Indian Institute of Technology, Kharagpur

Date of Exam.: .04.15 (FN/AN) Time: 3 Hrs. Full Marks: 50 No. of Students: 300
End (Spring) Semester Examination 2015 Department: Mathematics
Subject No. MA20106 Subject Name : Probability & Stochastic Processes
II yr B.Tech. EE/E&ECE/IE/MT

Instructions:
(i) Use of calculator and Statistical tables is allowed. All the notations are standard and no query or doubts will be entertained. If any data/statement is missing, identify it in your answer script.
(ii) Answer **All** questions.
(iii) All parts of a question **Must Be** answered at **One Place**.

1. Write **Only Answers on the First Page** of your answer script against each part of question 1. Detail working may be carried out on other pages. [1×8]

(_) Suppose that a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} 2\, e^{-(x+2y)} & \text{if } 0 < x, y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Calculate $P(X < Y)$.

&`' Suppose a bivariate random vector (X, Y) has the joint p.d.f. function

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(X|Y = y)$

&a'. X, Y are jointly distributed discrete random vector with p.m.f.

$$P(X = x, Y = y) = \begin{cases} \frac{2}{n(n+1)} & \text{if } x = 1, 2, \ldots, n; y = 1, 2, \ldots, x. \\ 0 & \text{otherwise} \end{cases}$$

Find $E(Y|X = x)$

&b' Suppose that a random vector (X, Y) has bivariate normal p.d.f with $\mu_X = 5$, $\mu_Y = 10$, $\sigma_X = 1$, $\sigma_Y = 5$ and $\rho > 0$. If $P(4 < X < 16|Y = 5) = 0.954$, find the constant ρ.

(c) Consider the following two state $\{0, 1\}$ markov chain with the transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Find $P(X_{n+3} = 0|X_{n+2} = 1, X_{n+1} = 1, X_n = 0)$.

(d) For the same markov chain in 1(c), find $P(X_{n+2} = 0, X_{n+1} = 1|X_n = 1)$.

(e) Consider a Markov chain with state space $S = \{0, 1, 2\}$ and transition probability matrix (tpm)

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{pmatrix}$$

If it is known that the process starts in state $X_0 = 1$, determine $P(X_0 = 1, X_1 = 0, X_2 = 2)$.

(f) With reference to the problem in 1(e), determine $P(X_2 = 2)$.

2. (a) In a particular dice game, a person bets Rupees 10 and rolls 5 balanced die simultaneously. If the 5 die all show the same number, the person wins Rupees 20; if the 5 die do not all show the same number, the person losses Rupees 10. Suppose that the person plays this game 5 consecutive times. Let the random variable X be the person's gain (in rupees) based on 5 plays of this game. Find the expected gain, that is $E(X)$.

(b) The acidity of a compound X depends on the proportion Y of a certain chemical present in the compound. Suppose $X = (1+Y)^2$. If the density of Y can be assumed to be

$$f(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the PDF of X. Also find $E(X)$ and $\text{Var}(X)$.

(c) Let the random variable X has a density function

$$f(x) = \begin{cases} cx, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the (a) value of c. (b) the CDF of X (c) mean and variance of X (d) median of X, that is find the value m such that $P(X \geq m) = P(X \leq m)$.

[2+2+2]

3. (a) If $X_i,\ i = 1, 2, \ldots, 50$ are independent random variables each having Poisson distribution with mean 0.03 and Let $S_n = \sum_{i=1}^{50} X_i$. Evaluate $P(S_n \geq 3)$ using central limit theorem. Compare your answer with the exact value of the probability.

(b) Let the random variables X has the Poisson distribution with parameter $\lambda = 1$. Find $E\{\frac{1}{(1+X)}\}$.

(c) If a random variable X has a Binomial distribution with parameters $n = 100$ and $p = 0.1$, find the approximate value of $P(12 < X < 14)$ using the normal approximation. Campare this value with the exact value of $P(12 < X < 14)$ by applying continuity correction.

[2+2+2]

4. (a) Let $X \sim NB(r, p)$ (negative binomial with parameters $r \geq 1$ and p, $0 < p < 1$.) Find the moment generating function and variance of X.

(b) State and prove the memoryless property of geometric distribution.

(c) If X is a standard normal variate then find the distribution of $Y = \frac{1}{2}X^2$.

(d) State the assumptions of the Poisson process and write the expression for probability distribution of number of occurrences in the interval [0,t].

[2+2+2+2]

5. (a) Suppose that a bivariate random vector (X, Y) has the bivariate normal distribution with parameters $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ and ρ. Write the expression for the followings. (i) $E(Y|X = x)$ and (ii) $V(Y|X = x)$.

(b) A certain group of college students takes both the SAT and IQ test. Let X and Y denote the students' score on the SAT and IQ tests, respectively. Assume that X and Y have a bivariate normal distribution with parameters $\mu_x = 980, \mu_y = 117, \sigma_x = 126, \sigma_y = 7.2$, and $\rho = 0.58$. Find $P(Y \leq 120|X = 1350)$.

(c) Let X and Y be independent random variables with common pdf $f(x) = e^{-x},\ x > 0$. Find the joint pdf of $U = \frac{X}{X+Y}$ and $V = X + Y$. Also find the marginal density function of U and V respectively.

[2+2+3]

6. (a) A particle moves on a circle through points which have been marked $0, 1, 2, 3, 4$ (in a clockwise order). At each step it has a probability p of moving to the right (clockwise) and $1 - p$ to the left (counterclockwise). Let X_n denote its location on the circle after the n^{th} step.

   (i) Find the one step TPM and equivalence classes.
   (ii) Find the stationary probability distribution.

(b) Three boys B_1, B_2 and B_3 are throwing a ball to each other. B_1 always throws the ball to B_2, and B_2 always throws the ball to B_3. But B_3 is just as likely to throw the ball to B_2 as to B_1. Find the one step TPM, classify the states, that is, find the equivalence classes, peroid of the states, states are recurrent or transient.

(c) Consider the Markov Chain having states $\{0, 1, 2, 3, 4\}$ TPM

$$\begin{bmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Find the equivalence classes, (ii) Find the period of the states, [3+3+3]
(iii) Whether states are transient or recurrent.

7. (a) If customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute.

   (i) Find the probability that exactly 3 customers will arrive during 10 minutes time period.
   (ii) Find the probability that the interval between 2 consecutive arrivals is between 1 and 2 minutes.

(b) A radioactive source emits particles (either with bluish or with white light) at a rate of 6 per minute in accordance with Poissson process. Particles that are emitted with bluish light has a probability $\frac{1}{3}$ and those emitted with white light has probability $\frac{2}{3}$. Find the probability that (i) 5 particles emit in a 7 minute peroid (ii) at least 2 particles emit with bluish light in a 5 minute peroid.

(c) Trials are performed in sequence. If the last two trials were successes, then the next trial is a success with probability 0.8; otherwise the next trial is a success with probability 0.5. In the long run, what proportion of trials are successes?

[2+2+2]

——-The End———