Department of Mathematics, Indian Institute of Technology, Kharagpur
Assignment 1, Probability and Stochastic Processes (MA 20106).

Due : Feb-10, 2015

1. A coin is tossed until for the first time the same result appear twice in succession. To an outcome requiring n tosses assign a probability 2^{-n}. Describe the sample space. Evaluate the probability of the following events:
a) A = The experiment ends before the sixth toss.
b) B = An even number of tosses are required.
c) $A \cup B, A \cap B, A \cap B', A' \cap B', A' \cap B$.

2. Three tickets are drawn randomly without replacement from a set of tickets numbered 1 to 100. Show that the probability that the number of selected tickets are in (i) arithmetic progression is $\frac{1}{66}$ and (ii) geometric progression is $\frac{105}{\binom{100}{3}}$.

3. Three players A, B and C play a series of games, none of which can be drawn and their probability of wining any game are equal. The winner of each game scores 1 point and the series is won by the player who first scores 4 points. Out of first three games A wins 2 games and B wins 1 game. What is the probability that C will win the series.

4. Urn I contains 3 black and 5 red balls and urn II contains 4 black and 3 red balls. One urn is chosen randomly and a ball is drawn randomly which is red. Find the probability that urn I was chosen.

5. A point P is randomly placed in a square with side of 1 cm. Find the probability that the distance from P to the nearest side does not exceed x cm.

6. If six dice are rolled find the probability that at least two faces are equal.

7. Suppose that in answering a question on multiple choice test an examinee either knows the answer or he/she guesses. Let p be the probability that he will know the answer and let $1 - p$ be the probability that he/she will guess. Assume that the probability of answering a question correctly is 1 for an examinee who knows the answer and $\frac{1}{m}$ for an examinee who guesses; here m is the multiple choice alternatives. Find the conditional probability that an examinee knew the answer to a question, given that he has correctly answered it.

8. Let A and B be two events. Assume $P(A) > 0$ and $P(B) > 0$. Prove that (i) if A and B are mutually exclusive ($A \cap B = \phi$), then A and B are independent; (ii) if A and B are independent then A and B are not mutually exclusive.

9. Let A, B and C be independent events. In terms of P(A), P(B) and P(C), express, for $k = 0, 1, 2, 3$
(i) P(exactly k of the events A, B, C will occur),
(ii) P(atleast k of the events A, B, C will occur),
(iii) P(atmost k of the events A,B,C will occur).

10. Let A and B be two independent events such that $P(A \cap B) = \frac{1}{6}$. (i) If P(neither of A and B occurs)=$\frac{1}{3}$, find P(A) and P(B) (ii) if P(A occurs and B does not occur)=$\frac{1}{3}$, find P(A) and P(B). For either part (i) and (ii), are P(A) and P(B) uniquely determined.

11. If A and B are independent events, show that $\bar{A}$ and $\bar{B}$ are also independent events.

12. Two fair dice labelled I and II are thrown simultaneously and outcomes of the top faces are observed. Let
A = Event that die I shows an even number
B = Event that die II shows an odd number
C = Sum of the two faces are odd.
Are A, B and C independent events?

13. Verify, whether or not the following functions can serve as p.m.f.
(a) $f(x) = \frac{x-2}{2}$ for $x = 1, 2, 3, 4$
(b) $f(x) = \frac{e^{\lambda}\lambda^{x}}{x}$ for $x = 1, 2, 3, ...$, (i) $\lambda > 0$, (ii) $\lambda < 0$

14. A battery cell is labelled as good if it works for at least 300 days in a clock, otherwise it is labelled as bad. Three manufactures, A, B and C make cells with probability of making good cell as 0.95, 0.90 and 0.80 respectively. Three identical clocks are selected and cells made by A, B and C are used in clock number 1,2 and 3 respectively. Let X be the total number of clocks working after 300 days. Find the probability mass function (p.m.f) ox X.

15. A fair die rolled independently three times. Define

$$X_i = \begin{cases} 1 & \text{if the i-th roll yields a perfect square} \\ 0 & \text{otherwise} \end{cases}$$

Find the p.m.f of X_i. Suppose $Y = X_1 + x_2 + X_3$. Find the p.m.f. of Y and also it's d.f. Find the mean and variance of Y. Verify Chebyshev's inequality in this case.

16. For what values of k, $f_X(x) = (1-k)k^x, x = 0, 1, 2, 3....$ can serve as a p.m.f. of a random variable X. Find the mean and variance of X.

17. Let

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - \frac{2}{3}e^{-\frac{x}{3}} - \frac{1}{3}e^{-[\frac{x}{3}]} & \text{for } x > 0 \end{cases}$$

where $[a]$ means the largest integer $\le a$. Show that $F(x)$ is a d.f. Determine (i) $P(X > 6)$, (ii) $P(X = 5)$, (iii) $P(5 \le X \le 8)$.

18. The daily water consumption X (in million of litres) is a random variable with p.d.f.

$$f_X(x) = \frac{x}{9}e^{-\frac{x}{3}}, \quad x > 0.$$

(a) Find the d.f., $E(X)$ and $V(X)$.
(b) Find the probability that on a given day, the water consumption is not more than 6 million litres.

19. A mode of a random variable X of the continuous and discrete type is value that maximizes the probability density function (p.d.f) or the probability mass function (p.m.f.) $f(x)$. If there is only one such x, it is called the mode of the distribution. Find the mode of each of the following distribution:
$f(x) = 12x^2(1-x), 0 < x < 1$ and 0 elsewhere.

20. If A_1, A_2, A_3, A_4, A_5 are independent events then show that $A_1 \cup A_3$, $A_2 \cap A_4$ and A_5^c are independent.

21. A secretary writes four letters and the corresponding addresses on envelopes. If he inserts the letters in the envelopes at random irrespective of address, then calculate the probability that all the letters are wrongly placed.

22. There are two identical urns containing 4 white and 3 red balls; 3 white and 7 red balls. A urn is chosen at random and a ball is drawn from it. Find the probability that the ball is white. What is the probability that it is from the first urn if the ball drawn is white?

23. Suppose a fair coin is tossed six times. What is the probability of the event that at least 5 heads will appear?

24. Suppose that five missiles were fired against an aircraft carrier in an ocean and it takes at least two direct hits to sink the ship. All the five missiles are on the correct trajectory but must get through the guns of the ship. It is known that the guns of the ship can destroy a missile with probability 0.9. What is the probability that the ship will still be afloat after the encounter?

25. A committee of three people is selected from a group of six people A, B, C, D, E, and F.Find the conditional probability that A and B are selected given that C and D are not selected.

26. An almirah contains four black,six brown and two black socks. Two socks are chosen at random from the almirah. Find the probability that the socks chosen will be of same colour.

27. Suppose a person makes 100 cheque transaction during a certain period. In balancing the cheque book, suppose he rounds off the entries to the nearest integer. Find an upper bound to the probability that total error committed exceeds 5 after 100 transactions.

28. In a college, there are 40% girls in the first year, 30% in the second year and 30% in the third year. It is observed that 30% in the third year. It is observed that 30% of all the first year girl students, 30% of the second year girl students and 50% of the third year girl students are enrolled in a computer course. What is the probability that a girl chosen at random is enrolled in a computer course?

29. A student taking a true-false test marks the correct answer to a question when he or she knows it decides true or false on the basis of tossing a fair coin when the answer is not known. Suppose the probability that the student knows the answer knows the answer for a question is $\frac{3}{5}$. What is the probability that the student knew the answer to a question to a correctly marked question?

30. Suppose a fair coin is tossed twice. Let E be the event "not more than one head" and F be the event "at least one of each face". Show that the events E and F are not independent.

31. If E and F are independent events, show that E and $\bar{F}$, $\bar{E}$ and F, and $\bar{E}$ and $\bar{F}$ from paris of independent events.

32. A shot is fired from each of three guns numbered 1,2 and 3 in a shooting competition. Let E_i be the event that the target is hit by the $i-$the gun. Suppose that $P(E_1) = 0.5, P(E_2) = 0.6$ and $P(E_3) = 0.8$. Assuming that the events $E_i, i = 1, 2, 3$ are independent, find the probability that exactly one gun hits the target.

33. Suppose the probability that an engine of an automobile does not work during any one hour period is $p = 0.02$. Find the probability that a given engine will work for 2 hours.

34. It was known that 60% of the household in region prefer to purchase a particular brand A of televisions. If a group of randomly selected households in the same region is interviewed, what is the probability that exactly five have to be interviewed to select the first house who prefer to purchase the television of brand A? What is the probability that at least five to be interviewed to select the first house who prefer to purchase the television of brand A?

35. Suppose that radioactive particles strike a target according to Poisson process at an average rate of 3 particles per minute. What is the probability that 10 or more particles will strike the target in particular 2-minute period?

36. A telephone exchange receives calls at an average rate of 16 per minute. If it can handle at most 24 calls per minute, what is the probability that in any one minute the switch board of the exchange will saturate?

37. Find the probability that in a family with five children girls out-number boys assuming that births are independent trials each with probability of boy equal to $\frac{1}{2}$.

38. In a 20-question true/false test, a student tosses a fair coin to determine the answer for each question. If the coin falls heads, he/she answers it is true and if it falls tails, he/she answers it as false. Find the probability that the student answers at least 12 questions correctly.

39. What is the probability of throwing exactly nine heads twice in five throws of ten fair coins?

40. Ten independent binary pulses per second arrive at a receiver. A zero received as a one and a one received as a zero is considered as an error. Suppose the probability of an error is .001. What is the probability of at least one error per second?

41. An oil company is allowed to drill a succession of holes in a given area to find a productive well. The probability for successful driling in a trial is 0.2. What is the probability that the third hole drilled is the first to give a productive well? If the company can drill a maximum of ten wells, what is the probability that it will fail to discover a productive well?

42. Suppose that the number of car accidents during the weekend at a certain intersection in a city has a Poisson distribution with mean 0.7. What is the probability for at least three accidents during a weekend?

43. Show that the function is a distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{x+2}{4} & \text{for } -1 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

44. Suppose that a random variable X has the following distribution function:

$$F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \frac{2}{5}x & \text{for } 0 < x \le 1 \\ \frac{3}{5}x - \frac{1}{5} & \text{for } 1 < x \le 2 \\ 1 & \text{for } x > 2 \end{cases}$$

Show that the random variable X is of continuous type and find its probability density function.

45. Buses arrive at a particular bus stop at 15-minute intervals starting at 8.00 a.m. If a passenger arrives at the stop at a time that is uniformly distributed between 8.00 a.m. and 8.30 a.m., find the probability that he or she waits for less than 5 minutes for the arrival of a bus.

46. Suppose the speed of a molecule in a uniform gas at equilibrium is a random variable with the probability density function

$$f(x) = \begin{cases} cx^2 e^{gx^2} & \text{for } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

where $g = \frac{m}{2kT}$ and k, T and m denote the Boltzmann constant, absolute temperature and the mass of the molecule respectively. Find the constant c in terms of g.

47. Let $0 < p < 1$. A $100p$-th percentile (quantile of order p) of the distribution of a random variable X is a value ψ_p such that $P(X \leq \psi_p) \geq p$ and $P(X \geq \psi_p) \geq 1 - p$. Find the 25-th percentile of the distribution with the probability with the probability density function

$$f(x) = \begin{cases} 4x^3 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

48. Compute the expectation and the variance of a random variable Y whose p.d.f. is given by

$$f(y) = \begin{cases} 1 - |y| & \text{if } -1 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

49. Determine the mean of a random variable U whose distribution function is given by

$$F(u) = \begin{cases} 1 - cosu & \text{for } 0 \leq u \leq \frac{\pi}{2} \\ 0 & \text{for } u < 0 \\ 1 & \text{for } u > \frac{\pi}{2} \end{cases}$$

50. Find the variance of a random variable X with the probability density function

$$f(u) = \begin{cases} \frac{3}{4} & \text{for } 0 \leq x \leq 1 \\ \frac{1}{4} & \text{for } 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

51. Let X be a random variable with the probability density function

$$f(x) = \begin{cases} \frac{2}{x^3} & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that the $E(X)$ is finite but the $Var(X)$ is not finite.

52. A *mode* of the distrbution of a random variable with the probability density function $f(x)$ is a value x_0 of x at which f is maximum. If there is only one such value x_0, then x_0 is called the mode of the density f and the density is said to be *unimodal*. Find the mode of the probability density function

$$f(x) = \begin{cases} \frac{1}{2}x^2 e^{-x} & \text{for } 0 \leq x \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

53. A *median* of the distribution of a random variable X is a value m such that $P(X \leq m) \geq \frac{1}{2}$ and $P(X \geq m) \geq \frac{1}{2}$. If there is only one such value m, then it is the unique median of the distribution. Find the median corresponding to the p.d.f.

$$f(x) = \frac{1}{\pi(1 + x^2)}, \qquad -\infty < x < \infty$$

54. Suppose X is a random variable with the m.g.f.

$$M_X(t) = e^{t^2 + 3t}, \qquad -\infty < t < \infty.$$

Find the mean and variance of X.

55. Find the probability density function of the random variable $Y = X^2$ when the random variable X has the uniform density on the interval [-1,1].

56. If X is a random variable with a distribution function $F(x)$ which is continuous and strictly increasing, show that $Y = F(X)$ has a uniform distribution on the interval [0,1].

57. If a random variable X has standard normal distribution, find $(a) P(0 \leq X \leq 0.87), (b) P(-2.64 \leq X \leq 0), (c) P(-2.13 \leq X \leq -0.56), (d) P(|X| > 1.39)$.

58. If the m.g.f of a random variable X is $M_X(t) = e^{-6t+32t^2}$, find $P(-4 \leq X < 16)$.

59. Show that if a random variable X has a normal distribution with mean zero and variance σ^2, so does -X.

60. If X has $N(\mu, \sigma^2)$ as its distribution, determine the probability density function of $Y = |X - \mu|$. Further prove that $E[Y] = \sigma\sqrt{\frac{2}{\pi}}$

61. If X has $N(\mu, \sigma^2)$ as its distribution, determine the probability density function of $Y = \frac{(X-\mu)^2}{\sigma^2}$.

62. If X has $N(5, 10)$ as its distribution, find $P(0.04 < (X - 5)^2 < 38.4)$.

63. A random variable X is said to have the *log-normal* distribution if $logX$ has the normal distribution. Suppose $logX$ has $N(\mu, \sigma^2)$ as its distribution. Show that $E[X] = \exp(\mu + \frac{\sigma^2}{2})$ and $Var(X) = \{\exp(\sigma^2) - 1\} \exp(2\mu + \sigma^2)$.

64. Suppose X is a random variable with Gamma probability density function with parameters α and λ and $E(X) = 2$ and $Var(X) = 7$. Find the parameters α and λ.

65. Let X_1, X_2, X_3, X_4 be i.i.d. $N(100, 25)$. What is the distribution of $Y = X_1 - 2X_2 + X_3 - 3X_4$?

66. Find the value of K such that

$$f(x) = \begin{cases} Kx(1-x) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function and compute $P(X > \frac{1}{2})$.

$$f(x) = \begin{cases} 0.5 & \text{if } x = -1 \\ \frac{1}{2}e^{-x} & \text{if } 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Determine the m.g.f of X