}$ is a poisson process and one event has taken place in the interval 0 to t. Show that Y, the random variable describing the time of occurrence of this Poisson event, has a continuous uniform distribution over the time interval $(0, t]$.

5. (a) Define an M/M/1 queuing system. State the condition for the existence of steady state solutions for this system. Write the steady state solution of this system.
   (b) A machine shop has two Machines (lathes). Each machine (lath) alternates between uptime when it is working condition and downtime when is being serviced. The uptimes can be assumed to have an exponential distribution with mean 2 hrs. and downtimes can be assumed to have an exponential distribution with mean 30 minutes. At the beginning of the day both machines are working. Assuming that they are used and serviced independently of each other, what is the probability that only 1 machine will be in a working condition at the end of an 8 hour workday?

6. (i) How do you define a Markov Chain?
   (ii) What do you mean by an irreducible Markov chain?
   (iii) Interpret the memoryless property of an exponential random variable.
   (iv) State the steady state condition of an M/M/1 queuing system.

7. (a) Let $t_1 < t_2 < .... < t_k < ...$, be successive occurrence times of events of a Poisson process and let interoccurrence times $\{\tau_n\}$ be defined by $\tau_1 = t_1, \tau_2 = t_2 - t_1, ..., \tau_k = t_k - t_{k-1}, ...$Show that $\{\tau_n\}$ are independent and identically distributed random variables.
(b)A shell landing and exploding at a certain position covers the target by homogeneous Poisson field of splinters with intensity $\lambda = 2.5$ splinters per square meter. The area of the projection of the target onto the plane on which the field of splinters falls is $S = 0.8m^2$. If a splinter hits the target it destroys it completely with probability 0.6. Find the probability that the target will not be destroyed.

8. Consider two independent Poisson processes, where the flow of events in the first process is type E and that of events in the second process is type F. The mean number of occurrences of these two processes are αt and βt respectively. Find the probability distribution of the number of occurrences of events E between two successive occurrences of F.

9. A city has its water supplied from a dam. Prompted by water scarcity in summer months, an investigation was made on the data and water content of the dam in the beginning of the summer. From these investigations it was found that if in the beginning of the one summer the dam was full, the probability that it would be full in the beginning of the next summer was 0.9, whereas if in the beginning one summer the dam was not full, the probability that it would end up being not full in the beginning of the next summer was found to be 0.4. Consider the two state space of the process as the dam was full or not full in the beginning of each summer.
(i) Find the transition probability matrix of the Markov chain.
(ii) Suppose that the beginning of last summer the dam was not full- What is the probability of finding it full after 5 years?
(iii) What is the expected length of an interval during which it remains not full?

10. There are 6 white balls in urn A and 4 red balls in urn B. At each step a ball is selected at random from each urn and the two balls are interchanged. Let X_n be the number of white balls in urn A after n transition. If $\{X_n\}$ forms a Markov Chain, obtain its transition probability matrix P.

11. Assume that the occupation of a son depends only on his father's occupation and not on his grandfather's occupation. Let the transition probability matrix be given by

$$\begin{array}{cc} & son's\ class \\ & \begin{array}{ccc} Lower & Middle & Upper \end{array} \\ Father's\ class \begin{array}{c} Lower \\ Middle \\ Upper \end{array} & \begin{pmatrix} 0.40 & 0.50 & 0.10 \\ 0.05 & 0.70 & 0.25 \\ 0.05 & 0.50 & 0.45 \end{pmatrix} \end{array}$$

For such a population what fractions of the people are in each class in the long run? If a man is in a lower income group, what is the probability that his great grandson will be in the upper income group?

12. Consider a gambler who at each play of the game has probability p of winning one unit and probability $1-p$ of losing one unit . Assuming that the successive plays of the game are independent , what is the probability that starting with i units, the gambler's fortune will reach N before reaching 0? (Here, 0 and N are taken as absorbing states, that is the game does not proceed any further.) Next, let $N = 4$ and $p = 0.4$ in the above problem. Starting with 2 units, determine (i) the expected amount of the time that the gambler has 3 units, (ii) the probability that the gambler ever has one unit.

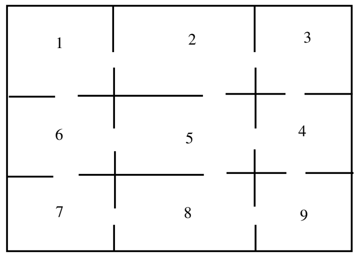
13. In a population of N people, some are infected with an infectious decease. When a sick person meets a healthy one, the latter is infected with probability α. All encounters between two persons. All possible encounters in pairs are equally likely. One such encounter takes place in every unit of time. Define a Markov chain for the spread of decease and find the transition probability matrix (TPM). Classify the states. Suppose $N = 4$ and $\alpha = 0.25$ and there is one infected person in the population. What is the probability that three persons will ever be infected.

14. A set of three cards are shuffled at each stage according to the following scheme: the top card is moved to the bottom with probability p and the top and middle cards are interchanged with probability $1-p$. Define the state space and find the TPM. Find the equivalence classes and the limiting probabilities of being in each state.

15. Ravi has Rs. 2000 and wants to increase it to Rs. 10000 in a hurry. He plays a game with the following rules: A fair coin is tossed. If Ravi calls correctly, he wins a sum of equal to his stake, otherwise loses the same amount. He stakes all his money whenever he has Rs. 5000 or less, otherwise stakes just enough to make it Rs. 10000 in case of a win. The game ends if either Ravi wins RS. 10000 or he goes bankrupt. Define the state space and find TPM. Starting with $X_0 = Rs.2000$, find the probability that Ravi will eventually win Rs. 10000.

16. Each morning Madhu leaves her house and goes for a run. She is equally likely to leave either from the front or back door. Upon leaving the house, she chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which she departed). On the return, she is equally likely to enter, and leave her running shoes, either by the front or the back door. If she owns a total k pairs of running shoes, what proportion of the time does she run barefoot?

17. A white rat is put into the maze shown. The rat moves through the compartments at random, that is, if there are k ways to leave the compartment, he chooses each of these with probability $1/k$. He makes one change of compartment at each instant of time. The state of the system is the number of the compartment the rat is in. Find TPM and classify the states into equivalence classes.



18. Find equivalence classes and recurrent/transient sates associated with the following TPM:

(i)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

(iii)

$$\begin{pmatrix} 0.4 & 0.3 & 0.3 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0.4 \end{pmatrix}$$

(iv)

$$\begin{pmatrix} 0.1 & 0 & 0 & 0.4 & 0.5 & 0 \\ 0.1 & 0.2 & 0.2 & 0 & 0.5 & 0 \\ 0 & 0.1 & 0.3 & 0 & 0 & 0.6 \\ 0.1 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

19. For the Markov chain with state 1, 2, 3,4 whose TPM is as specified below, find the expected number of time periods that the chain is in state 3 given that it started in state i, $i = 1, 2, 3$. Also find the probability

that it ever makes a transition to state 3 given that it started in state $i,\ i = 1, 2, 3.$

$$\begin{pmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.5 & 0 & 0.5 & 0 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

20. An urn contains N balls-some white and some black balls. At each stage, a coin having probability p, $0 < p < 1$, of landing heads is flipped. If head appears, then a ball is chosen randomly from the urn and is replaced by a white ball; if tail appears, then a ball is chosen randomly from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the nth stage. Find the state space and transition probabilities for all states i, j. Determine the equivalence classes and classify the states as recurrent or transient. For $N = 3$, find the proportion of times that the process is in each state.

21. A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4. Find the TPM for the chain. Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability that he is at hotel B at time 3. Further in the long run what proportion of time he spends in each of three locations.

22. A criminal named Ajit and a policeman named Amar move between three possible hideouts according to Markov chains X_n and Y_n with TPM

$$P_{Ajit} = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$$

and

$$P_{Amar} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

When both reach the same location Amar catches Ajit and the chase ends. Describe the chase using Markov chain and find the expected time of the chase.

23. Consider the MC with state space {1, 2, 3, 4} and TPM

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ p & 0 & q & 0 \end{pmatrix}$$

Classify the states and find the equivalence classes. If the initial state is 2 find the expected amount of time the chain is in (i) state 3, (ii) state 4. If the initial state state is 3 find the probability that the chain ever enters (i) state 2, (ii) state 4.

24. Trials are performed in sequence. If the last two trials were successes, then the next trial is a success with probability 0.8; otherwise the next trial is a success with probability 0.5. In the long run, what proportion of trials are successes?

25. In a test paper the questions are arranged so that 3/4 of the time a TRUE answer is followed by a TRUE, while 2/3 of time a FALSE answer is followed by a FALSE. You are confirmed with a 100 question test paper. Approximately what fraction of the answers will be true?

26. Traffic at a point on a highway follows a Poisson process with rate 40 vehicles per hour. 10% of the vehicles are trucks and the other 90% are cars. What is the probability that at least one truck passes in one hour? Given that 10 trucks have passed by in an hour, what is the expected number of vehicles that have passed by? Given that 50 vehicles have passed by in an hour, what is the probability there were exactly 5 trucks and 45 cars?

27. In a Poisson process with rate λ, if an event occurs at time s it is classified as a type i event with probability $P_i(s)$, $i = 1, 2, \ldots, k$ where $\sum_{i=1}^{k} P_i(s) = 1$. Let $N_i(t)$, $i = 1, 2, \ldots, k$ represents type i events occurring by time t. Prove that $N_i(t)$, $i = 1, 2, \ldots, k$ are independently distributed Poisson random variables. Find the mean functions.

28. Prove that in a one dimensional random walk problem all states are recurrent or transient according as the walk is symmetric or not. Find the probability of ever returning to any given state.

29. Let Y_n denote the sum of n independent rolls of fair die. Find $\lim_{x\to\infty} P(Y_n$ is multiple of 13).

30. (a) Derive waiting time distribution in the waiting line for an $M/M/1$ queueing system.
(b) The capacity of a communication line is 2 Kb/second. This line is used to transmit eight-bit characters, so that the maximum rate is 250 characters/second. The applications calls for traffic from many devices to be sent on the line with a total number of 12,000 characters per minute.

    (i) Find the probability that the line is busy.

    (ii) Find the average numbers of characters waiting to be transmitted.

    (iii) Find that value of the time for which 90% of the characters spend less than this time in the system.

31. Suppose that there are five types of breakfast cereal, which we call A, B, C, D and E. Customer tend to stick to the same brand. Those who choose type A choose it the next time around with the probability 0.8; those who choose type B choose it the next time with the probability 0.9. The probabilities for type C, D and E are given by 0.7, 0.8 and 0.6 respectively. When customer do change the brand, they choose one of the other four equally probably. Explain how this may be modelled by a Markov chain and give the TPM.

32. Consider a store that sells TV sets. If at the end of the day there is one zero sets left, then that evening, after the store has closed, the shopkeeper brings in enough new sets so that the number of sets in stock for the next day is equal to 5. This means that each morning, at store opening time, there are between two and five TV sets available for sale. Such a policy is said to be an (s, S) inventory control policy. Let $s = 1$ and $S = 5$. The shopkeeper knows from the experience that the probabilities of selling 0 to 5 sets on any given day are 0.4, 0.3, 0.15, 0.15, 0.0 and 0.0.

    Explain how this scenario may be modelled by Markov chain $\{X_n,\ n = 1, 2....\}$, where X_n is the random variable that defines the number of TV sets left at the end of the n^{th} day. Write down the structure of the TPM.

33. The tpm of a dtmc is given by

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.8 & 0.2 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 & 0 \end{pmatrix}$$

    Draw all sample paths of length 4 that begins in state 1. What is the probability of being in each of the states 1 through 5 after four steps beginning in state 1?

34. Consider a dtmc consisting of four states a, b, c and d whose tpm is given by

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0.4 & 0.6 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \end{pmatrix}$$

    Compute the following probabilities
    (a) $P(X_4 = c, X_3 = c, X_2 = c, X_1 = c | X_0 = a)$
    (b) $P(X_6 = d, X_5 = c, X_4 = b | X_3 = a)$
    (c) $P(X_5 = c, X_6 = a, X_7 = c, X_8 = c | X_4 = b, X_3 = d)$

35. A Markov chain with two states a and b has the following conditional probabilities: If it is in state a at time step n, $n = 0, 1, 2, ...$, then it stays in state a with probability $0.5(0.5)^n$. If it is in state b at time step n, $n = 0, 1, 2, ...$, then it stays in state b with probability $0.75(0.25)^n$. If the MC begins in state a at time step $n = 0$, compute the probabilities of the following sample paths:

$$a \longrightarrow b \longrightarrow a \longrightarrow b \text{ and } a \longrightarrow a \longrightarrow b \longrightarrow b$$

36. The following matrix is one step tpm of a dtmc which describes the weather. State 1 represents a sunny day, state 2 a cloudy day, and state 3 a rainy day.

$$P = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}$$

(a) What is the probability of a sunny day being followed by two cloudy day?
(b) Give that today is rainy, what is the probability that the sun will shine the day after tomorrow?
(c) What is the mean length of rainy day?

37. Consider the four state dtmc whose tpm at time step n, $n = 0, 1, 2, ...,$ is

$$P(n) = \begin{pmatrix} 0 & 0.6 & 0.4 & 0 \\ 0.8 & 0 & 0 & 0.2 \\ 0 & 0 & 0.5(0.5)^n & 1-0.5(0.5)^n \\ 0 & 0 & 0.8(0.8)^n & 1-0.8(0.8)^n \end{pmatrix}$$

What is the probability distribution after two steps if the MC is initiated (i) state 1 (ii) state 4?

38. Give as precise a classification as possible to each of the states of the dtmc with state space $S = \{1, 2, ..., 10\}$ whose tpm is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

39. Find the mean recurrence time of the state 2 and the mean first passage time from state 1 to state 2 in dtmc whose tpm is

$$P = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$

40. Consider the dtmc with transition probabilities given by

$$p_{ij} = e^{-\lambda} \sum_{k=0}^{j} \binom{i}{k} p^k q^{i-k} \frac{\lambda^{j-k}}{(j-k)!}$$

where $p + q = 1, 0 \leq p \leq 1$ and $\lambda > 0$.
(a) Is the chain reducible/irreducible? Explain.
(b) Is the chain periodic/aperiodic?

41. Consider a MC having state space $S = \{0, 1, 2\}$ and tpm

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

Show that this chain has unique stationary distribution π. Find π.