(1) Joint p.d.f (X, Y) is

$$f_{X,Y}(x, y) = \frac{1}{4} e^{-\frac{(x+y)}{2}} \quad , \quad x > 0, y > 0$$

Consider

$$U = \frac{X - Y}{2} \quad , \quad V = Y$$

$$u = \frac{x-y}{2} \quad , \quad v = y \Rightarrow x = 2u + v \; , \; y = v$$

$$J = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 \quad ; \quad |J| = 2$$

(a) Joint p.d.f of (U, V) is

$$f_{U,V}(u, v) = f_{X,Y}(x, y) \, |J|$$

$$= \begin{cases} \frac{1}{2} e^{-(u+v)} & ; \; v > -2u, \; v > 0, -\infty < \\ 0 & ; \; \text{elsewhere} \end{cases}$$

(b) Marginal distribution of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v) \, dv$$

$$= \begin{cases} \int_{-2u}^{\infty} \frac{1}{2} e^{-(u+v)} \, dv & \text{if } u < 0 \\ \int_{0}^{\infty} \frac{1}{2} e^{-(u+v)} \, dv & \text{if } u > 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{u} & \text{if } u < 0 \\ \frac{1}{2} e^{-u} & \text{if } u \geq 0 \end{cases}$$

$$= \frac{1}{2} e^{-|u|} \quad , \quad -\infty < u < \infty$$

(2)

$(X, Y) \sim BVN(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\} \quad \text{--- (1)}$$

Comparing (1) with given density

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{3}} \exp\left\{-\frac{2}{3}\left[(x-1)^2 + \frac{1}{4}(y+1)^2 - \frac{1}{2}(x-1)(y+1)\right]\right\}$$

So we have

(a) $\rho = \frac{1}{2}$, $\mu_1 = 1$, $\mu_2 = -1$, $\sigma_1 = 1$, $\sigma_2 = 2$

$X \sim N(1, 1)$; $Y \sim N(-1, 4)$ [By properties of BVN distributions]

$E(X) = 1$, $V(X) = 1$; $E(Y) = -1$, $V(Y) = 4$

ρ = Correlation coefficient between X and Y $= \frac{1}{2}$

(b) Since $Y \mid X = x \sim N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \sigma_2^2(1-\rho^2)\right)$

[By properties of BVN distribution]

$$E(Y \mid x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$

$$= -1 + \frac{1}{2} \times \frac{2}{1}(x-1) = x-2$$

$$V(Y \mid x) = \sigma_2^2(1-\rho^2) = 4\left(1-\frac{1}{4}\right) = 3$$

$\therefore\ [Y \mid X = x] \sim N(x-2, 3)$

2 (c)

$$Y \mid X = 4 \sim N(2, 3)$$

$$\therefore P(4 \le Y \le 6 \mid X = 4)$$

$$= P\left(\frac{2}{\sqrt{3}} \le Z \le \frac{4}{\sqrt{3}}\right)$$

$$= \phi\left(\frac{4}{\sqrt{3}}\right) - \phi\left(\frac{2}{\sqrt{3}}\right)$$

$$= \phi(2.3) - \phi(1.2)$$

$$= 0.9892 - 0.8849$$

$$= 0.1043$$

(3) (a) X is a random variable with m.g.f

$$M_X(t) = e^{32t^2 - 6t} \qquad -\infty < t < \infty$$

So $E(X) = -6$

$Var(X) = 64$

$X \sim N(-6, 64)$

$$P(-4 \leq X \leq 16)$$

$$= P\left[\frac{-4+6}{8} \leq \frac{X+6}{8} \leq \frac{16+6}{8}\right]$$

$$= P(0.25 \leq Z \leq 2.75)$$

$$= \Phi(2.75) - \Phi(0.25)$$

$$= 0.39831$$

b) $X_1 + X_2 + X_3 \sim N(40+50+60, 10+12+14)$

$$= N(150, 36)$$

$$P(X_1 + X_2 + X_3 \leq 160)$$

$$= P\left(Z \leq \frac{10}{6}\right)$$

$$= \Phi(1.67) = 0.95254$$

$X_1 + X_2 - 2X_3 \sim N(40+50-120, 10+12+4\times 14)$

$$= N(-30, 78)$$

$$P(X_1 + X_2 - 2X_3 \geq 0)$$

$$= 1 - P\left(Z \leq \frac{30}{\sqrt{78}}\right)$$

$$= 1 - \Phi(3.396)$$

$$= 1 - 0.99966$$

$$= 0.00034$$