Department of Mathematics, Indian Institute of Technology, Kharagpur
Assignment 2-3, Probability and Statistics, March 2015.
Due:-March 25, 2015.

1. Show that the function

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{x+2}{4} & \text{for } -1 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

is a distribution function of a random variables X. Sketch the graph of F and compute the following : (a) $P(-\frac{1}{2} < X \le \frac{1}{2})$ (b) $P(X = 0)$ (c) $P(X = 1)$ (d) $P(2 < X \le 3)$.

2. Suppose that a random variable X has the probability density function $f(x) = \frac{1}{2}e^{-|x|}, -\infty < x < \infty$. Find the values x_0 such that $F(x_0) = 0.5$.

3. Suppose that a random variable X has the following distribution function:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

Show that the random variable X is of continuous type and find its probability density function.

4. A random variable X has the distribution function

$$F(x) = \frac{1}{\pi}(\frac{\pi}{2} + tan^{-1}x), -\infty < x < \infty$$

where $tan^{-1}x$ is taken in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Find the probability density function of X and compute the probability $P(|X| < 1)$.

5. Consider the function

$$f(x) = \begin{cases} c(2x - x^3) & \text{for } 0 < x < \frac{5}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Check whether the function f is a probability density function? If yes, calculate the constant c.

6. Consider the distribution function

$$F(x) = \frac{1}{1 + e^{-x}}, \quad -\infty < x < \infty.$$

Find its probability density function and show that the graph of the probability density function is symmetric about 0.

7. Find the expected value of a random variable Y with the probability density function

$$f(y) = \frac{1}{2}e^{-|y|}, \quad -\infty < y < \infty.$$

8. Let $0 < p < 1$. A $100p$-th percentile (quantile of order p) of the distribution of a random variable X is a values ψ_p such that $P(X \le \psi_p) \ge p$ and $P(X \ge \psi_p) \ge 1 - p$. Find the 25-th percentile of the distribution with the probability density function

$$f(x) = \begin{cases} 4x^3 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

9. Let X be a random variable with the probability density function f. If m is the unique median of X, show that

$$E(|X-a|) = E(|X-m|) + 2\int_m^a (a-x)f(x)dx$$

for any real number a provided the expectations exist. Hence prove that $E(|X-a|)$ is minimum when $a=m$.

10. Suppose X has the uniform distribution on the interval $[\alpha, \beta]$. Let μ be its mean. Find $E(X^r)$ and $E(X-\mu)^r$ for positive integers $r \geq 1$.

11. Let X be a random variable with the probability density function

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 2-x & \text{for } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Determine the m.g.f of X whenever it exists.

12. Let X be a random variable with the probability density function

$$f(x) = \begin{cases} xe^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine the m.gf. of X whenever it exists.

13. Suppose X is a random variable with the m.g.f

$$M_X(t) = e^{t^2+3t}, \quad -\infty < t < \infty.$$

Find the mean and variance of X.

14. Find the probability density function of the random variable $Y = X^2$ when the random variable X has the uniform density on the interval $[-1, 1]$.

15. Suppose a random variable X has the probability density function

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = X(2-X)$. Find the probability density function of Y.

16. Suppose a random variable θ is uniformly distributed on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Find the probability density function of $R = asin\theta$ where a is a constant.

17. Suppose a random variable X has the standard uniform distribution. Let $F(x)$ be a distribution function which is continuous and strictly increasing. Let F^{-1} denote the inverse of F. Show that $Y = F^{-1}(X)$ has the distribution function $F(x)$.

18. Suppose a random variable X has the probability density function

$$f_X(x) = \frac{e^{-x}}{(1-e^{-x})^2}, \quad -\infty < x < \infty.$$

Show that $Y = \frac{1}{1+e^{-X}}$ has the uniform probability density function on the interval $[0, 1]$.

19. If the m.g.f of a random variable X is $M_X(t) = e^{-6t+32t^2}$, find $P(-4 \leq X < 16)$.

20. If X has $N(\mu, \sigma^2)$ as its distribution, determine the probability density function of $Y = |X - \mu|$. Further prove that $E[Y] = \sigma\sqrt{\frac{2}{\pi}}$.

21. Suppose that the telephone calls arrive at a telephone exchange following a exponential distribution with parameter $\lambda = 5$ per hour. What are the probabilities that the waiting time for a call at the exchange is (a) at least 15 minutes (b) not more than 10 minutes and (c) exactly 5 minutes ?

22. Let X be a random variable with the Gamma probability density function with parameters α and λ. Compute $E[X^m]$ for all $m \geq 1$ and hence find $E[X]$ and $Var(X)$.

23. Suppose X is a random variable with the standard normal distribution. Show that $Y = X^2$ has the Gamma distribution with the parameters $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{2}$.

24. Determine the constant c such that the function

$$f(x) = \begin{cases} cx^3(1-x)^6 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function.

25. Suppose the bivariate probability distribution of random vector (X, Y) is given according to the entries in the following table,

	X		
Y	0	1	2
0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$
1	$\frac{2}{9}$	$\frac{2}{9}$	0
2	$\frac{1}{9}$	0	0

Show that the random variables X and Y are not independent and find $P(X = 0|Y = 2), P(X = 1|Y = 2), P(X = 2|Y = 2)$.

26. Suppose that a bivariate random vector (X, Y) has the joint probability density function

$$f(x, y) = \begin{cases} 2e^{-x}e^{2y} & 0 < x, y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density functions $f_X(x)$ and $f_Y(y)$ of X and Y respectively also find $P[X < Y]$

27. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x, y) = \begin{cases} 2 & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$ of X and Y respectively. Hence find the conditional probabililty density functions $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$. Also find $P(0 < X < \frac{1}{2}|Y = \frac{3}{4})$.

28. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x, y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$ of X and Y respectively. Hence show that X, Y are not independent random variables. Also find $E[X], E[Y], Var(X), Var(Y)$. Hence find $Cov(X, Y)$.

29. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} 1 & \text{if } -y < x < y,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$ of X and Y respectively. Also show that $Cov(X, Y) = 0$.

30. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional probabililty density functions $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$. Hence find $E[Y|X = x]$ and $E[X|Y = y]$.

31. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} 1 & \text{if } -y < x < y,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that the random variable X has linear regression on Y but the random variable Y does not have the linear regression on X.

32. Consider an electronic system with two components. Suppose the system is such that one component is on the reserve and it is activated only if the other component fails. The system fails if and only if both the components fail. Let X and Y denote the life times of these components. Suppose (X, Y) has the joint probalility density function

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that the system will last for more than 500 hours?

33. Suppose a random vector (X, Y) has the joint probability distribution function

$$f(x,y) = \begin{cases} (1 - e^{-\lambda x})(1 - e^{-\lambda y}) & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint probability density function of (X, Y)

34. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} y^2 e^{-y(x+1)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine the marginal density functions of X and Y.

35. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} 6x & 0 < x < 1, 0 < y < 1 - x \\ 0 & \text{otherwise} \end{cases}$$

Determine the marginal density functions of X and Y.

36. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} \frac{12}{5} x(2 - x - y) & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional probability density function of X given $Y = y$ for $0 < y < 1$.

37. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} c(x+y^2) & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the conditional probability density function of X given $Y = y$ for $0 < y < 1$.
(b) Compute $P(X < \frac{1}{2} | Y = \frac{1}{2})$.

38. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

Compute $P(Y < 1 | X < 1)$.

39. Suppose that the conditional probability density function of Y given $X = x$ and the marginal density function of X are given by

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y+4x}{1+4x} & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_X(x) = \begin{cases} \frac{1+4x}{3} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

respectively. Determine the marginal density function of Y.

40. The joint probability density function random vector (X, Y) is given by

$$f(x,y) = \begin{cases} 2e^{-(x+y)} & 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional distribution function of Y given $X = x$.

41. Let X denote the percentage of marks obtained by a student in Mathematics and Y denote the percentage in English in the final examinations. Suppose that the random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 < x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What percentage of the students obtain more than 80% in Mathematics?
(b) If a student has obtained 30% in English, what is the probability that he or she gets more than 80% in Mathematics?
(c) If a student has obatained 30% in Mathematics, what is the probability that he or she gets more than 80% in English?

42. The joint probability density function of a random vector (X, Y) is given by

$$f(x,y) = \begin{cases} 12xy(1-y) & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that X and Y are independent random variables.

43. The joint probability density function random vector (X, Y) is given by

$$f(x,y) = \begin{cases} 4x(1-y) & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine $P(0 < X < \frac{1}{3}, 0 < Y < \frac{1}{3})$.

44. The joint probability density function random vector (X, Y) is given by

$$f(x,y) = \begin{cases} c(x^2 - y^2)e^{-x} & -x \leq y \leq x, 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the constant c and (b) check whether X and Y are independent random variables.

45. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} xe^{(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

46. Determine the correlation coefficient of (X, Y) when the random vector (X, Y) has the joint probability density function is given by

$$f(x,y) = \begin{cases} 2 & x \geq 0, y \geq 0, x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

47. Show that $Cov(aX + b, cY + d) = acCov(X, Y)$ for any random vector (X, Y) with finite covariance and for arbitary constants a, b, c and d.

48. Suppose that U is a random variable uniformly distributed on the interval $[0, 2\pi]$. Define $X = cosU$ and $Y = sinU$. Show that X and Y are uncorrelated. Are X and Y independent.

49. If X has the standard normal distribution and $Y = a + bX + cX^2$ with $b \neq 0$ or $c \neq 0$ then show that $\rho_{X,Y} = \frac{b}{\sqrt{b^2+2c^2}}$

50. If X and Y are independent random variables and g and h are functions such that $E[g(X)]$ and $E[h(Y)]$ are finite, show that $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.

51. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} x + y & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the regression function of Y on X.

52. If (X, Y) is a bivariate normal random vector, then show that $E(Y|X = x) = \mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(x - \mu_X)$ where $\mu_X = E[X].\mu_Y = E[y], \sigma_X^2 = Var(X), \sigma_Y^2 = Var(Y)$.

53. Show that if X and Y are independent random variables, then $E[X|Y = y] = E[X]$.

54. Suppose that a random vector (X, Y) has the bivariate normal probability density function with $\mu_X = 5, \mu_Y = 10, \sigma_X = 1, \sigma_Y = 5$ and $\rho > 0$. If $P(4 < Y < 16|X = 5) = 0.954$, find the constant ρ.

55. Suppose X_1 and X_2 are independent random variables with a common probability density function

$$f(x) = \begin{cases} \frac{1}{2}e^{\frac{-x}{2}} & if 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the joint probability density function of (X_1, X_2). Also find distribution function of $Z_1 = \frac{X_1 - X_2}{2}$.

56. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} 2(x+y) & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the probability density function of $Z = X + Y$.

$$f(x,y) = \begin{cases} (x+y) & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the probability density function of $Z = X + Y$.

57. Suppose that the bivariate random vector (X, Y) has the uniform probability density function on the unit square $[0,1] \times [0,1]$. Find the probability density function of $Z = \frac{X}{Y}$.

58. Suppose that X_1 and X_2 are independent random variables with a common probability density function $f(x)$. Let $F(x)$ be the corresponding distribution function. Find the probability density function of $Z = min(X_1, X_2)$.

59. Suppose that X_1 and X_2 are independent random variables with the Gamma probability density functions $f_i(x), i = 1, 2$ given by

$$f_i(x_i) = \begin{cases} \frac{x_i^{\alpha_i - 1} e^{-x_i}}{\Gamma(\alpha_i)} & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $Z_1 = X_1 + X_2$ and $Z_2 = \frac{X_1}{X_1+X_2}$. Show that Z_1 and Z_2 are independent random variables. Find the distribution functions of Z_1 and Z_2.

60. Suppose a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} 4xy & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Define $Z_1 = \frac{X}{Y}$ and $Z_2 = XY$. Determine the joint probability density function of (Z_1, Z_2).

61. The ideal size of the first year class of students in a college is 150. It is known from an earlier data that on the average only 30% of those accepted for admission to the colege will actually join. Suppose the college accepts admission of 450 students. What is the probability that more than 150 students join in the frist year class?

62. If X is a random variable with $E[X] = 3$ and $E[X^2] = 13$, find a lower bound for $P(-2 \le X \le 8)$.

63. Defects in a particular kind of metal sheet occur at an average rate of one per 100 square meters. Find the probability that two or more defects occur in a sheet of size 40 square meters.

64. In a particular book 520 pages, 390 printing errors were there. Where is the probability that a page selected from this book at random will contain no errors?

65. In a large population, the proportion of people having a certain disease is 0.01. Find the probability that at least four will have the disease in random group of 200 people.

66. If a random variable X has the Binomial distribution with parameters $n = 100$ and $p = 0.1$, find the approximate value of $P(12 < x < 14)$ using (a) the normal approximation, and (b) the Poisson approximation.

67. Two jobs for the execution of some projects are randomly allotted to three companies A, B and C. Let X denotes the number of jobs allotted to A and Y denote the number of jobs allotted to B. Find the joint probability function of the bivariate random vector (X,Y).

68. Let X denote the amount of time a child watches televisions and Y denote the amount spent by the child on studies in a day. Suppose (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} xye^{-\lambda(x+y)}, & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that a child chosen at random spends at least twice as much time watching television as he or she does on studies.

69. The joint probability density function of a random vector (X, Y) is given by

$$f(x,y) = \begin{cases} c(x^2 - y^2)e^{-x} & -x \leq y \leq x, \ 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional distribution function of Y given $X = x$.

70. Prove that $E[E(Y|X)] = E[Y]$ for any random vector (X, Y) whenever the expectations exist.

71. Suppose that a bivariate random vector (X, Y) has the joint probability density function

$$f(x,y) = \begin{cases} y^2e^{-y(x+1)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the regression function of Y on X.

72. For any bivariate random vector (X, Y) and for any two functions $g(\cdot)$ and $h(\cdot)$, prove that

$$E[g(X)h(Y)|X = x] = g(x)E[h(Y)|X = x]$$

with probability one whenever the expectations exist.

73. Suppose that a random vector (X, Y) has the bivariate normal probability density function with $\mu_X = 5, \mu_Y = 10, \sigma_X = 1, \sigma_Y = 5$ and $\rho > 0$. If $P(4 < Y < 16|X = 5) = 0.954$, find the constant ρ.

74. Suppose that a random vector (X, Y) has the bivariate normal density with $\sigma_X = \sigma_Y$. Show that the random variables $X + Y$ and $X - Y$ are independent.

75. Suppose that a random vector (X, Y) has the bivariate probability density function defined by

$$f(x,y) = \frac{1}{2\pi} exp[-\frac{1}{2}(x^2 + y^2)]\{1 + xy \ exp[x^2 + y^2 - 2]\}$$

This function is not a bivariate normal probability density function. Show that the marginal probability density functions of X and Y are normal probability density functions thus establishing the fact that the marginal density functions of X and Y being normal does not imply that the joint distribution is bivariate normal.

76. Suppose that a random vector (X, Y) has the bivariate probability density function defined by

$$f(x,y) = c \ exp[-(\frac{x^2}{2} + xy + y^2 - x - 2y)]$$

Evaluate the constant c.

77. A large lot of items manufactured by a company contains 20% with just one defect, 10% with more than one defect and the rest with no defects. Suppose that n items are randomly selected from the lot. If X_1 denote the number of items with one defect and X_2 denotes the number of items with more than one defect in the sample, the repair costs are $X_1 + 2X_2$. Find the mean and the variance of the repair costs.

78. Let (X, Y) be a bivariate random vector with the uniform distribution on the unit square $[0, 1] \times [0, 1]$. The joint distribution of (X, Y) is

$$f(x, y) = \begin{cases} 1 & \text{if } 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the distribution function of $Z = g(X, Y) = XY$.

79. Let X and Y be two independent random variables with the probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the distribution function of $Z = X + Y$.

80. Suppose X and Y are two independent random variables with the probability distribution function F. Find the distribution function of $Z = max(X, Y)$.

81. Suppose X_1 and X_2 are independent random variables with a common probability density function

$$f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & \text{if } 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the distribution function of $Z_1 = \frac{1}{2}(X_1 - X_2)$.

82. Suppose that the bivariate random vector (X, Y) has the joint probability density function

$$f(x, y) = \begin{cases} 2(x + Y) & \text{for } 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the probability density function of $Z = X + Y$.

83. Suppose that the bivariate random vector (X, Y) has the joint probability density function

$$f(x, y) = \begin{cases} x + y & \text{for } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the probability density function of $Z = X + Y$.

84. Suppose that the bivariate random vector (X, Y) has the uniform probability density function on the unit square $[0, 1] \times [0, 1]$. Find the probability density function of $Z = \frac{X}{Y}$.

85. Suppose that X_1 and X_2 are independent random variables with a common probability density function $f(x)$. Let $F(x)$ be the corresponding distribution function. Find the probability density function of $Z = min(X_1, X_2)$.

86. Suppose that X_1 and X_2 are independent random variables with the Gamma probability density functions $f_i(x)$, $i = 1, 2$ given by

$$f_i(x_i) = \begin{cases} \frac{x_i^{\alpha_i - 1} e^{-x_i}}{\Gamma(\alpha_i)} & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $Z_1 = X_1 + X_2$ and $Z_2 = \frac{X_1}{X_1 + X_2}$. Show that Z_1 and Z_2 are independent random variables. Find the distribution functions of Z_1 and Z_2.

87. Let X_1 and X_2 be two independent random variables uniformly distributed on the interval $[0, 1]$. Define

$$Z_1 = (-2\log X_1)^{\frac{1}{2}} cos(2\pi X_2)$$
$$Z_2 = (-2\log X_1)^{\frac{1}{2}} sin(2\pi X_2)$$

Show that the random variables Z_1 and Z_2 are independent standard normal random variables. This transformation gives a method for generating observations from a standard normal distribution from those of a standard uniform distribution.

88. Let X_1 and X_2 be independent standard normal random variables. Define $Z_1 = \frac{X_1}{X_2}$. Show that the random variable Z_1 has the standard Cauchy probability density function defined by

$$f_{Z_1}(z_1) = \frac{1}{\pi(1+z_1^2)}, \quad -\infty < z_1 < \infty.$$

89. Suppose that the bivariate random vector (X, Y) has the joint probability density function

$$f(x, y) = \begin{cases} 4xy & \text{for } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Define $Z_1 = \frac{X}{Y}$ and $Z_2 = XY$. Determine the joint probability density function of (Z_1, Z_2).

90. If X is a random variable with $E(X) = \mu$ and $Var(X) = 0$, show that $P(X = \mu) = 1$.

91. If X is a random variable with $E(X) = \mu$ and $Var(X) = \sigma^2$, find an upper bound for $P(|X - \mu| \geq 3\sigma)$.

92. If X is a random variable with $E(X) = 3$ and $E(X^2) = 13$, find a lower bound for $P(-2 \leq X \leq 8)$.

93. Suppose that X is a random variable with the exponential probability density function given by

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Compute $E(X) = \mu$ and $Var(X) = \sigma^2$. Compute an upper bound for $P(|X - \mu| \geq 2\sigma)$ using the Chebyshev's inequality and compare it with the exact probability obtained from the distribution of X.

94. How large the size of a random sample should be, from a population with mean μ and finite variance σ^2, in order that the probability that the sample mean will be within 2σ limits of the population mean μ is at least 0.99?

95. Defects in a particular kind of a metal sheet occur at an average rate of one per 100 square meters. Find the probability that two or more defects occur in a sheet of size 40 square meters.

96. In a particular book of 520 pages, 390 printing errors were there. What is the probability that a page selected from this book at random will contain no errors?

97. In a large population, the proportion of people having a certain disease is 0.01. Find the probability that at least four will have the disease in a random group of 200 people.

98. If ten fair dice are rolled, find the approximate probability that the sum of the numbers observed is between 30 and 40.

99. Suppose $X_i,\ 1 \leq i \leq 10$ are independent random variables each uniform on $[0,1]$. Determine an approximation to $P(X_1 + \cdots + X_{10} > 6)$.

100. Show that if a random variable Y has the Poisson distribution with parameter λ, then $\frac{Y-\lambda}{\sqrt{\lambda}}$ has approximately a standard normal distribution when λ is large.

101. Let $X_1, ..., X_{15}$ be a random sample of size 15 from a population with the probability density function

$$f(x) = \begin{cases} 3(1-x)^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the approximate probability for the event $\frac{1}{8} < \bar{X} < \frac{3}{8}$.

102. If a random variable X has the Binomial distribution with $n = 100$ and $p = \frac{1}{2}$, find an approximation for $P(X = 50)$.

103. If a random variable X has the Binomial distribution with parameters n and $p = 0.55$, determine the smallest value of n for which $P(X > \frac{n}{2}) \geq 0.95$ approximately.

104. If a random variable X has the Binomial distribution with parameters $n = 100$ and $p = 0.1$, find the approximate value of $P(12 < X < 14)$ using (a) the normal approximation, and (b) the Poisson approximation.