Shortest Paths I: Properties, Dijkstra's Algorithm

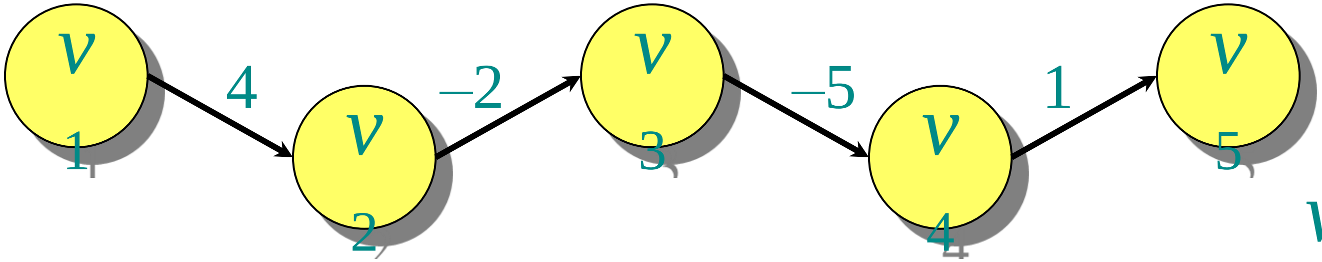
Lecture 14

Paths in graphs

Consider a digraph $G = (V, E)$ with edge-weight function $w : E \to \mathbb{R}$. The ***weight*** of path $p = v_1 \to v_2 \to \cdots \to v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

Example:



$w(p) = -2$

Shortest paths

A ***shortest path*** from u to v is a path of minimum weight from u to v. The ***shortest-path weight*** from u to v is defined as

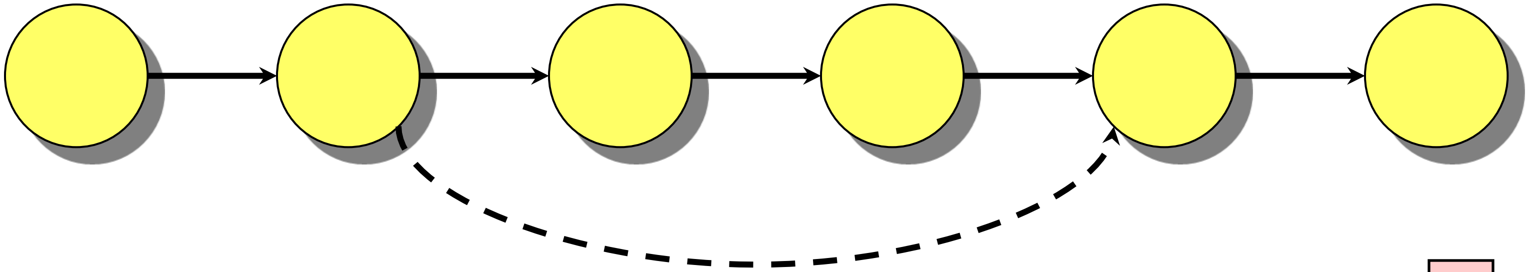
$$\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}.$$

Note: $\delta(u, v) = \infty$ if no path from u to v exists.

Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.

Proof. Cut and paste:



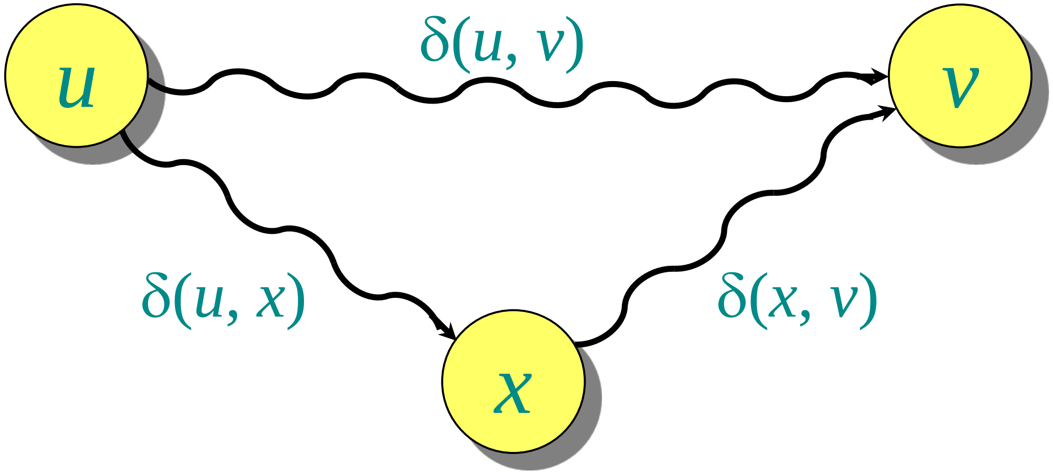
□

Triangle inequality

Theorem. For all $u, v, x \in V$, we have

$$\delta(u, v) \le \delta(u, x) + \delta(x, v).$$

Proof.

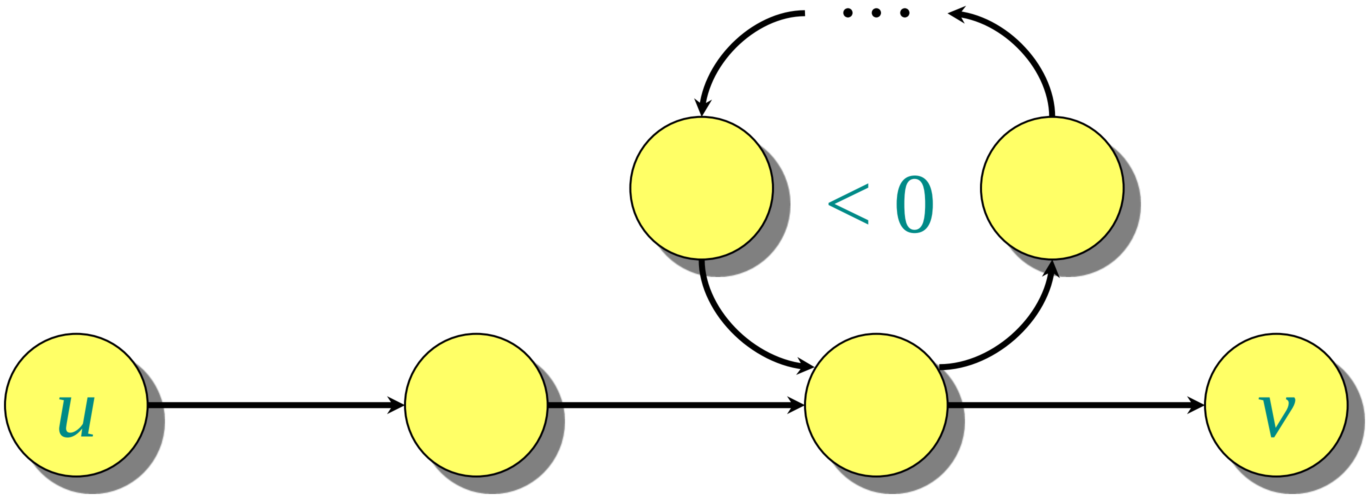


□

Well-definedness of shortest paths

If a graph G contains a negative-weight cycle, then some shortest paths may not exist.

Example:



Single-source shortest paths

Problem. From a given source vertex $s \in V$, find the shortest-path weights $\delta(s, v)$ for all $v \in V$.

If all edge weights $w(u, v)$ are *nonnegative*, all shortest-path weights must exist.

IDEA: Greedy.

1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step add to S the vertex $v \in V - S$ whose distance estimate from s is minimal.
3. Update the distance estimates of vertices adjacent to v.

Dijkstra's algorithm

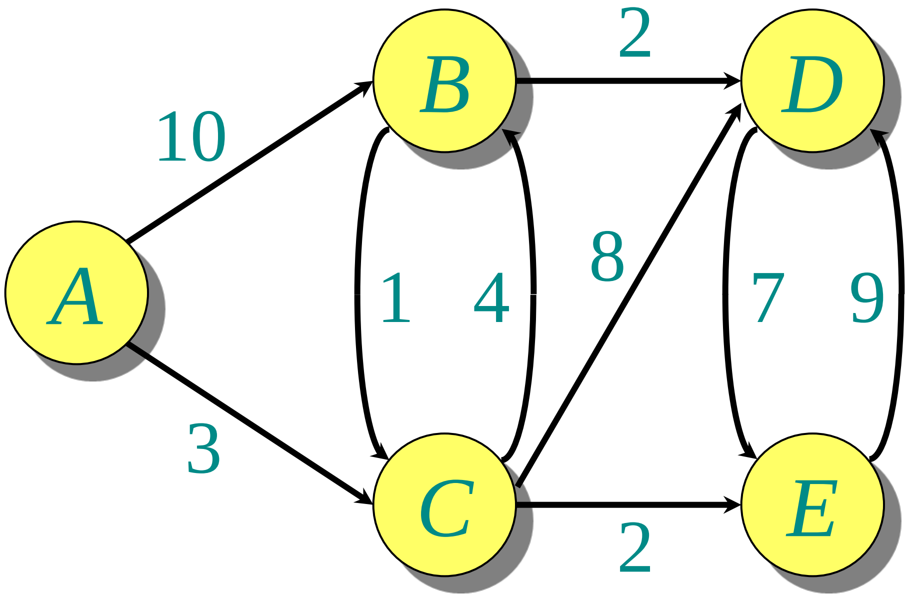
```
$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$
$S \leftarrow \emptyset$
$Q \leftarrow V$            ⊳ $Q$ is a priority queue maintaining $V - S$
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN$(Q)$
       $S \leftarrow S \cup \{u\}$
       for each $v \in Adj[u]$
           do if $d[v] > d[u] + w(u, v)$
                 then $d[v] \leftarrow d[u] + w(u, v)$
```

relaxation step

Implicit DECREASE-KEY

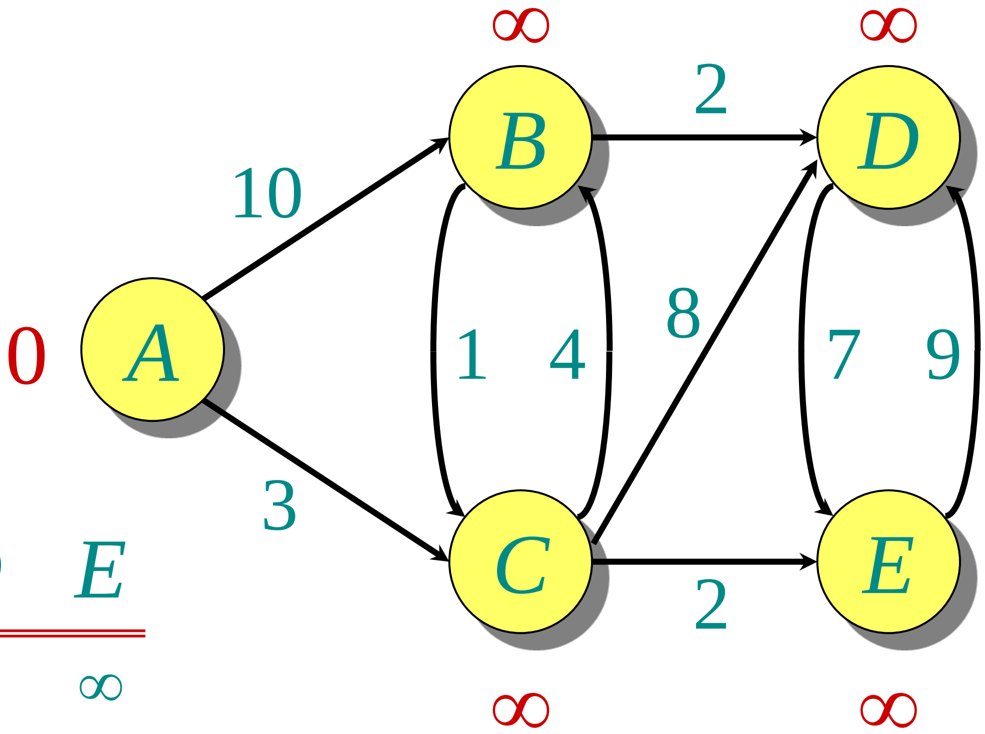
Example of Dijkstra's algorithm

Graph with nonnegative edge weights:



Example of Dijkstra's algorithm

Initialize:

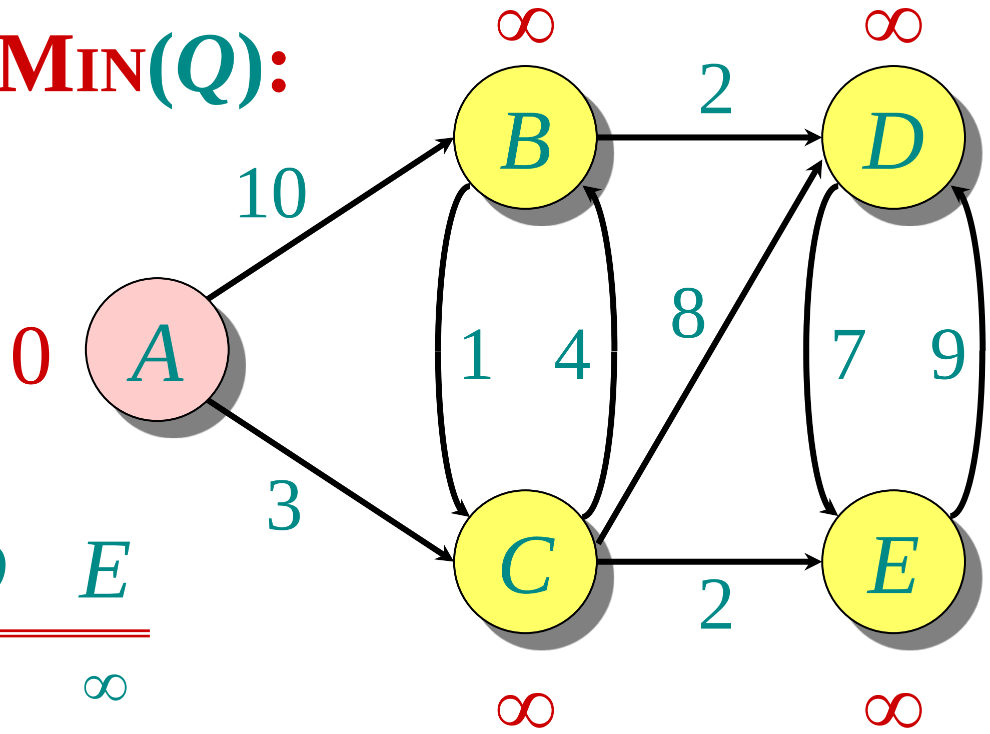


Q:	A	B	C	D	E
	0	∞	∞	∞	∞

S: { }

Example of Dijkstra's algorithm

"*A*" ← EXTRACT-MIN(*Q*):

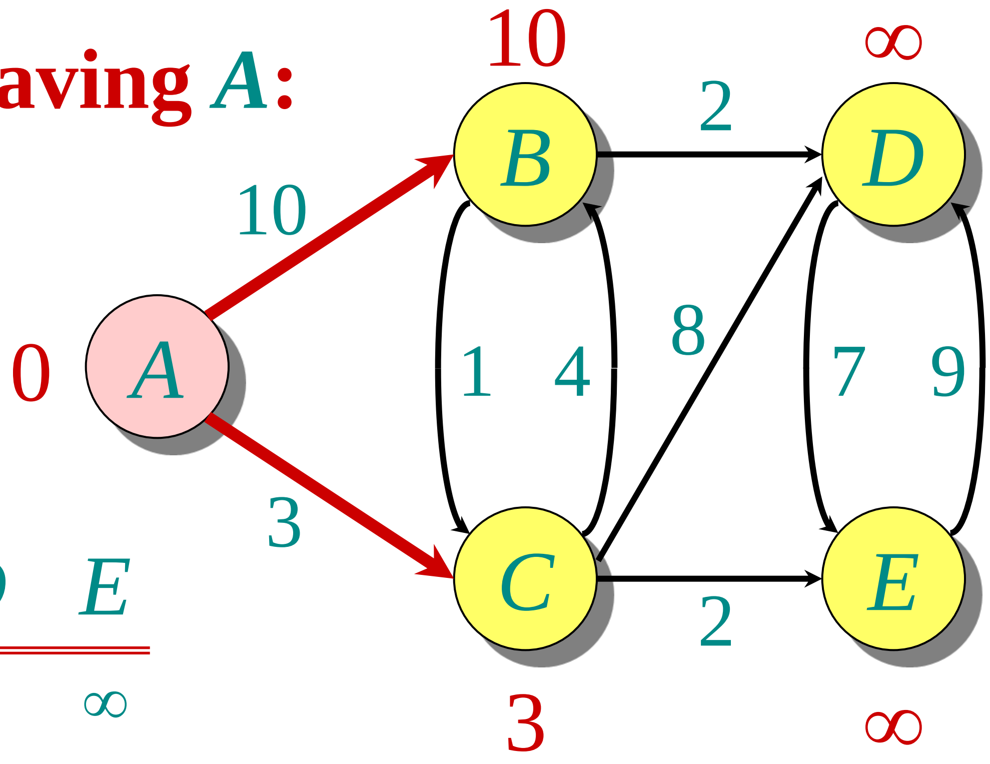


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞

S: { *A* }

Example of Dijkstra's algorithm

Relax all edges leaving *A*:

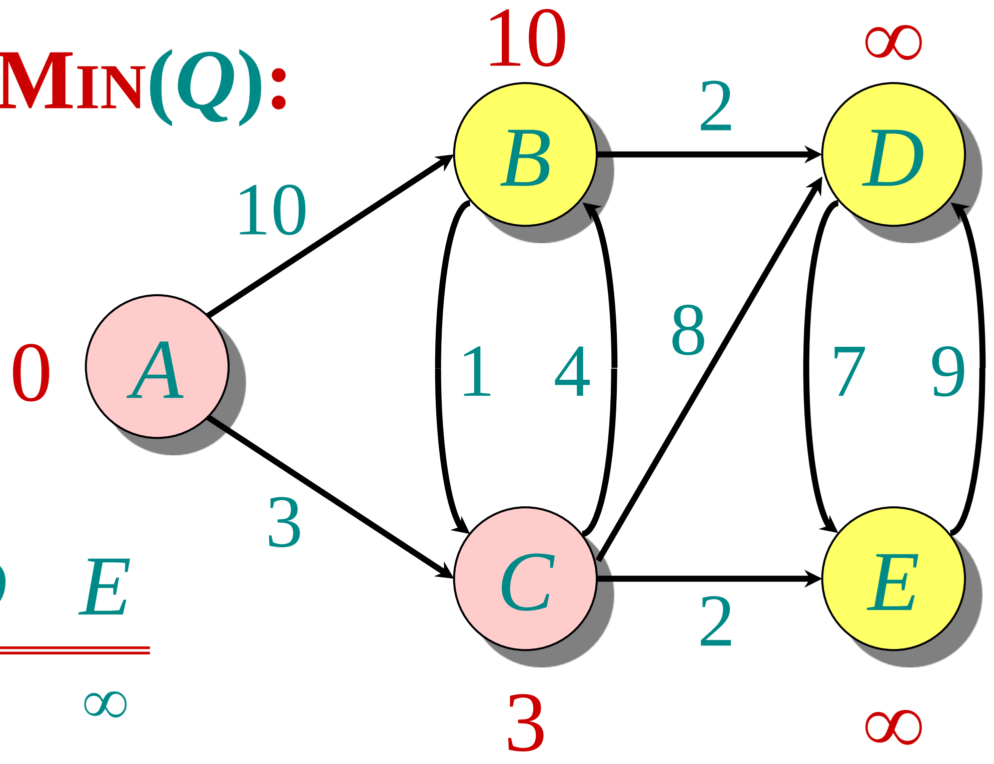


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–

S: { *A* }

Example of Dijkstra’s algorithm

“*C*” ← **Extract-Min**(*Q*):

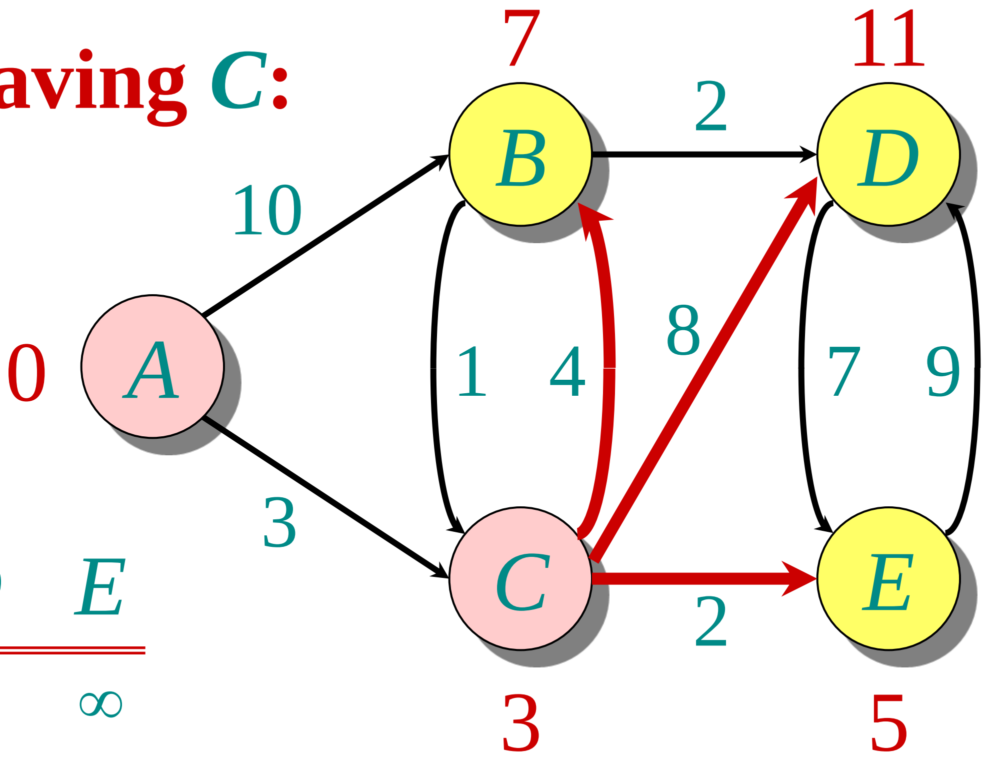


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–

S: { *A*, *C* }

Example of Dijkstra's algorithm

Relax all edges leaving *C*:

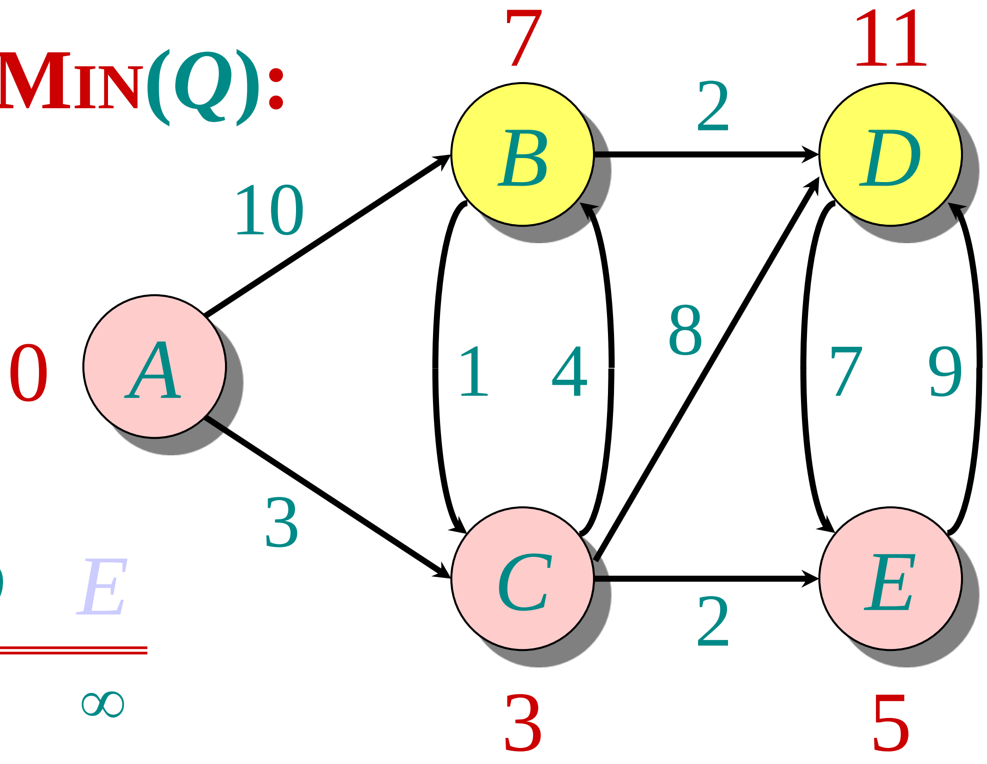


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5

S: { *A*, *C* }

Example of Dijkstra's algorithm

"E" ← **EXTRACT-MIN(*Q*):**

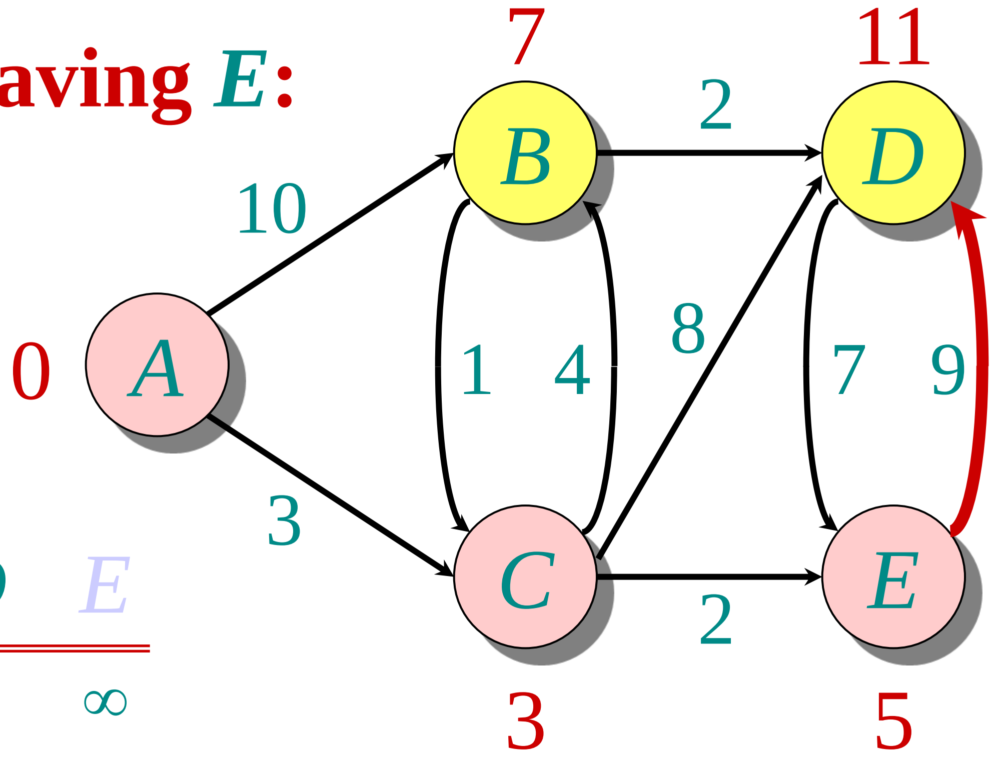


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	–	–
		7		11	5

S: { *A*, *C*, *E* }

Example of Dijkstra’s algorithm

Relax all edges leaving *E*:

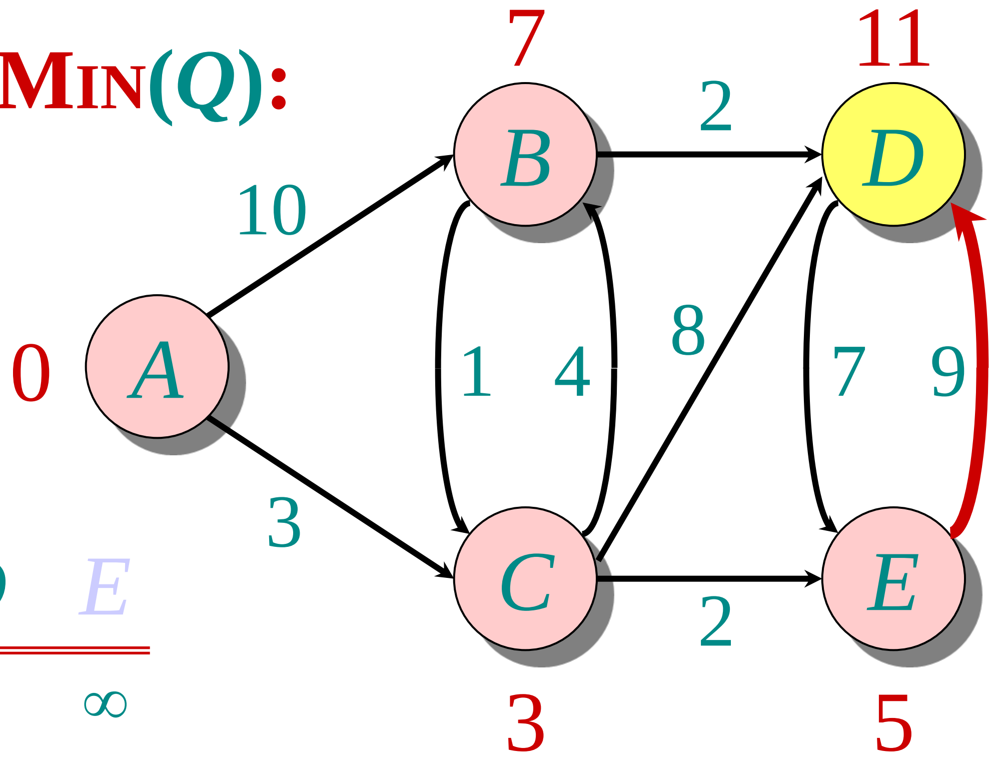


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E* }

Example of Dijkstra’s algorithm

“*B*” ← **Extract-Min(*Q*):**

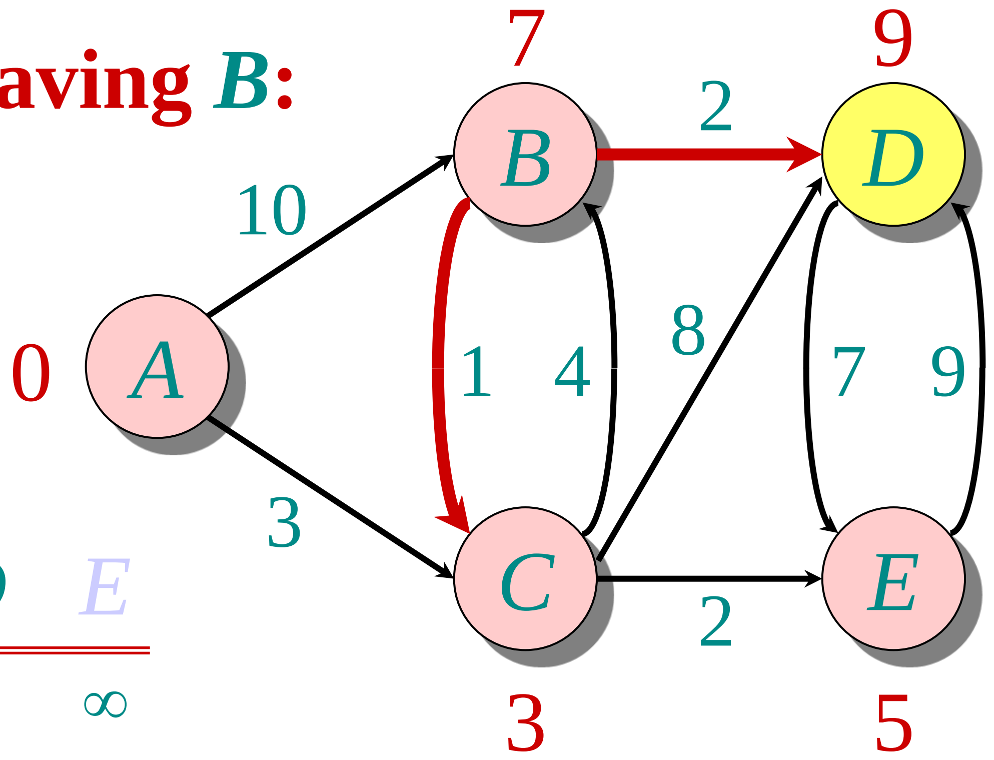


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E*, *B* }

Example of Dijkstra’s algorithm

Relax all edges leaving *B*:

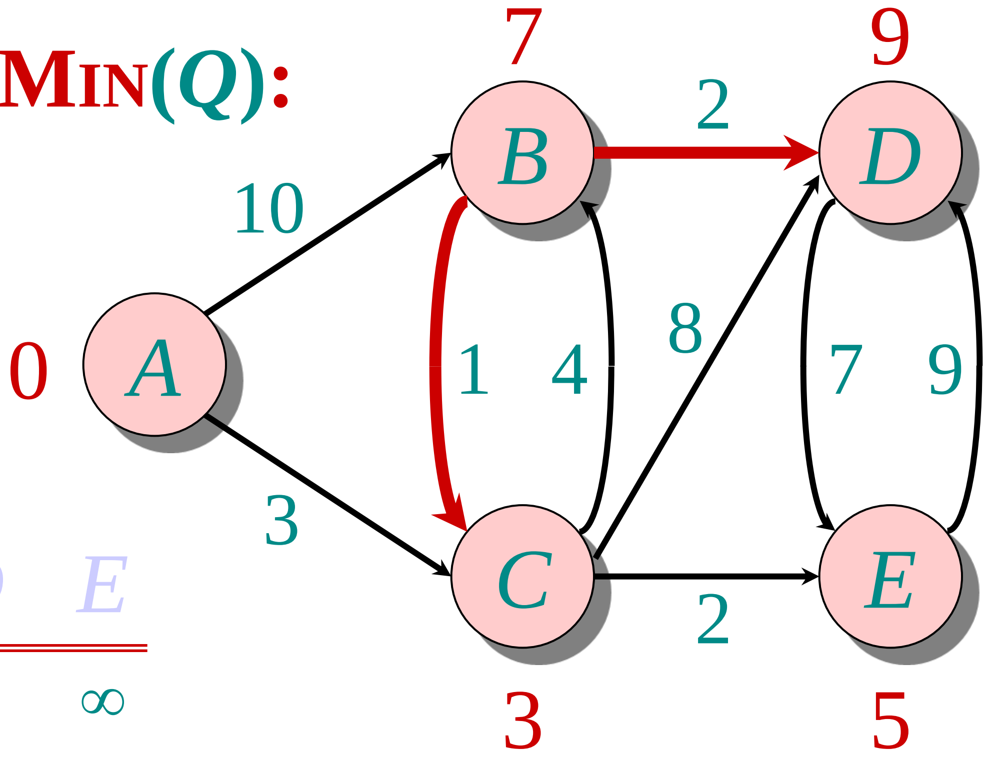


Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { *A*, *C*, *E*, *B* }

Example of Dijkstra's algorithm

"D" ← **EXTRACT-MIN(*Q*):**



Q:	*A*	*B*	*C*	*D*	*E*
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { *A*, *C*, *E*, *B*, *D* }

Correctness — Part I

Lemma. Initializing $d[s] \leftarrow 0$ and $d[v] \leftarrow \infty$ for all $v \in V - \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps.

Proof. Suppose not. Let v be the first vertex for which $d[v] < \delta(s, v)$, and let u be the vertex that caused $d[v]$ to change: $d[v] = d[u] + w(u, v)$. Then,

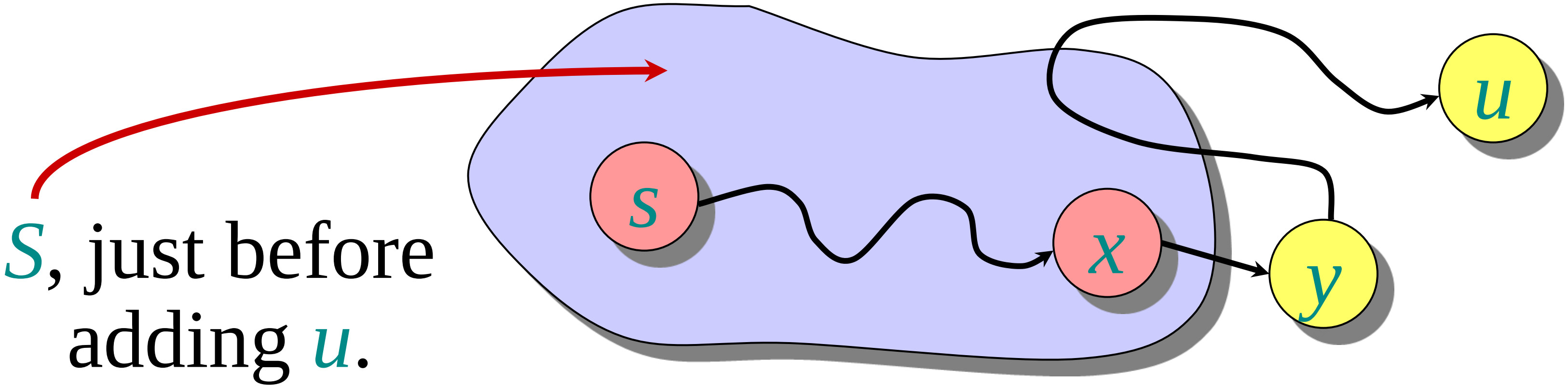
$$\begin{aligned} d[v] &< \delta(s, v) && \text{supposition} \\ &\leq \delta(s, u) + \delta(u, v) && \text{triangle inequality} \\ &\leq \delta(s,u) + w(u, v) && \text{sh. path} \leq \text{specific path} \\ &\leq d[u] + w(u, v) && v \text{ is first violation} \end{aligned}$$

Contradiction. □

Correctness — Part II

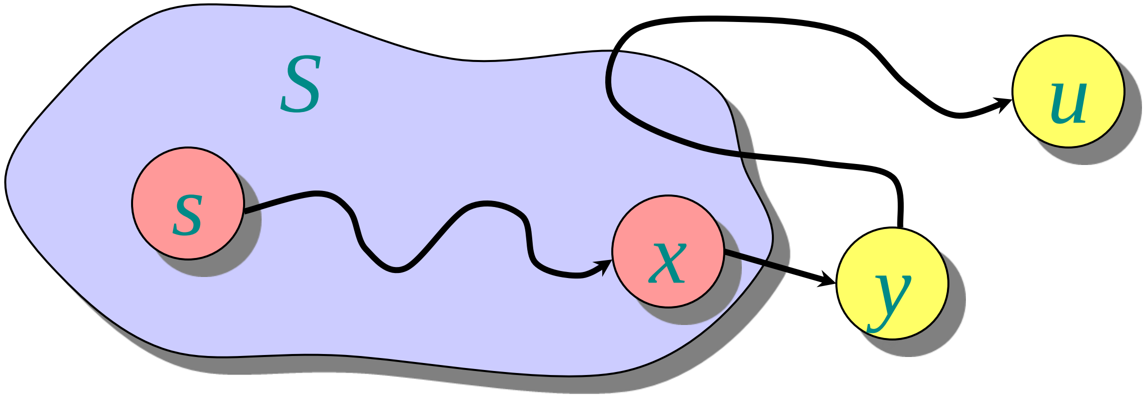
Theorem. Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.

Proof. It suffices to show that $d[v] = \delta(s, v)$ for every $v \in V$ when v is added to S. Suppose u is the first vertex added to S for which $d[u] \neq \delta(s, u)$. Let y be the first vertex in $V - S$ along a shortest path from s to u, and let x be its predecessor:



S, just before adding u.

Correctness — Part II (continued)



Since u is the first vertex violating the claimed invariant, we have $d[x] = \delta(s, x)$. Since subpaths of shortest paths are shortest paths, it follows that $d[y]$ was set to $\delta(s, x) + w(x, y) = \delta(s, y)$ when (x, y) was relaxed just after x was added to S. Consequently, we have $d[y] = \delta(s, y) \leq \delta(s, u) \leq d[u]$. But, $d[u] \leq d[y]$ by our choice of u, and hence $d[y] = \delta(s, y) = \delta(s, u) = d[u]$. Contradiction. □

Analysis of Dijkstra

$|V|$ times

$degree(u)$ times

```
while $Q \neq \emptyset$
    do $u \leftarrow$ EXTRACT-MIN($Q$)
       $S \leftarrow S \cup \{u\}$
       for each $v \in Adj[u]$
           do if $d[v] > d[u] + w(u, v)$
                 then $d[v] \leftarrow d[u] + w(u, v)$
```

Handshaking Lemma $\Rightarrow$ $\Theta(E)$ implicit DECREASE-KEY's.

Time $= \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Note: Same formula as in the analysis of Prim's minimum spanning tree algorithm.

Analysis of Dijkstra (continued)

Time = $\Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case