# *Augmenting Data Structures, Dynamic Order Statistics, Interval Trees*

***Lecture 11***

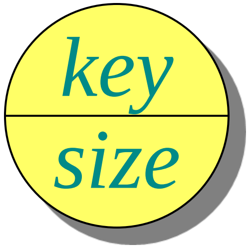
# Dynamic order statistics

OS-SELECT$(i, S)$: returns the $i$ th smallest element in the dynamic set $S$.

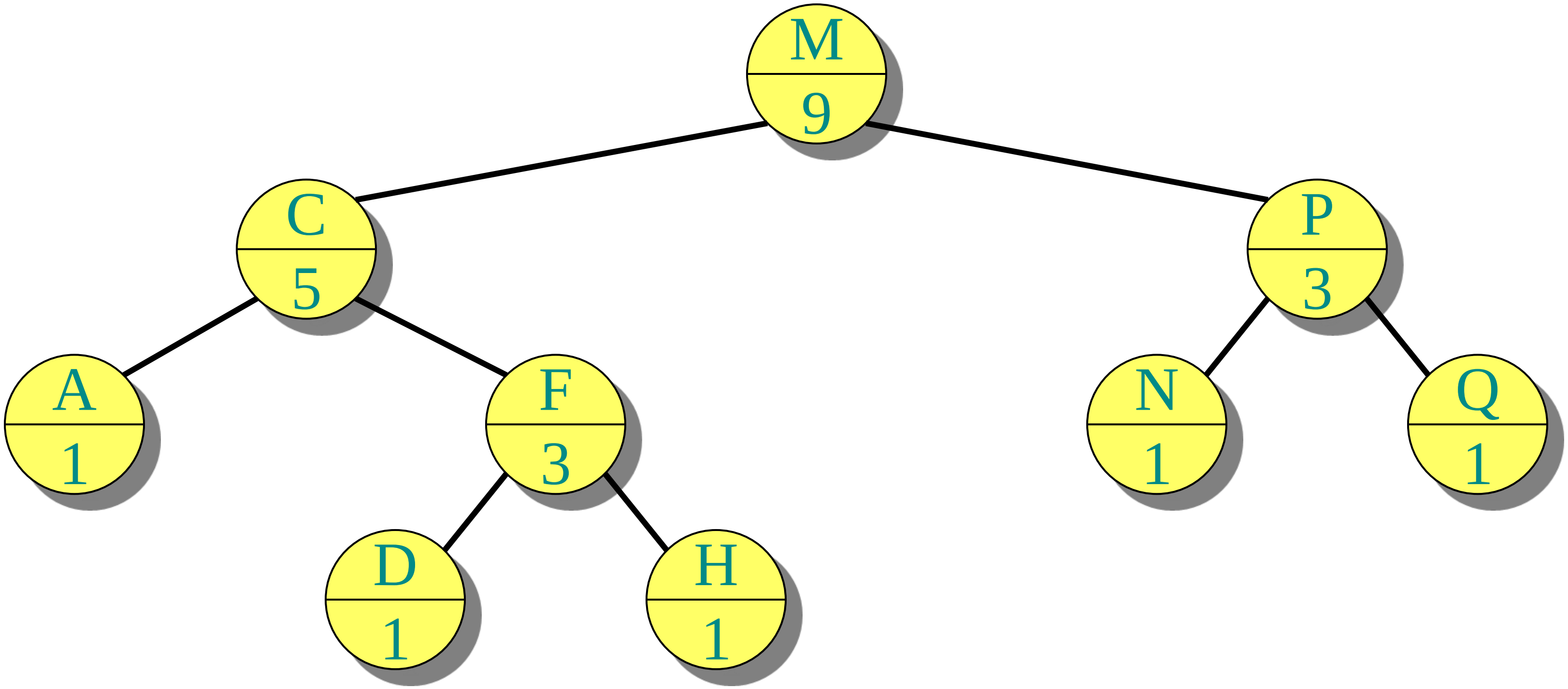
OS-RANK$(x, S)$: returns the rank of $x \in S$ in the sorted order of $S$'s elements.

**IDEA:** Use a red-black tree for the set $S$, but keep subtree sizes in the nodes.

Notation for nodes:



# Example of an OS-tree




$$size[x] = size[left[x]] + size[right[x]] + 1$$

# Selection

**Implementation trick:** Use a ***sentinel*** (dummy record) for NIL such that $size[\text{NIL}] = 0$.

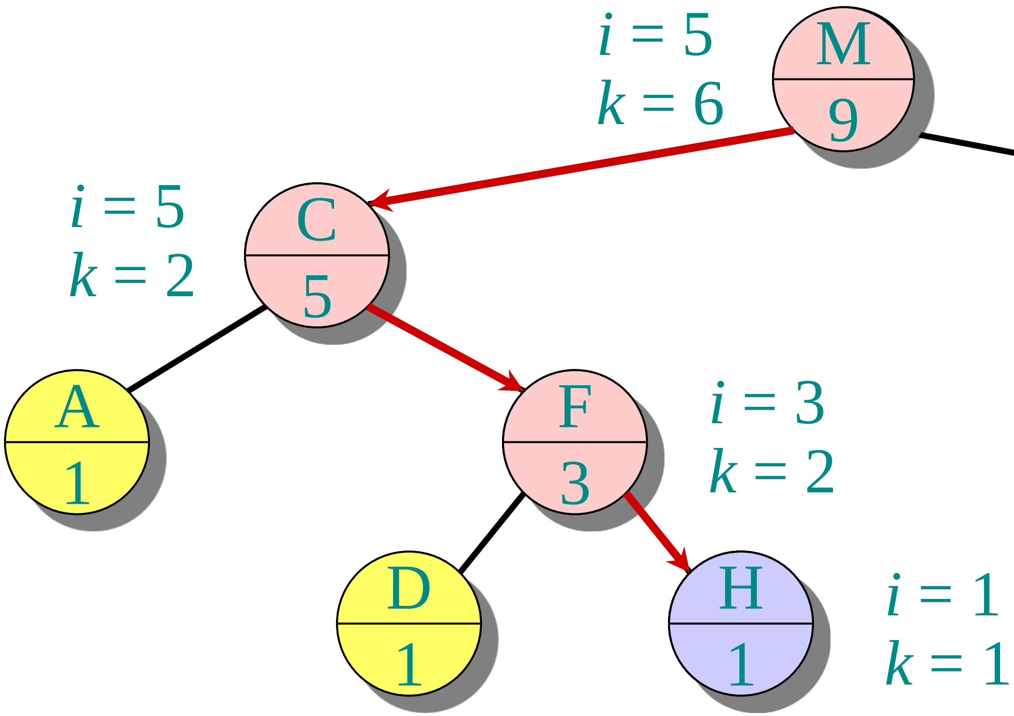
OS-SELECT$(x, i)$ ⊳ $i$th smallest element in the subtree rooted at $x$

$k \leftarrow size[left[x]] + 1$ ⊳ $k = \text{rank}(x)$
**if** $i = k$ **then return** $x$
**if** $i < k$
 **then return** OS-SELECT$(left[x], i)$
 **else return** OS-SELECT$(right[x], i - k)$

(OS-RANK is in the textbook.)

# Example

OS-SELECT(*root*, 5)




Running time = $O(h) = O(\lg n)$ for red-black trees.

# Data structure maintenance

**Q.** Why not keep the ranks themselves in the nodes instead of subtree sizes?

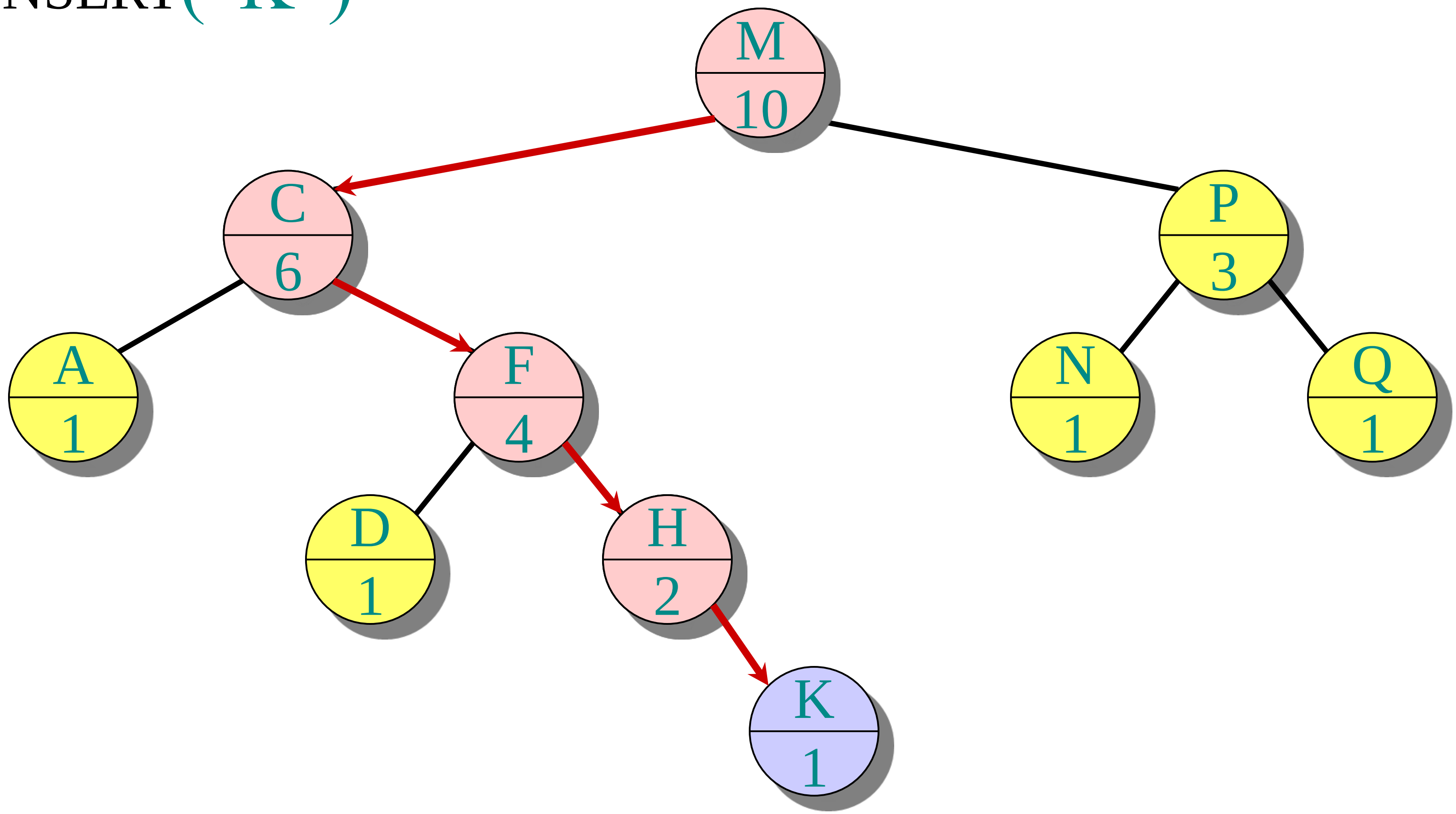
**A.** They are hard to maintain when the red-black tree is modified.

**Modifying operations:** INSERT and DELETE.

**Strategy:** Update subtree sizes when inserting or deleting.

# Example of insertion

INSERT("K")

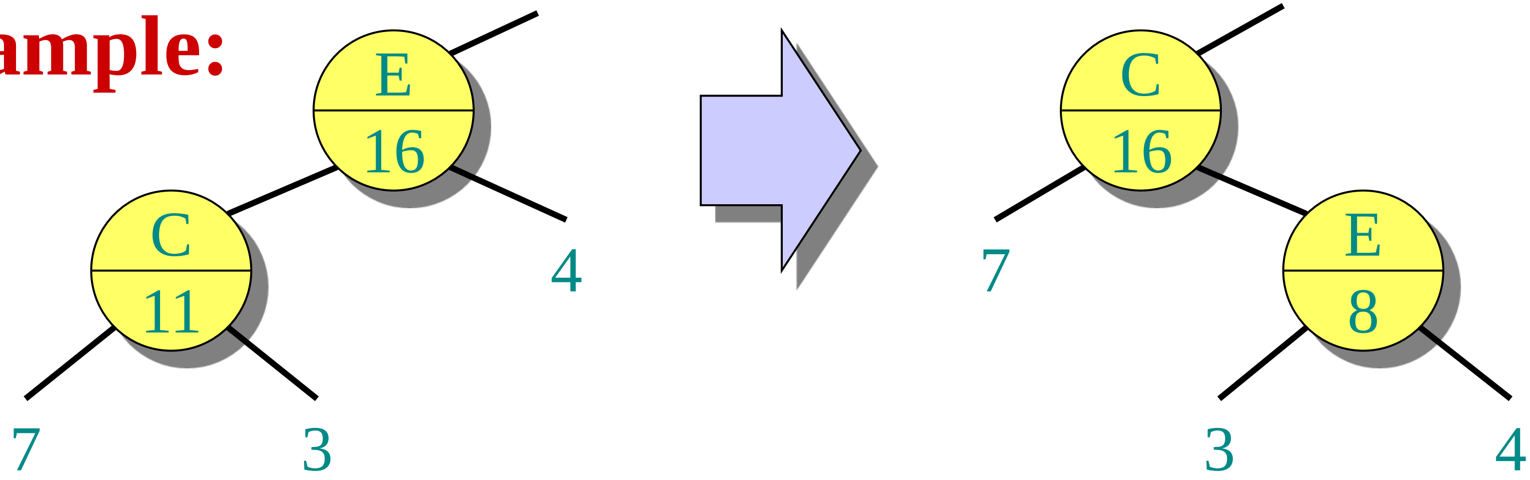


# Handling rebalancing

Don't forget that RB-INSERT and RB-DELETE may also need to modify the red-black tree in order to maintain balance.

- *Recolorings*: no effect on subtree sizes.
- *Rotations*: fix up subtree sizes in $O(1)$ time.

**Example:**




∴ RB-INSERT and RB-DELETE still run in $O(\lg n)$ time.

# Data-structure augmentation

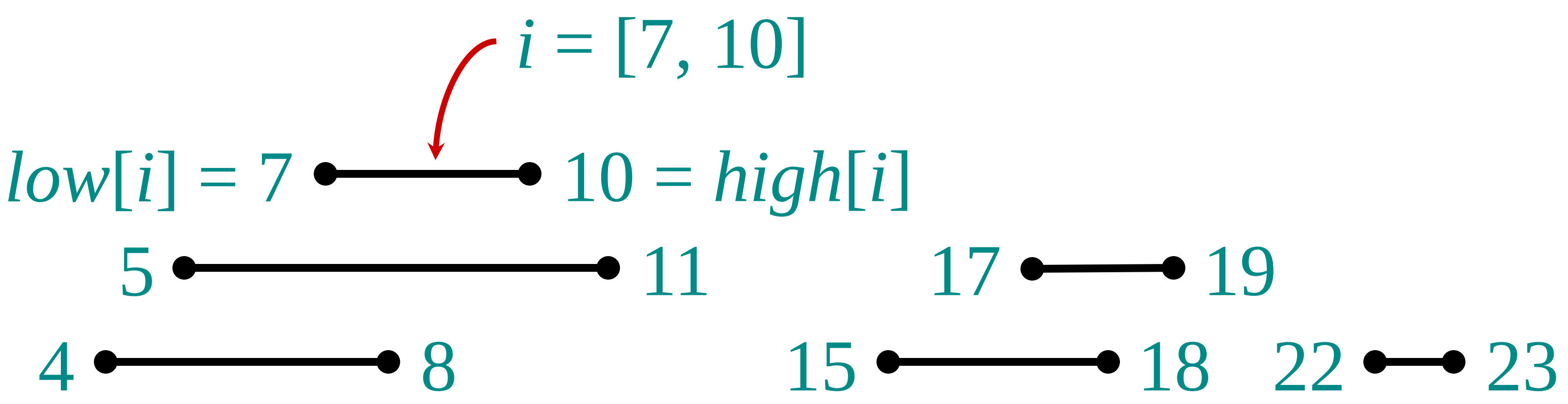
**Methodology:** (*e.g., order-statistics trees*)

1. Choose an underlying data structure (*red-black trees*).
2. Determine additional information to be stored in the data structure (*subtree sizes*).
3. Verify that this information can be maintained for modifying operations (*RB-INSERT, RB-DELETE — don't forget rotations*).
4. Develop new dynamic-set operations that use the information (*OS-SELECT and OS-RANK*).

These steps are guidelines, not rigid rules.

# Interval trees

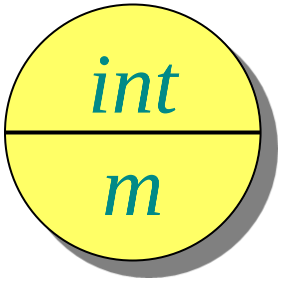
**Goal:** To maintain a dynamic set of intervals, such as time intervals.



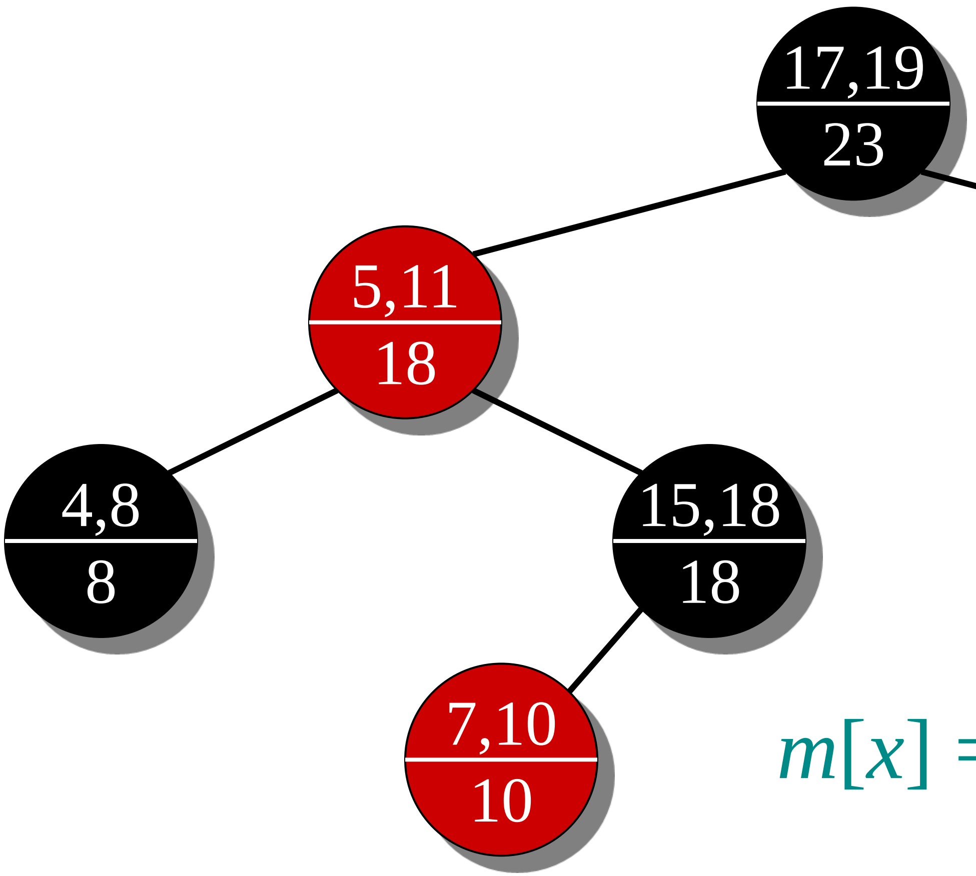

**Query:** For a given query interval $i$, find an interval in the set that overlaps $i$.

# Following the methodology

1. *Choose an underlying data structure.*
   - Red-black tree keyed on low (left) endpoint.
2. *Determine additional information to be stored in the data structure.*
   - Store in each node $x$ the largest value $m[x]$ in the subtree rooted at $x$, as well as the interval $int[x]$ corresponding to the key.



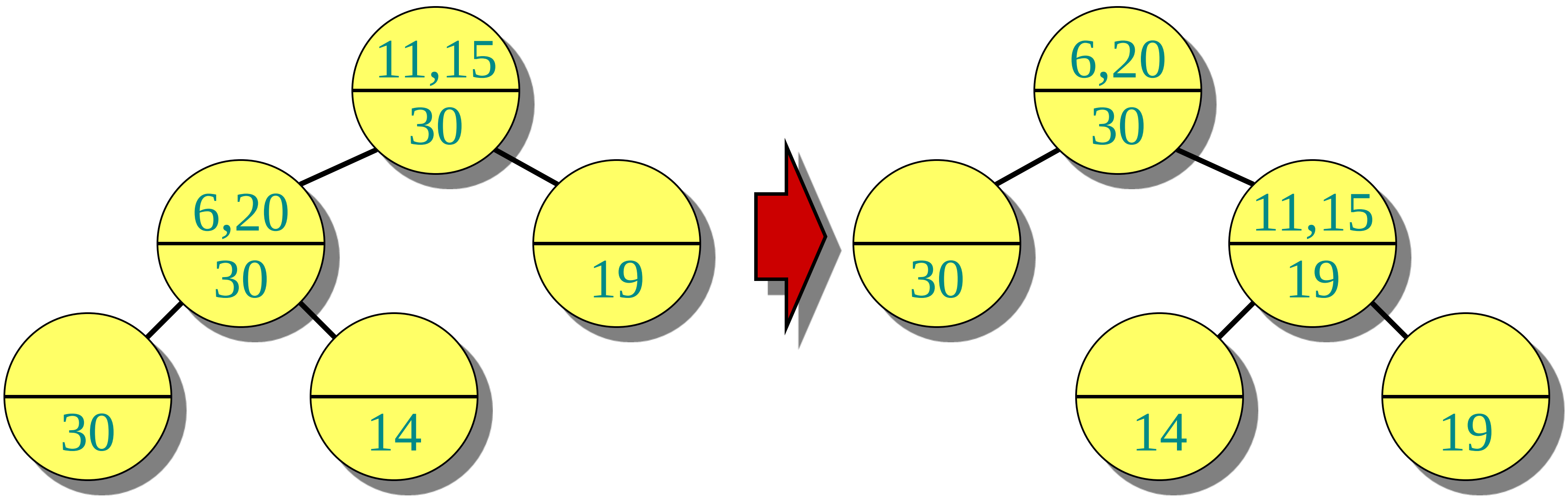
# Example interval tree




$$m[x] = \max \begin{cases} high[int[x]] \\ m[left[x]] \\ m[right[x]] \end{cases}$$

# Modifying operations

3. *Verify that this information can be maintained for modifying operations.*
   - INSERT: Fix *m*'s on the way down.
   - Rotations — Fixup = $O(1)$ time per rotation:



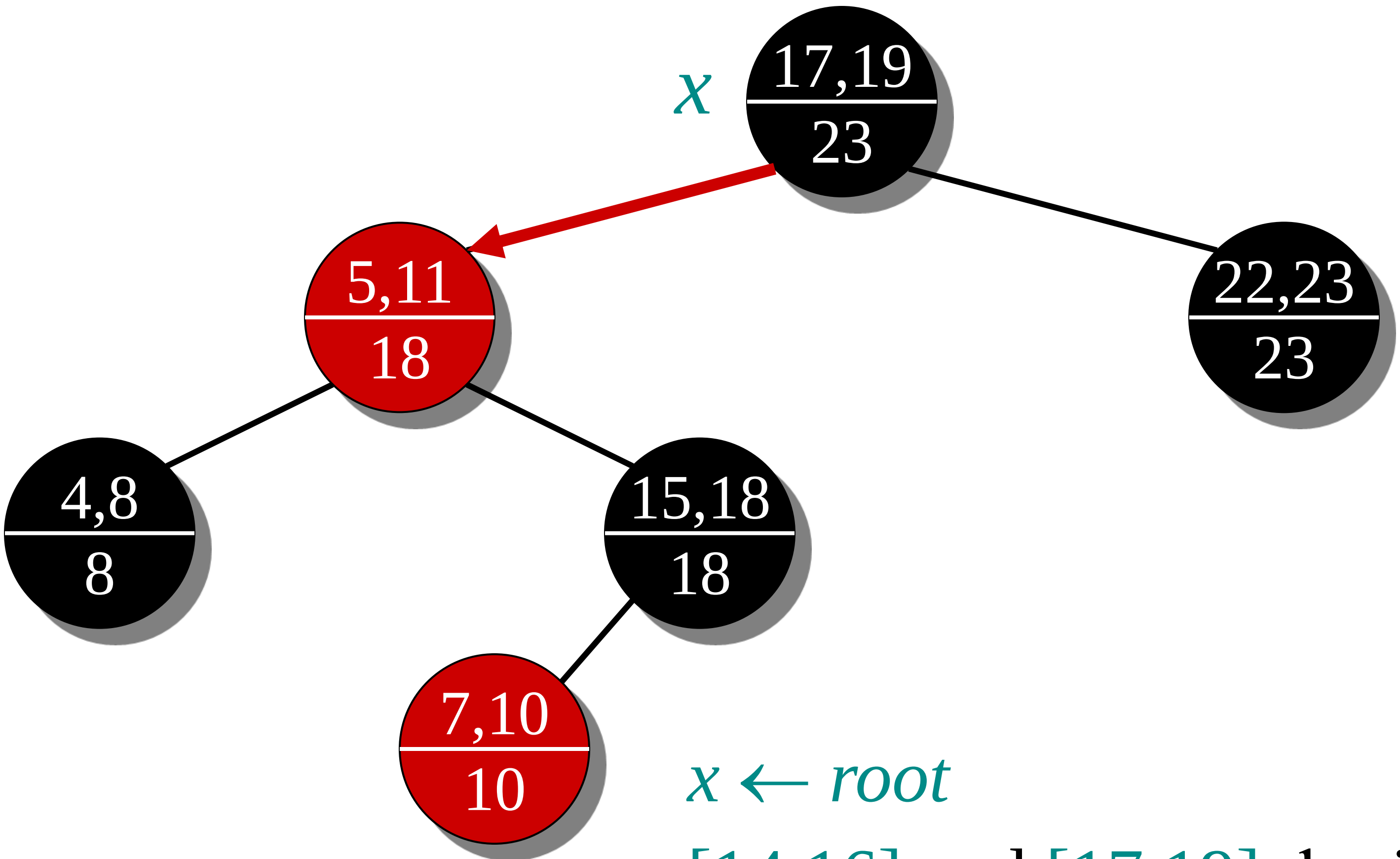

Total INSERT time = $O(\lg n)$; DELETE similar.

# New operations

4. Develop new dynamic-set operations that use the information.

```
INTERVAL-SEARCH(i)
    x ← root
    while x ≠ NIL and (low[i] > high[int[x]]
                        or low[int[x]] > high[i])
        do ▷ i and int[x] don't overlap
            if left[x] ≠ NIL and low[i] ≤ m[left[x]]
                then x ← left[x]
                else x ← right[x]
    return x
```

# **Example 1:** INTERVAL-SEARCH([14,16])

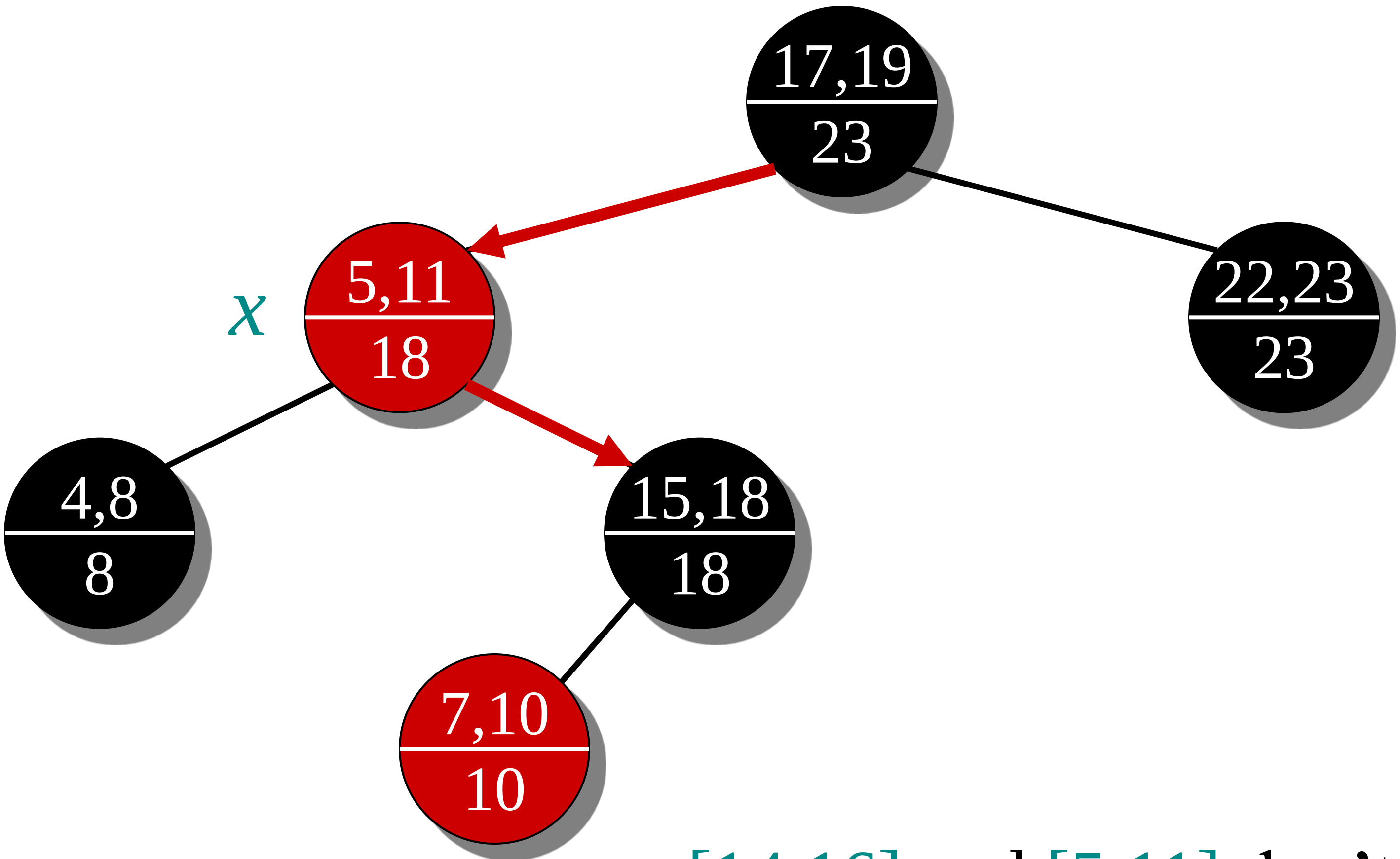



$x \leftarrow root$

[14,16] and [17,19] don't overlap

$14 \leq 18 \Rightarrow x \leftarrow left[x]$

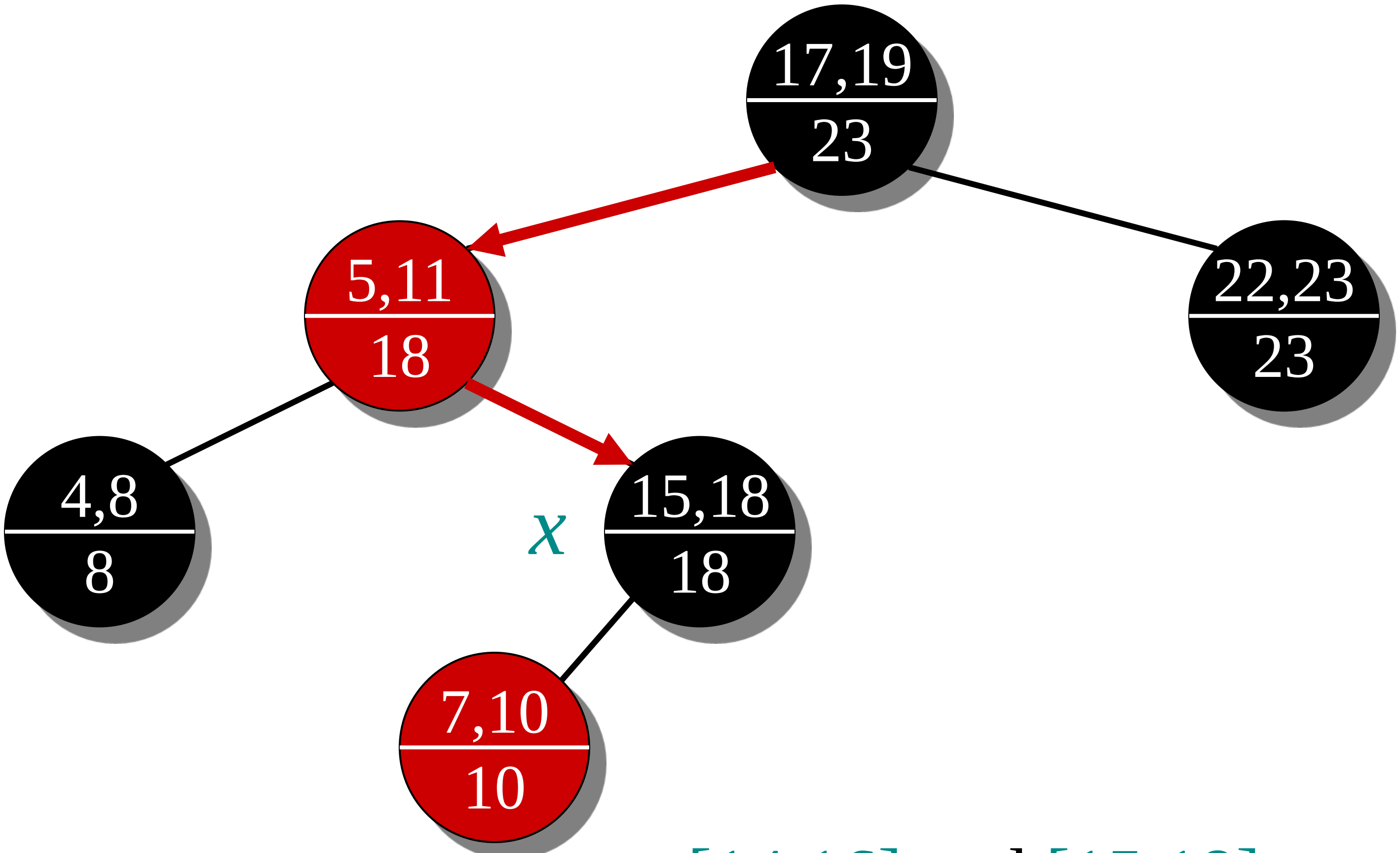
# Example 1: INTERVAL-SEARCH([14,16])




[14,16] and [5,11] don't overlap

$14 > 8 \Rightarrow x \leftarrow right[x]$

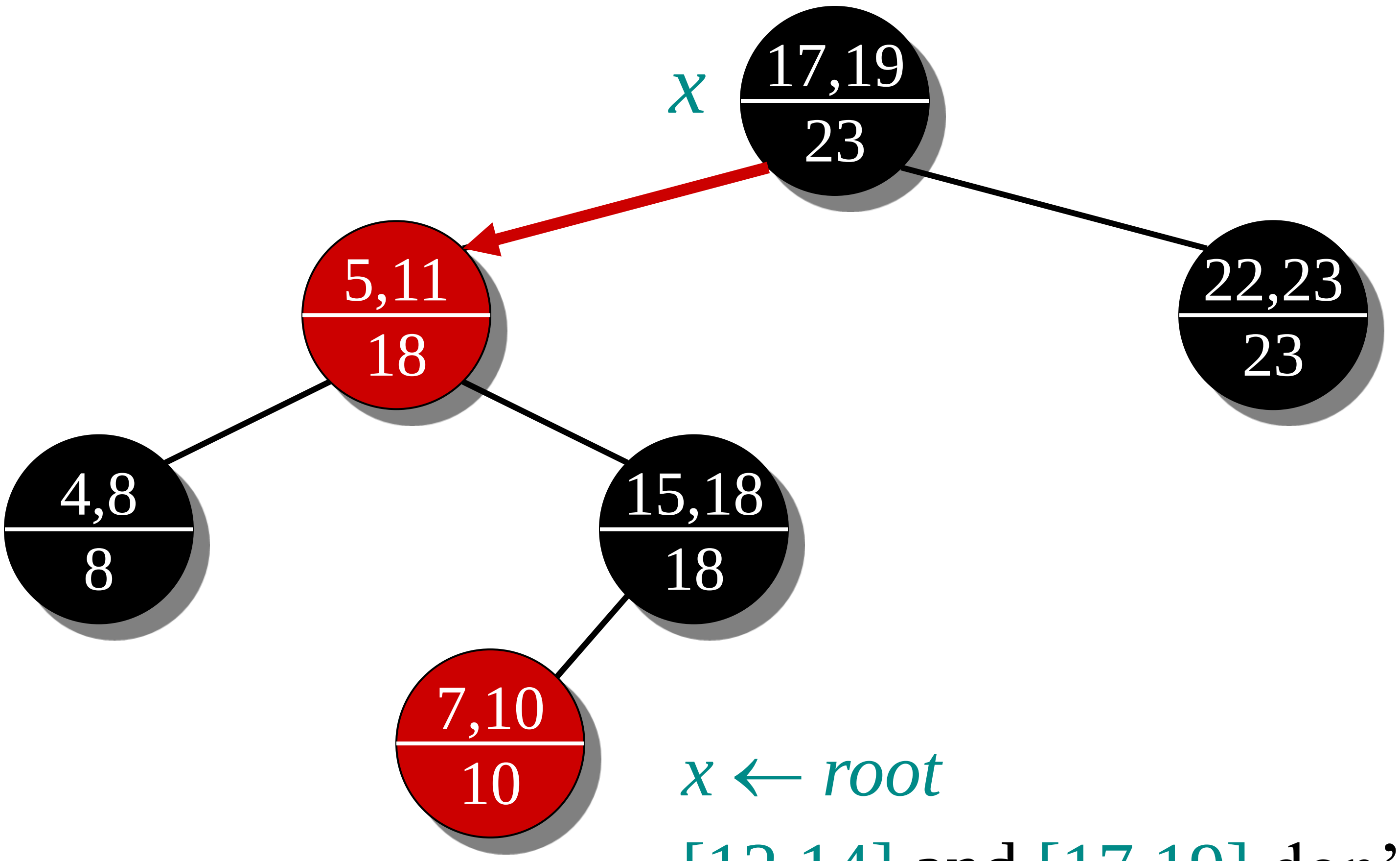
# Example 1: INTERVAL-SEARCH([14,16])




[14,16] and [15,18] overlap

**return** [15,18]

# **Example 2:** INTERVAL-SEARCH([12,14])

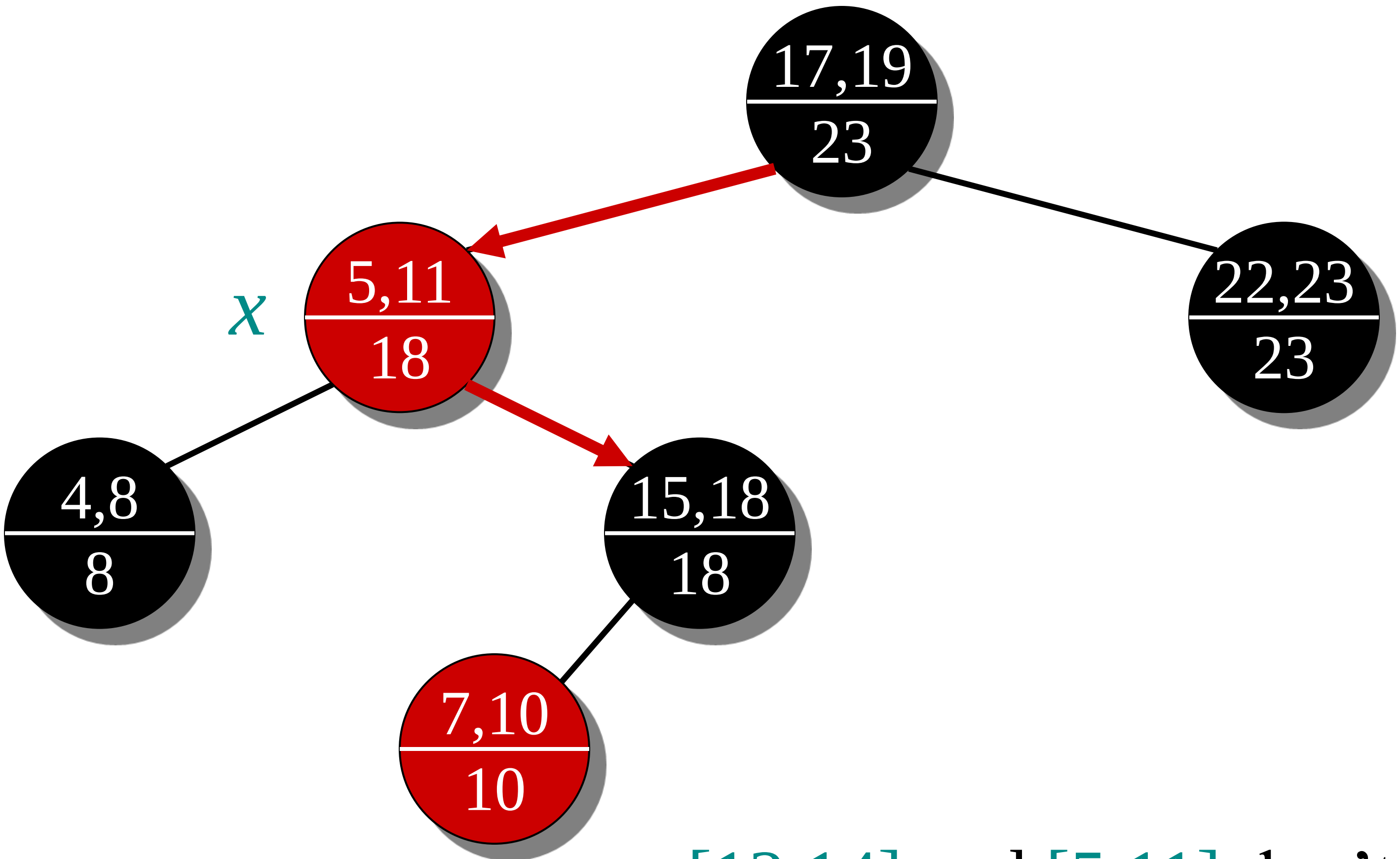



$x \leftarrow root$

[12,14] and [17,19] don't overlap

$12 \leq 18 \Rightarrow x \leftarrow left[x]$

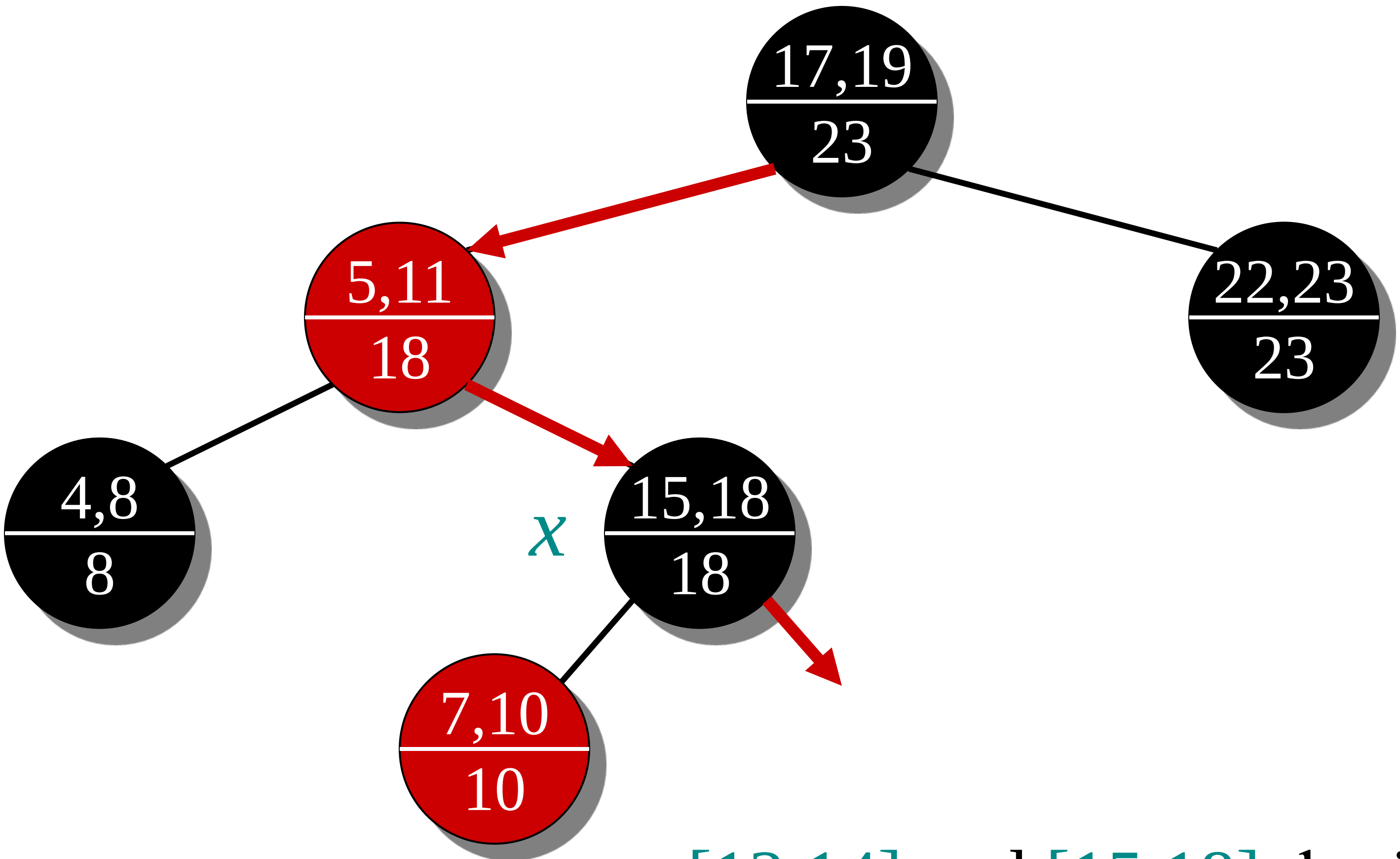
# Example 2: INTERVAL-SEARCH([12,14])




[12,14] and [5,11] don't overlap

$12 > 8 \Rightarrow x \leftarrow right[x]$

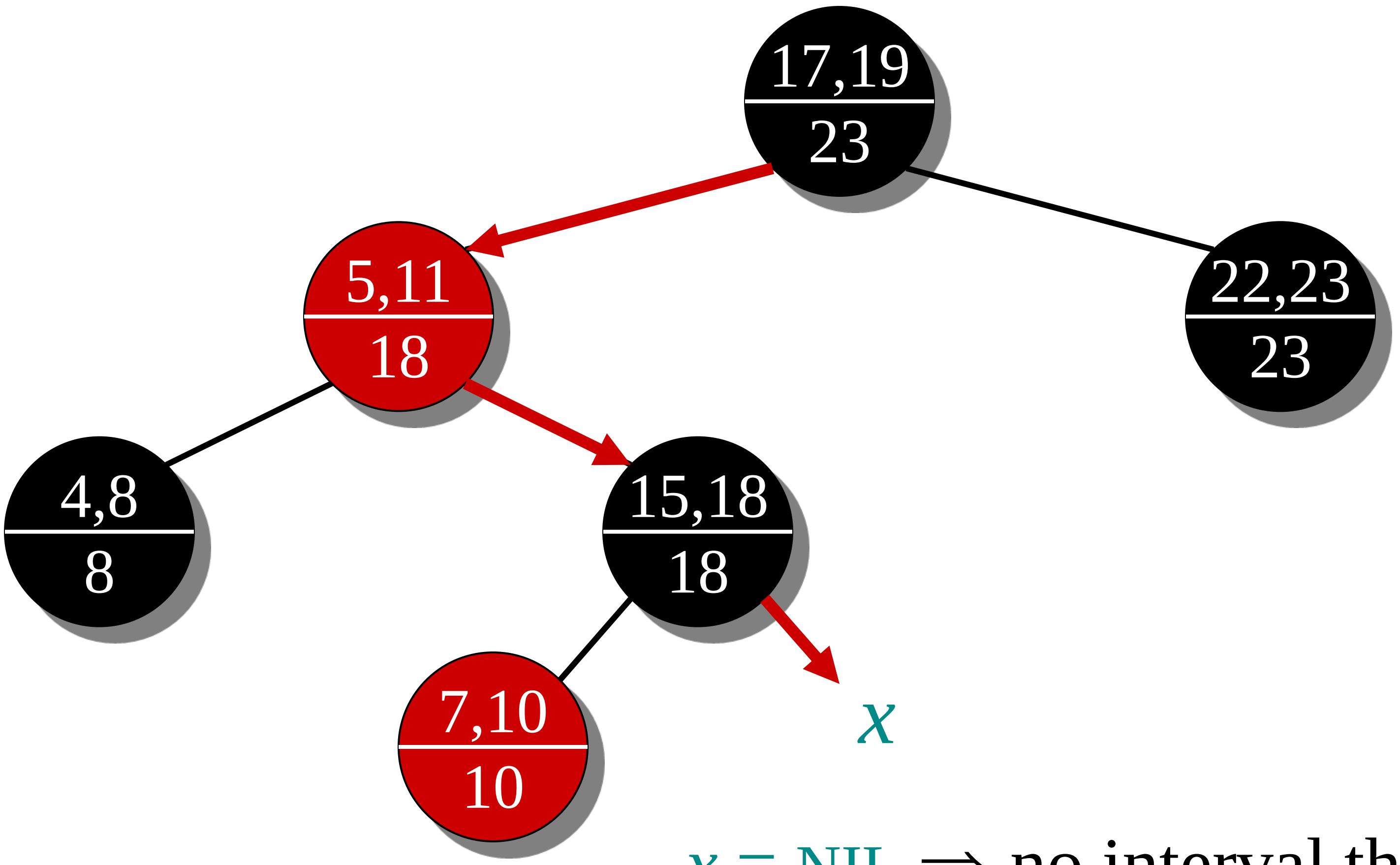
# **Example 2:** INTERVAL-SEARCH([12,14])




[12,14] and [15,18] don't overlap

$12 > 10 \Rightarrow x \leftarrow right[x]$

# Example 2: INTERVAL-SEARCH([12,14])




$x$ = NIL ⇒ no interval that overlaps [12,14] exists

# Analysis

Time $= O(h) = O(\lg n)$, since INTERVAL-SEARCH does constant work at each level as it follows a simple path down the tree.

List *all* overlapping intervals:

- Search, list, delete, repeat.
- Insert them all again at the end.

Time $= O(k \lg n)$, where $k$ is the total number of overlapping intervals.

This is an ***output-sensitive*** bound.

Best algorithm to date: $O(k + \lg n)$.

# Correctness

**Theorem.** Let $L$ be the set of intervals in the left subtree of node $x$, and let $R$ be the set of intervals in $x$'s right subtree.

- If the search goes right, then

$$\{ i' \in L : i' \text{ overlaps } i \} = \varnothing.$$

- If the search goes left, then

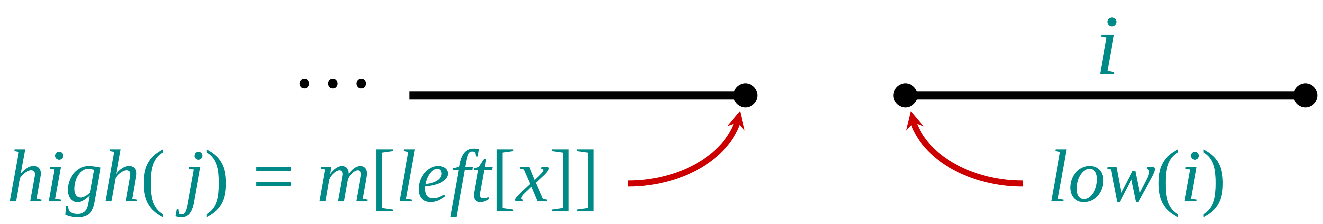
$$\{ i' \in L : i' \text{ overlaps } i \} = \varnothing \Rightarrow \{ i' \in R : i' \text{ overlaps } i \} = \varnothing.$$

*In other words, it's always safe to take only 1 of the 2 children: we'll either find something, or nothing was to be found.*

# Correctness proof

*Proof.* Suppose first that the search goes right.

- If $left[x] = \text{NIL}$, then we're done, since $L = \emptyset$.
- Otherwise, the code dictates that we must have $low[i] > m[left[x]]$. The value $m[left[x]]$ corresponds to the right endpoint of some interval $j \in L$, and no other interval in $L$ can have a larger right endpoint than $high(j)$.




- Therefore, $\{i' \in L : i' \text{ overlaps } i\} = \emptyset$.

# Proof (continued)

Suppose that the search goes left, and assume that

$\{i' \in L : i' \text{ overlaps } i\} = \varnothing$.

- Then, the code dictates that $low[i] \le m[left[x]] = high[j]$ for some $j \in L$.
- Since $j \in L$, it does not overlap $i$, and hence $high[i] < low[j]$.
- But, the binary-search-tree property implies that for all $i' \in R$, we have $low[j] \le low[i']$.
- But then $\{i' \in R : i' \text{ overlaps } i\} = \varnothing$. ∎