Red-black Trees, Rotations, Insertions, Deletions

Lecture 10
Balanced search trees

**Balanced search tree:** A search-tree data structure for which a height of $O(lg \ n)$ is guaranteed when implementing a dynamic set of $n$ items.

**Examples:**
- AVL trees
- 2-3 trees
- 2-3-4 trees
- B-trees
- Red-black trees
Red-black trees

This data structure requires an extra one-bit color field in each node.

**Red-black properties:**

1. Every node is either red or black.
2. The root and leaves (NIL’s) are black.
3. If a node is red, then its parent is black.
4. All simple paths from any node $x$ to a descendant leaf have the same number of black nodes $= \text{black-height}(x)$. 
Example of a red-black tree

$h = 4$
Example of a red-black tree

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Example of a red-black tree

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### Height of a red-black tree

**Theorem.** A red-black tree with \( n \) keys has height
\[
h \leq 2 \lg(n + 1).
\]

**Proof.** (The book uses induction. Read carefully.)

**Intuition:**
- Merge red nodes into their black parents.
Theorem. A red-black tree with n keys has height

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Theorem. A red-black tree with $n$ keys has height $h \leq 2 \lg(n + 1)$.

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Height of a red-black tree

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**Intuition:**
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Height of a red-black tree

**Theorem.** A red-black tree with $n$ keys has height

$$h \leq 2 \log_2(n + 1).$$

**Proof.** (The book uses induction. Read carefully.)

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Theorem. A red-black tree with $n$ keys has height

$$h \leq 2 \lg(n + 1).$$

Proof. (The book uses induction. Read carefully.)

Intuition:

- Merge red nodes into their black parents.
- This process produces a tree in which each node has 2, 3, or 4 children.
- The 2-3-4 tree has uniform depth $h'$ of leaves.
Proof (continued)

• We have \( h' \geq h/2 \), since at most half the leaves on any path are red.

• The number of leaves in each tree is \( n + 1 \)
  \[ \Rightarrow n + 1 \geq 2^{h'} \]
  \[ \Rightarrow \lg(n + 1) \geq h' \geq h/2 \]
  \[ \Rightarrow h \leq 2 \lg(n + 1). \]
Query operations

**Corollary.** The queries `SEARCH`, `MIN`, `MAX`, `SUCCESSOR`, and `PREDECESSOR` all run in \(O(\lg n)\) time on a red-black tree with \(n\) nodes.
Modifying operations

The operations \texttt{INSERT} and \texttt{DELETE} cause modifications to the red-black tree:

- the operation itself,
- color changes,
- restructuring the links of the tree: "rotations".
Rotations

Rotations maintain the inorder ordering of keys:
• \( a \in \alpha, \ b \in \beta, \ c \in \gamma \Rightarrow \ a \leq A \leq b \leq B \leq c. \)

A rotation can be performed in \( O(1) \) time.
**Insertion into a red-black tree**

**Idea:** Insert $x$ in tree. Color $x$ red. Only red-black property 3 might be violated. Move the violation up the tree by recoloring until it can be fixed with rotations and recoloring.

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**Example:**
- Insert $x = 15$.
- Recolor, moving the violation up the tree.
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Pseudocode

**RB-INSERT**(\(T, x\))

**TREE-INSERT**(\(T, x\))

\(color[x] \leftarrow \text{RED} \quad \triangleright \text{only RB property 3 can be violated}\)

\(\text{while } x \neq \text{root}[T] \text{ and } color[p[x]] = \text{RED}\)

\(\text{do if } p[x] = \text{left}[p[p[x]]] \quad \triangleright y = \text{aunt/uncle of } x\)

\(\text{then } y \leftarrow \text{right}[p[p[x]]]\)

\(\text{if } color[y] = \text{RED}\)

\(\text{then } \langle \text{Case 1} \rangle\)

\(\text{else if } x = \text{right}[p[x]]\)

\(\text{then } \langle \text{Case 2} \rangle \quad \triangleright \text{Case 2 falls into Case 3}\)

\(\langle \text{Case 3} \rangle\)

\(\text{else } \langle \text{“then” clause with “left” and “right” swapped} \rangle\)

\(color[root[T]] \leftarrow \text{BLACK}\)
Graphical notation

Let ▲ denote a subtree with a black root.

All ▲’s have the same black-height.
Case 1

(Or, children of $A$ are swapped.)

Push $C$’s black onto $A$ and $D$, and recurse, since $C$’s parent may be red.
Case 2

L\textsc{eft-R\textsc{otate}}(A)

Transform to Case 3.
Case 3

\[ \text{RIGHT-ROTATE}(C) \]

Done! No more violations of RB property 3 are possible.
Analysis

• Go up the tree performing Case 1, which only recolors nodes.
• If Case 2 or Case 3 occurs, perform 1 or 2 rotations, and terminate.

**Running time:** $O(\lg n)$ with $O(1)$ rotations.

**RB-DELETE** — same asymptotic running time (see textbook).