Shortest Paths III: All-pairs Shortest Paths, Dynamic Programming, Matrix Multiplication, Floyd-Warshall, Johnson

Lecture 16

Shortest paths

Single-source shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm — $O(E + V \lg V)$
- General
  - Bellman-Ford — $O(VE)$
- DAG
  - One pass of Bellman-Ford $O(V + E)$

All-pairs shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm $|V|$ times — $O(VE + V^2 \lg V)$
- General
  - Three algorithms today.

All-pairs shortest paths

Input: Digraph $G = (V, E)$, where $|V| = n$, with edge-weight function $w : V \to \mathbb{R}$.

Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.

IDEA #1:

- Run Bellman-Ford once from each vertex.
- Time $= O(V^2 E)$.
- Dense graph $\Rightarrow$ $O(V^4)$ time.

Good first try!

Dynamic programming

Consider the $n \times n$ adjacency matrix $A = (a_{ij})$ of the digraph, and define

$d_{ij}^{(m)}$ = weight of a shortest path from i to j that uses at most m edges.

Claim: We have

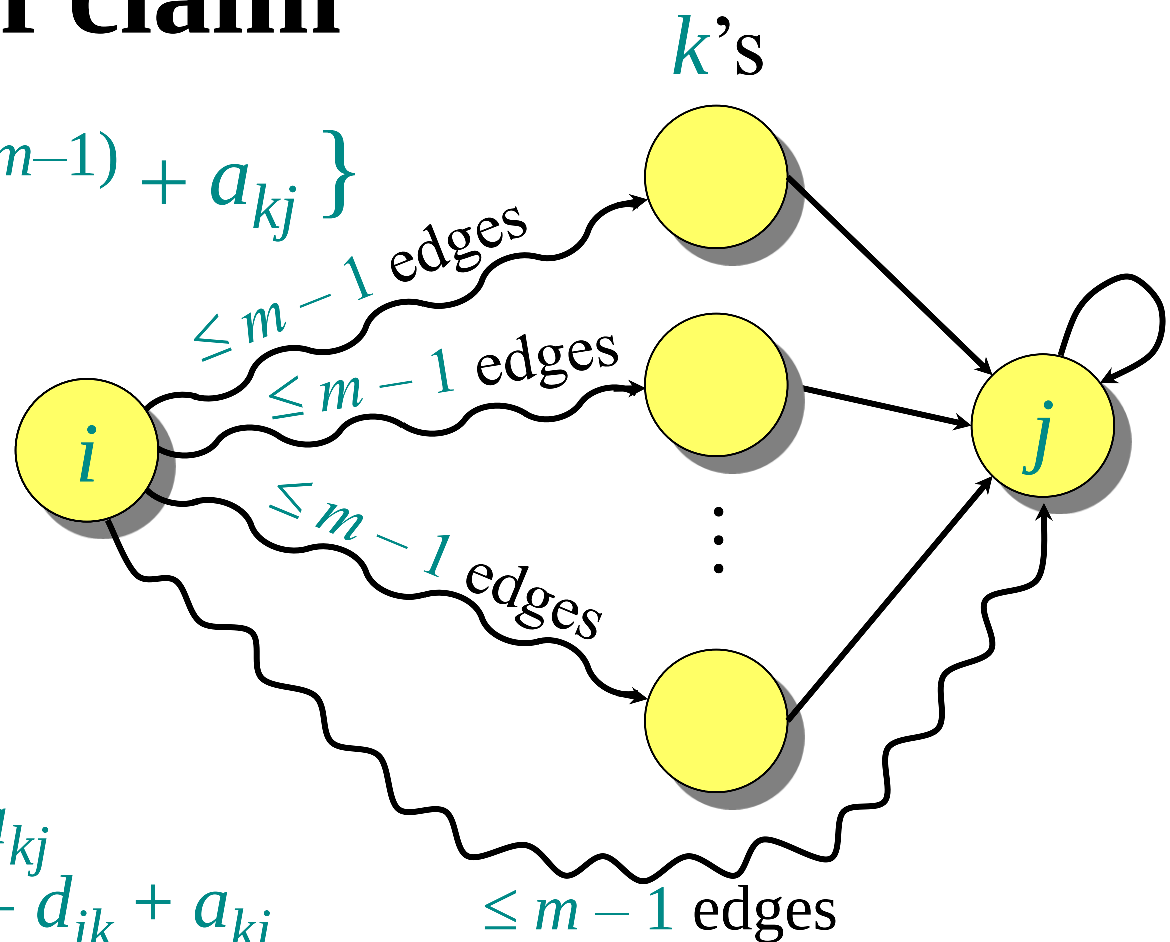
$$d_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j, \\ \infty & \text{if } i \neq j; \end{cases}$$

and for $m = 1, 2, \ldots, n - 1$,

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}.$$

Proof of claim

$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}$



Relaxation!

for $k \leftarrow 1$ **to** n

  do if $d_{ij} > d_{ik} + a_{kj}$

    then $d_{ij} \leftarrow d_{ik} + a_{kj}$

Note: No negative-weight cycles implies

$$\delta(i, j) = d_{ij}^{(n-1)} = d_{ij}^{(n)} = d_{ij}^{(n+1)} = \cdots$$

Matrix multiplication

Compute $C = A \cdot B$, where C, A, and B are $n \times n$ matrices:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

Time $= \Theta(n^3)$ using the standard algorithm.

What if we map "$+$" $\rightarrow$ "min" and "$\cdot$" $\rightarrow$ "$+$"?

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\}.$$

Thus, $D^{(m)} = D^{(m-1)}$ "$\times$" A.

Identity matrix $= \mathrm{I} = \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} = D^0 = (d_{ij}^{(0)})$.

Matrix multiplication (continued)

The (min, +) multiplication is ***associative***, and with the real numbers, it forms an algebraic structure called a ***closed semiring***.

Consequently, we can compute

$$
\begin{aligned}
D^{(1)} &= D^{(0)} \cdot A = A^1 \\
D^{(2)} &= D^{(1)} \cdot A = A^2 \\
&\vdots \\
D^{(n-1)} &= D^{(n-2)} \cdot A = A^{n-1},
\end{aligned}
$$

yielding $D^{(n-1)} = (\delta(i, j))$.

Time $= \Theta(n \cdot n^3) = \Theta(n^4)$. No better than $n \times$ B-F.

Improved matrix multiplication algorithm

Repeated squaring: $A^{2k} = A^k \times A^k$.

Compute $\underbrace{A^2, A^4, \ldots, A^{2^{\lceil \lg(n-1) \rceil}}}_{O(\lg n) \text{ squarings}}$.

Note: $A^{n-1} = A^n = A^{n+1} = \cdots$.

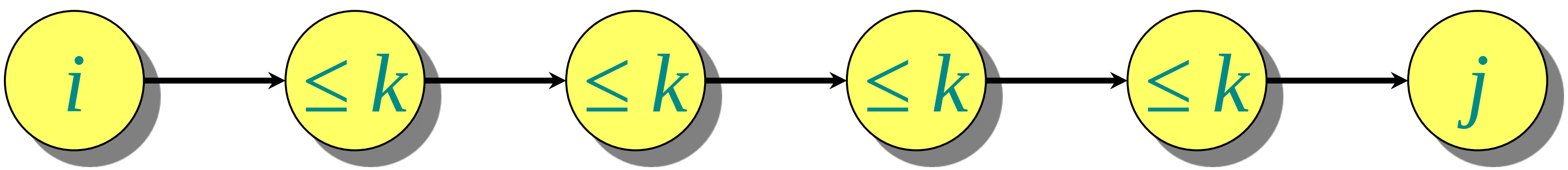
Time $= \Theta(n^3 \lg n)$.

To detect negative-weight cycles, check the diagonal for negative values in $O(n)$ additional time.

Floyd-Warshall algorithm

Also dynamic programming, but faster!

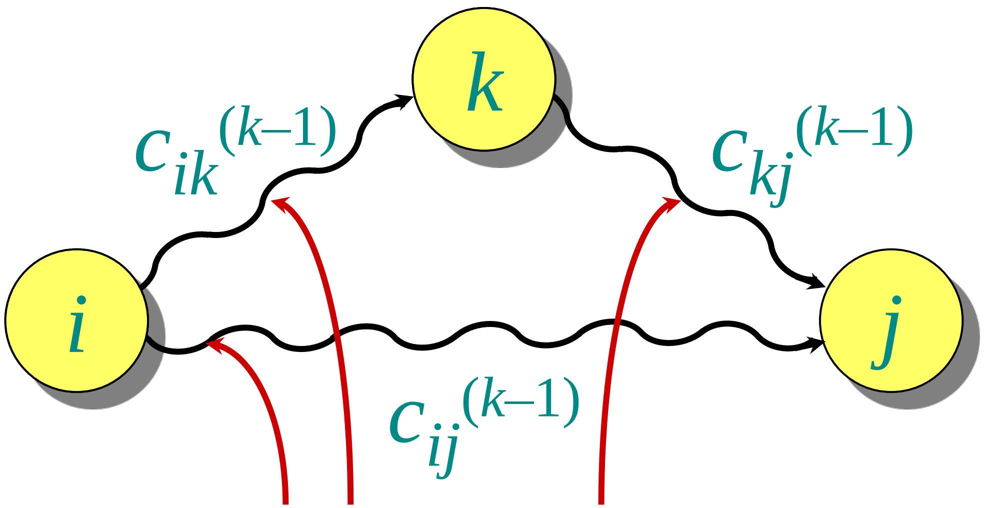
Define $c_{ij}^{(k)}$ = weight of a shortest path from i to j with intermediate vertices belonging to the set $\{1, 2, \ldots, k\}$.



Thus, $\delta(i, j) = c_{ij}^{(n)}$. Also, $c_{ij}^{(0)} = a_{ij}$.

Floyd-Warshall recurrence

$$c_{ij}^{(k)} = \min_k \{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$



intermediate vertices in $\{1, 2, \ldots, k\}$

Pseudocode for Floyd-Warshall

for $k \leftarrow 1$ **to** n
    do for $i \leftarrow 1$ **to** n
        do for $j \leftarrow 1$ **to** n
            do if $c_{ij} > c_{ik} + c_{kj}$
                then $c_{ij} \leftarrow c_{ik} + c_{kj}$ } ***relaxation***

Notes:

- Okay to omit superscripts, since extra relaxations can't hurt.
- Runs in $\Theta(n^3)$ time.
- Simple to code.
- Efficient in practice.

Transitive closure of a directed graph

Compute $t_{ij} = \begin{cases} 1 & \text{if there exists a path from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$

IDEA: Use Floyd-Warshall, but with $(\vee, \wedge)$ instead of $(\min, +)$:

$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}).$$

Time $= \Theta(n^3)$.

Graph reweighting

Theorem. Given a label $h(v)$ for each $v \in V$, ***reweight*** each edge $(u, v) \in E$ by

$$\hat{w}(u, v) = w(u, v) + h(u) - h(v).$$

Then, all paths between the same two vertices are reweighted by the same amount.

Proof. Let $p = v_1 \to v_2 \to \cdots \to v_k$ be a path in the graph.

Then, we have
$$\begin{aligned}
\hat{w}(p) &= \sum_{i=1}^{k-1} \hat{w}(v_i, v_{i+1}) \\
&= \sum_{i=1}^{k-1} \left(w(v_i, v_{i+1}) + h(v_i) - h(v_{i+1}) \right) \\
&= \sum_{i=1}^{k-1} w(v_i, v_{i+1}) + h(v_k) - h(v_1) \\
&= w(p) + h(v_k) - h(v_1). \quad \square
\end{aligned}$$

Johnson's algorithm

1. Find a vertex labeling h such that $\hat{w}(u, v) \geq 0$ for all $(u, v) \in E$ by using Bellman-Ford to solve the difference constraints
$$h(v) - h(u) \leq w(u, v),$$
or determine that a negative-weight cycle exists.
   - Time $= O(VE)$.
2. Run Dijkstra's algorithm from each vertex using $\hat{w}$.
   - Time $= O(VE + V^2 \lg V)$.
3. Reweight each shortest-path length $\hat{w}(p)$ to produce the shortest-path lengths $w(p)$ of the original graph.
   - Time $= O(V^2)$.

Total time $= O(VE + V^2 \lg V)$.