REVIEW OF A THERMODYNAMIC THEORY
FOR GRANULAR MATERIALS

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ABSTRACT

A new thermodynamic theory for frictional material has been reviewed. The model, though does not require any a priori yield criteria, degenerates to different elastic-plastic theories as special cases showing prominent yield surface(s) and satisfying Drucker's postulates. The model takes into account initial anisotropy, stress-induced anisotropy, compressibility and the stress-history dependency often exhibited by frictional materials, like soil. A complete derivation of the theory from the principles of thermodynamics has been given and material parameters for an ordinary sand and a cemented sand have been obtained from laboratory test results to highlight different features of the model and its flexibility to model different material behavior.

INTRODUCTION

In most of the granular soils and concrete where contact friction between grains obey Coulomb's law, Hill's principle of maximum work and Drucker's postulate do not apply. Mandel (1964), Rice (1971) and others have proven a long time ago that Drucker's postulate is a sufficient condition of stability and not a necessary one. Recently, Lade (1987) reported results of some experiments that were stable yet violated Drucker's stability conditions both in the large and in the small. It has also been observed in cyclic tests that negative plastic work must be done upon a stress reversal if the plastic strain is not zero (i.e., the plastic strain increment is in the opposite direction

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from the stress). Theories based on the fundamental laws of thermodynamcis, instead of Drucker's postulate, circumvent the above mentioned difficulties often encountered in modeling behavior of frictional materials like soils.

In this paper a thermodynamic model based on second law of thermodynamics has been reviewed. The model (Valanis and Peters, 1988) does take into account initial anisotropy, stress-induced anisotropy, compressibility (pressure dependency) and the stress-history dependency often exhibited by materials like soils. Also, unlike many plasticity theories, load reversal in the thermodynamic theory under investigation comes about naturally - a major advantage of the theory. The thermodynamic theory under consideration differs from the Endochronic theory of Valanis and Read [32] in many respects. Endochronic theory was developed for metals and concrete and thus permitted only shear-induced contraction. The theory was not suitable for frictional materials, like soils, which show dilatancy during shearing. The present theory models contraction as well as dilatancy exhibited by most of the soils. The original Endochronic theory was criticized because it showed openness of the hysteresis loop and did not satisfy Drucker's postulates. In the present theory above problems have been rectified by introducing a weakly singular kernel function in the formulation. The new theory not only exhibits hysteresis loop closure but also satisfies Drucker's stability postulates under certain situations. The greatest triumph of the present theory is that it does not require any a priori definition of yield function, yet depending on the chosen material parameters, it automatically degenerates to various established elasto-plastic constitutive theories showing prominent yield surfaces. Commonly observed behaviors of soils like dilation, strain-softening, etc., are admitted by the theory due to the very fact that the second law of thermodynamics is not violated under these situations as long as the entropy of the whole system remains positive. Thus the thermodynamic theory under investigation possesses the ability to model the constitutive behaviors of a large number of different types of material. The theory attempts to unify different prevailing plasticity theories in the sense that they may be obtained and justified as special cases of the present theory.

A complete derivation of the theory from the basic principles of thermodynamics has been shown. Procedures for the material parameters determination have been delineated with two different examples. Attempts have been made to explain the physical meaning of each material parameter and its relations to traditional soil mechanics principles.

THE THERMODYNAMIC THEORY

The theory under consideration is different from the classical theories of plasticity in that it is based on irreversible thermodynamic principles instead of conservation of mechanical energy. The theory is guided by the requirement of positive entropy production to satisfy Clausius-Duhem inequality.

Attempts to introduce thermodynamics to describe the behavior of different materials are not new. In the mid-1950's, Biot published a series of papers (1954, 1955, 1956) and showed how irreversible thermodynamics can be combined with a mechanical theory to form a basis for a thermomechanical theory of small visco-elastic deformation. Truesdall (1965) carried on Biot's work further and developed axiomatic approach to describe complex nonlinear behavior of solids. In 1964, Coleman developed a thermodynamic theory of a simple material. He presented a series of axioms based on the principles of fading memory to show stress, heat flow, internal energy and entropy as functional of the history of the deformation gradient and temperature as well as functions of current temperature.

Kestin and Rice (1971) in their papers discussed the difficulties encountered in extending thermostatics to the description of irreversible processes in solid materials. They examined the role of internal variables in bringing inelastic behavior within the framework of the classical theory. They proved the existence of a potential function of stress for rate dependent materials, at each set of internal variables from which the inelastic strain rate may be derived. Rice (1971) has discussed the role of internal variables by which inelastic structural rearrangements of a rate dependent material can be related to its macroscopic deformation. He applied this kind of theory to metals deforming plastically through dislocation motion. Oden (1973) and Perzyna (1968) have also discussed the use of internal variables to describe the material behavior.

It is to be noted that a constitutive relation based on thermodynamic laws has certain restrictions. For a constitutive law of this type to hold good, it is essential that for the material under investigation a free energy density function, ψ, should exist and should be differentiable at every stage. In general ψ is a function of strains, ϵ_{ij}, temperature, θ and n internal variables, $q_{ij}^{(r)}$, which may be infinite in number and are representative of the material's internal structure. If ψ exists and is differentiable then the following thermodynamic relations will hold good:

$$\sigma_{ij} = \frac{\partial \psi}{\partial \varepsilon_{ij}} \tag{1}$$

$$\eta = -\frac{\partial \psi}{\partial \theta} \tag{2}$$

$$\theta_o \dot{\eta} = -\frac{\partial \psi}{\partial q_{ij}^{(r)}} \dot{q}_{ij}^{(r)} \geq 0$$

$$Q_{ij}^{(r)} = -\frac{\partial \psi}{\partial q_{ij}^{(r)}} \qquad h_i = K_{ij}\,\theta_{,j} \qquad h_i\,\theta_{,i} \leq 0 \qquad (3)$$

where, σ_{ij} are the stress tensors, η is the entropy density, $Q_{ij}^{(r)}$ are the internal forces, h_i are the heat flux vectors, θ_o is the reference temperature and K_{ij} are the thermal conductivity tensors.

In 1967 Valanis proposed that the internal forces, $Q_{ij}^{(r)}$, are proportional to the intrinsic time rate of change of $q_{ij}^{(r)}$, that is,

$$Q_{ij}^{(r)} = b_{ijkl}^{(r)} \frac{dq_{kl}^{(r)}}{dZ} \qquad (4)$$

where Z is an intrinsic time measure and dZ is the infinitesimal increment of Z. b_{ijkl} is a fourth order positive definite tensor (also known as "resistance" tensor) and is a material property in the sense that in general it will vary from material to material.

In the original theory, Valanis defined incremental intrinsic time measure as

$$dZ^2 = P_{ijkl}\,d\varepsilon_{ij}\,d\varepsilon_{kl} \qquad (5)$$

where P_{ijkl} is a fourth order positive definite tensor. Above definition shows nonelastic unloading-reloading and openness of hysteresis loop. So it was later redefined as the length of the path in plastic strain space instead of total strain space. Mathematically it may be written as

$$dZ^2 = P_{ijkl}\,d\varepsilon_{ij}^p\,d\varepsilon_{kl}^p \qquad (5a)$$

where $d\varepsilon_{ij}^p$ and $d\varepsilon_{kl}^p$ are the plastic strain vectors.

If the definition of internal forces, $Q_{ij}^{(r)}$ is substituted in the equation (3), it is possible to write

$$\frac{\partial \psi}{\partial q_{ij}^{(r)}} + b_{ijkl}^{(r)} \frac{dq_{kl}^{(r)}}{dZ} = 0 \qquad r = 1, 2, \ldots, n.$$

For isotropic materials, Valanis and Read [32] have shown that it is possible to have uncoupled deviatoric and hydrostatic responses. If the subscript D and H refer to the deviatoric and hydrostatic parts, respectively, then it is possible to express ψ as follows:

$$\psi = \psi_D + \psi_H \qquad (6)$$

ψ_D and ψ_H can be further partitioned into elastic (denoted by superscript e) and plastic (denoted by superscript p) components such that

$$\psi_D = \psi_D^e + \psi_D^p \tag{7}$$

and

$$\psi_H = \psi_H^e + \psi_H^p \tag{8}$$

where,

$$\psi_D^e = \psi_D^e(\tilde{e}^e) \tag{9}$$

$$\psi_D^p = \psi_D^p(\tilde{e}^p, \tilde{q}^{(r)}) \tag{10}$$

and

$$\psi_H^e = \psi_H^e(\varepsilon^e) \tag{11}$$

$$\psi_H^p = \psi_H^p(\varepsilon^p, p^{(r)}) \tag{12}$$

$\tilde{e}^e$ are elastic (deviatoric) strains, e^p are plastic (deviatoric) strains, ϵ^e are elastic (hydrostatic) strains, ϵ^p are plastic (hydrostatic) strains, $\tilde{q}^{(r)}$ are deviatoric components of internal variables and $p^{(r)}$ are the hydrostatic components of the internal variables.

From the theory of irreversible thermodynamics, it is possible to write

$$\tilde{s} = \frac{\partial \psi_D^p}{\partial \tilde{e}^p} \tag{13}$$

with

$$-\frac{\partial \psi_D^p}{\partial \tilde{q}^{(r)}} \, d\tilde{q}^{(r)} > 0, \qquad \|d\tilde{q}^{(r)}\| > 0 \tag{14}$$

and

$$\sigma = \frac{\partial \psi_H^p}{\partial \varepsilon^p} \tag{15}$$

with

$$-\frac{\partial \psi_H^p}{\partial p^{(r)}} \, dp^{(r)} > 0, \qquad |dp_{(r)}| > 0 \tag{16}$$

where $\tilde{s}$ and σ are the deviatoric and hydrostatic stresses, respectively.

It has been shown [32] that if quadratic forms for the free energies are assumed then the expressions for the free energy are reducible to the following canonical forms given below

$$\psi_D^p = \frac{1}{2} \sum_{r=1}^{n} A^{(r)} \|e_{ij}^p - q_{ij}^{(r)}\|^2 \tag{17}$$

$$\psi_H^p = \frac{1}{2} \sum_{r=1}^{n} B^{(r)} (\varepsilon_{kk}^p - p_{kk}^{(r)})^2 \tag{18}$$

where $A^{(r)}$ and $B^{(r)}$ are fourth order positive definite tensors to be defined later.

So, it is possible to write

$$\frac{\partial \psi_D^p}{\partial \tilde{q}^{(r)}} + b_{11}^{(r)} \frac{d\tilde{q}^{(r)}}{dZ} = 0 \tag{19}$$

$$\frac{\partial \psi_H^p}{\partial p^{(r)}} + b_{22}^{(r)} \frac{dp^{(r)}}{dZ} = 0 \tag{20}$$

where $b_{11}^{(r)}$ and $b_{22}^{(r)}$ are the resistance coefficients in shear and hydrostatic deformation, respectively. Both $b_{11}^{(r)}$ and $b_{22}^{(r)}$ are functions of intrinsic time scale and can be expressed as

$$b_{11}^{(r)}(Z) = b_{11}^{o(r)} f_D(Z) \tag{21}$$

and

$$b_{22}^{(r)}(Z) = b_{22}^{o(r)} f_H(Z) \tag{22}$$

where, $b_{11}^{o(r)}$ and $b_{22}^{o(r)}$ are constants and $f_D(Z)$ and $f_H(Z)$ are the hardening parameters for deviatoric and hydrostatic conditions, respectively. If f_D and f_H increase with Z, the material will harden, while if they decrease with Z, the material will exhibit softening behavior. But under no circumstances they can be negative quantities.

It is possible now for one to introduce a material time scale as

$$dZ^2 = dZ_D^2 + k^2 dZ_H^2 \tag{23}$$

where

$$dZ_D = \frac{dZ}{f_D} = \|d\bar{\varepsilon}^p\| \tag{24}$$

$$dZ_H = \frac{dZ}{\because f_H} = |d\varepsilon^p| \tag{25}$$

and k is a coupling term. $\|d\bar{\varepsilon}^p\|$ denotes norm of the incremental plastic deviatoric strain tensor and $|d\varepsilon^p|$ represents the absolute value of plastic hydrostatic strain.

With the above definitions it is possible to write

$$\frac{\partial \psi_D^p}{\partial \tilde{q}^{(r)}} + b_{11}^{o(r)} \frac{d\tilde{q}^{(r)}}{dZ_D} = 0 \tag{26}$$

$$\frac{\partial \psi_H^p}{\partial p^{(r)}} + b_{22}^{o(r)} \frac{dp^{(r)}}{dZ_H} = 0 \tag{27}$$

Substituting the expression for ψ_D^p and ψ_H^p into the above equations and defining internal forces $\tilde{Q}_r$ and P_r as

$$\tilde{Q}_r = -\frac{\partial \psi_D^p}{\partial \tilde{q}^{(r)}} \tag{28}$$

$$P_r = -\frac{\partial \psi_H^p}{\partial p^{(r)}} \tag{29}$$

it is possible to write equations (26) and (27) as

$$\alpha_r \tilde{Q}_r + \frac{d\tilde{Q}_r}{dZ_D} = A_r \frac{d\tilde{e}^p}{dZ_D} \tag{30}$$

and

$$\beta_r P_r + \frac{dP_r}{dZ_H} = B_r \frac{d\varepsilon^p}{dZ_H} \tag{31}$$

where,

$$\alpha_r = \frac{A_r}{b_{11}^{o(r)}} \tag{32}$$

and

$$\beta_r = \frac{B_r}{b_{22}^{o(r)}} \tag{33}$$

If it is assumed that, at $Z_H = Z_D = 0$

$$\tilde{Q}_r(0) = \tilde{Q}_r^o \tag{34}$$

and

$$P_r(0) = P_r^o \tag{35}$$

integration of the equations (30) and (31) yields

$$\bar{s} = \int_0^{Z_D} \rho(Z_D - Z) \frac{\partial \tilde{e}^p}{\partial Z} \, dZ \tag{36}$$

and

$$\sigma = \int_0^{Z_H} \phi(Z_H - Z) \frac{\partial \varepsilon^p}{\partial Z} \, dZ \tag{37}$$

where,

$$\rho(Z_D) = \sum_r A_r \, e^{-\alpha_r Z_D}$$

$$\phi(Z_H) = \sum_r B_r \, e^{-\beta_r Z_H}$$

In order to satisfy the Clausius-Duhem inequality, it is necessary that $\alpha_r \geq 0$, $\beta_r \geq 0$ and $A_r \geq 0$, $B_r \geq 0$. Moreover, to ensure that $\rho(Z_D)$ and $\phi(Z_H)$ are singular at the origin and integrable over a finite domain, we must have

$$\rho(0) = \sum_{r=1}^{\infty} A_r = \infty \tag{38}$$

$$\phi(0) = \sum_{r=1}^{\infty} B_r = \infty \tag{39}$$

and

$$\sum_{r=1}^{\infty} \frac{A_r}{\alpha_r} < \infty, \qquad \sum_{r=1}^{\infty} \frac{B_r}{\beta_r} < \infty \tag{40}$$

It may be noted that while the sums of the internal forces $\tilde{Q}_r$ and P_r are assumed to be zero initially (i.e., at $Z_H = Z_D = 0$), they do not require the individual internal forces to be identically zero initially. Thus the above equations have implications regarding initial anisotropy as discussed by Valanis and Peters [31].

The soils, unlike metals, are in general compressible materials. The response of cohesionless soils, subjected to shear, depends on the initial relative density of the soil. If the initial relative density is less than a critical value, that is, a loose soil, the shear stress typically increases monotonically with increasing shear strain to a limiting shear stress, the value of which depends on the confining pressure. The volume of the soil will contract up to a certain limit after which the soil starts to behave like a dense incompressible material. If, however, the initial relative density is greater than critical, that is a dense soil, the stress-strain curve will first increase to a peak stress and then decrease towards a finite limiting stress depending on the confining pressure. The volume of the soil may first experience contraction due to initial densification of the soil particles but later on shows large dilation with the emergence of cracks and sometimes shear bands.

In the Endochronic theory of Valanis and Read (eqns. (36) and (37)), coupling between shear and hydrostatic components is achieved through the definition of the intrinsic time scale (refer to eqn. (23)) which permits only shear-induced contraction. To take care of the dilation part during shearing, the theory needs to be modified. To incorporate such behavior, a coupling term into the hydrostatic rate equations has been introduced giving rise to a coupling between shear and hydrostatic components that can produce either contraction or dilation.

It has been assumed that though the deviatoric plastic work is a cause of

hydrostatic plastic strain, it is external to the hydrostatic process in the sense that it is not a hydrostatic mechanism. Thus it qualifies as a thermodynamic internal force of the first kind in the sense of Valanis [31].

Thus, the thermodynamic equations appropriate to represent the plastic behavior of a frictional material, like soil, are given by

$$\frac{\partial \tilde{\psi}_D^p}{\partial \tilde{q}^{(r)}} + b_{11}{}^{(r)} \frac{d\tilde{q}^{(r)}}{dZ} = 0 \tag{41}$$

$$\frac{\partial \tilde{\psi}_H^p}{\partial p^{(r)}} + b_{22}{}^{(r)} \frac{dp^{(r)}}{dZ} = R_r \tag{42}$$

where

$$R_r = -b_{21}{}^{(r)} \tilde{s} \cdot \frac{\partial \tilde{e}^p}{dZ} \tag{43}$$

It may be noted that equation (41) is the same as equation (26). But unlike equation (27), equation (42) contains an additional term R_r given by equation (43) through which the coupling between the hydrostatic and deviatoric strain rates and the corresponding internal forces are achieved. In the above equation ψ_D^p and ψ_H^p are given by equations (17) and (18), and $b_{11}^{(r)}$, $b_{22}^{(r)}$ and $b_{21}^{(r)}$ are known as resistance coefficients or Onsager coefficients. $b_{11}^{(r)}$ and $b_{22}^{(r)}$ are not constants but are related to the hardening functions f_D and f_H by equations (21) and (22). The coefficient b_{22} in a frictional material will depend very strongly on the prevailing hydrostatic stress, σ, as well as on the porosity (or equivalently the density or plastic volumetric strain, ϵ^p) and the dependence is likely to be substantially additive.

Like before if it is assumed that, at $Z_H = Z_D = 0$

$$\tilde{Q}_r(0) = 0 \tag{44}$$

and

$$P_r(0) = 0 \tag{45}$$

it is possible to express the deviatoric and hydrostatic stresses as

$$\tilde{s} = \int_0^{Z_D} \rho(Z_D - Z) \frac{\partial \tilde{e}^p}{\partial Z} dZ \tag{46}$$

and

$$\sigma = \frac{\sigma_{kk}}{3} = \int_0^{Z_H} \phi(Z_H - Z) \frac{\partial \epsilon^p}{\partial Z} dZ + \int_0^{Z_H} \Gamma(Z_H - Z) \tilde{s} \cdot \frac{\partial \tilde{e}^p}{\partial Z} dZ \tag{47}$$

where,

$$\rho(Z_D) = \sum_{r=1}^{N} A_r e^{-\alpha_r Z_D} \tag{48}$$

$$\phi(Z_H) = \sum_{r=1}^{N} B_r e^{-\beta_r Z_H} \tag{49}$$

$$\Gamma(Z_H) = \sum_{r=1}^{N} \beta_r \frac{b_{21}^{(r)}}{f_H} e^{-\beta_r Z_H} \tag{50}$$

If it is assumed that the effect of the deviatoric plastic work rate is distributed uniformly among all the hydrostatic mechanisms (Valanis and Peters, 1988) then,

$$b_{21}^{o\,(r)} = b_{21}^{o} \tag{51}$$

and

$$\Gamma(Z_H) = b_{21}^{o} \phi(Z_H) \tag{52}$$

Equations (46) and (47), which represent the constitutive relations of the new thermodynamic theory, may or may not show a yield surface. Whether they will show any particular yield shape or not depends very much on the choice of the internal variables. A yield surface is a mathematical consequence of representing at least one of the exponential terms of the kernel function with a Dirac-delta function.

It may be shown that the above two integral equations (eqns. (46) and (47) may be alternatively expressed as

$$(\tilde{s} - \tilde{r}) = \rho_o f_D \frac{d\tilde{e}^p}{dZ} \tag{53}$$

and

$$(\sigma - r') = \phi_o \frac{d\varepsilon^p}{dZ_H} + \Gamma_o \tilde{s} \cdot \frac{d\tilde{e}^p}{dZ} - \frac{1}{\phi_o} \frac{d\sigma}{dZ_H} \tag{54}$$

where,

$$\tilde{r} = \int_0^{Z_D} \rho_1(Z_D - Z) \frac{\partial \tilde{e}^p}{\partial Z} dZ \tag{55}$$

and

$$r' = \int_0^{Z_H} \phi_1(Z_H - Z) \frac{\partial \varepsilon^p}{\partial Z} dZ + \int_0^{Z_H} \Gamma_1(Z_H - Z) \tilde{s} \cdot \frac{\partial \tilde{e}^p}{\partial Z} dZ \tag{56}$$

ϕ_o, ρ_0, Γ_o are material parameters defined later and ϕ_1, ρ_1 and Γ_1 are functions like ϕ, ρ and Γ respectively. Thus the equation for the yield surface can be written as

$$\|\tilde{s} - \tilde{r}\|^2 + \left(\frac{\rho_o f_D}{k\phi_o f_H}\right)^2 \left[(\sigma - r') - \frac{\Gamma_o}{\rho_o f_D} \tilde{s} \cdot (\tilde{s} - \tilde{r})\right]^2 - \rho_o^2 f_D^2 = 0 \tag{57}$$

Above equation describes a yield behavior that includes both kinematic and isotropic hardening. If f_D and f_H are constant, the behavior is purely kinematic. For incompressible material, like metals, where $\phi_o f_H \rightarrow \infty$, the deviatoric projection of the yield surface is circular with radius $\rho_o f_D$ showing Von Mises type of yield surface. While for compressible materials like soils and concrete, where $\Gamma \neq 0$, the projection in the deviatoric plane is non-circular.

The Mohr-Coulomb type of behavior may be accomplished by assuming that the hydrostatic response may be represented by the summation of a series containing only a single exponential term, thus

$$\phi(Z_H) = \phi_o \delta(Z_H) \tag{58}$$

and

$$\Gamma(Z_H) = \Gamma_o f_H \delta(Z_H) \tag{59}$$

where,

$$\delta = \text{Dirac-Delta function}$$

$$\Gamma_o = \sum_{r=1}^{N} b_{21}^{o\,(r)} \tag{60}$$

For this particular case, the hydrostatic stress may be expressed as :

$$\sigma = \frac{\sigma_{kk}}{3} = \phi_o\, f_H\, k\, \frac{d\varepsilon^p}{dZ} + \Gamma_o\, \tilde{s} \cdot \frac{d\tilde{e}^p}{dZ} - \frac{1}{\phi_o}\, \frac{d\sigma}{dz_H} \tag{61}$$

For purely hydrostatic conditions ($\tilde{s} = 0$, $\tilde{e}^p = 0$)

$$\sigma_o = \text{a function of } \varepsilon^p \tag{62}$$

for all σ_o.

To achieve a Mohr-Coulomb type of behavior the deviatoric hardening parameter should be expressed in the following form

$$f_D = \delta_o + \eta\sigma \tag{63}$$

where δ_o and η are material parameters and σ is the mean stress.

It has been shown (Valanis and Peters, 1988 [31]) that under constant hydrostatic stress condition, deviatoric and hydrostatic plastic strains may be related by the following equation

$$k\frac{d\varepsilon^p}{de^p} = \frac{1 - \Gamma_o^2\, S^{*-2}}{f_H \Gamma_o S^* + \sqrt{f_H^2 + \Gamma_o^2 S^{*-2} - 1}} \tag{64}$$

with

$$f_H = (1 - \varphi_1)e^{\beta e^p} + \varphi_1 \tag{65}$$

and

$$S^* = \frac{s}{\sigma_o} \tag{66}$$

In the above equation Γ_0 is a coupling term which relates the shear-induced volume change to the deviatoric plastic work. β and ϕ_1 are material parameters and ϕ_0 is any suitable large number needed for the closure of hysteresis loops under hydrostatic loading cycles.

DETERMINATION OF MATERIAL PARAMETERS

Material parameters for constitutive models derived from principles other than traditional elasticity and plasticity theories are often criticized for not having any physical meaning to go with them. Based on the above theory, material parameters have been derived for an ordinary sand and a cemented sand to illustrate the fact that the material parameters for the thermodynamic theory do possess physical meaning and they can be related to prevailing elastic and plastic material parameters. It has been shown here that all the parameters can be obtained from a series of consolidated drained triaxial test results. There are more than one way to determine material parameters, as will be shown below, depending on the material behaviors to be modelled and availability of data. In this paper only the material parameters have been derived for the above two materials. The actual predictions of the model along different stress paths will be discussed in a different place.

A detailed description of the sand which has been used here for verification purposes may be obtained from the proceedings of the International Workshop on Constitutive Equations for Granular Non-Cohesive Soils (1987). The information contains results of hydrostatic tests and triaxial compression and extension tests performed at different initial spherical stresses. Out of the two type of sands utilized in the conference, data on the sand referred to as Hostun sand have been utilized for the present investigation. The data presented in the workshop may be divided into two large categories: one obtained from a hollow cylinder device (like Tests HH1, PHH1, PHH3, etc.) and the other obtained from a cubical (true triaxial) device (like Tests CH1, CH2, CHC, etc.)

For the material parameter determination, the data of the hollow cylinder test (labelled as HH1 in this paper) performed under an initial spherical stress of 203 KPa has been utilized. Material parameters for Hostun sand are determined according to the guideline provided by Peters (1987). Figures 1 and 2 show the triaxial and the hydrostatic test results (HH1 and HH6) as obtained from the proceedings. From the test results, Young's modulus and bulk modulus are estimated as 260,000 KPa and 225,000 KPa, respectively. Under purely hydrostatic conditions like in Test HH6, the hydrostatic response is given by equation (62). The plastic volumetric strains may be calculated using the relation
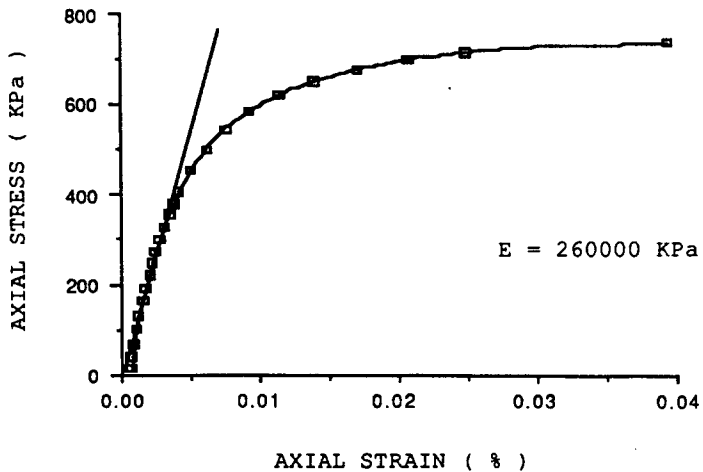
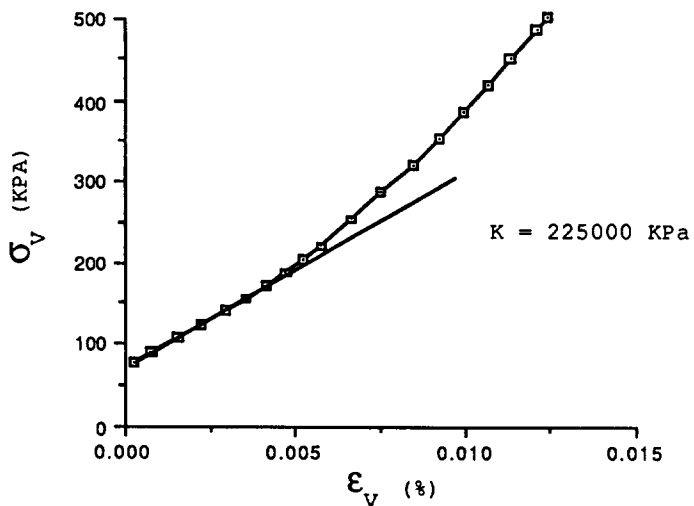Figure 1 Determination of Young's Modulus

Figure 2 Determination of Bulk Modulus

$$\varepsilon_v^P = \varepsilon_v - \frac{\sigma_v}{K} \tag{67}$$

One suitable form for $\phi_o(\epsilon^P)$ in equation (62) which fits the experimental data may be expressed as

$$\sigma = \varphi_o(\varepsilon^P) = \frac{C_1 \, e^{\beta_1 \, \epsilon^P}}{1 + C_2 \, e^{-\beta_2 \epsilon^P}} \tag{68}$$

In the above equation, $\phi_o(\epsilon^P)$ may be viewed as a strain dependent limiting pressure for hydrostatic loading. The constants C_1, β_1, C_2 and β_2 may be readily obtained by a curve fitting the results of the hydrostatic test. A typical virgin hydrostatic compression curve for soils is initially convex in shape and quickly turn concave in shape as ϵ^P becomes greater than zero. In equation (68), C_2 and β_2 control the initial hydrostatic behavior of a material during which $\epsilon^P \leq 0$ and the curve is convex. Once ϵ^P becomes positive and the hydrostatic curve becomes concave, the denominator of the above equation becomes unity and the other two parameters, C_1 and β_1, control the behavior of the material along the concave portion of the hydrostatic curve. Direct fit of ϕ_o to the hydrostatic loading data in the present case poses some difficulties. The difficulties arise from the fact that for Hostun sand the hydrostatic test is conducted at low pressures while it needs very high pressure for the sand to yield. Thus, the hydrostatic data does not represent the virgin consolidation curve. Based on past knowledge of behavior of sand and using the fact that the predictions do not require an accurate response to changes to σ, it has been assumed that ϕ_o is a constant. In which case $\beta_1 = \beta_2 = C_2 = 0$ and $C_1 > \sigma$ in the equation (68). Thus ϕ_o may be considered as an additional constant to be determined from equation (64). Equation (64) gives the stress-dilatancy relationship which may be conveniently expressed as

$$k \frac{d\varepsilon^P}{de^P} = \frac{\sigma^2 - \Gamma_o^2 s^2}{f_{II} \, \Gamma_o \, s + \sigma \, \sqrt{f_H^2 + \Gamma_o^2 s^2 - \sigma^2}} \tag{69}$$

where,

$$f_{II} = (1 - \varphi_1) \varphi_o + \varphi_1 \, \sigma \tag{70}$$

The above equations allow one to determine material parameters k, Γ_o, ϕ_1 and ϕ_o from a single triaxial test result. Figure 3 shows the shear-volume relationship for the Hostun sand. Plots of (s/σ) versus $(d\epsilon_v^P/de^P)$ for different test (loading part only) results performed at different confining pressure have been shown for comparison purposes. It may be seen from the figure that the test data of HH1 ($\sigma_3 = 203$ KPa) is quite scattered and needs some kind of judgement in fitting procedure. Data for the tests performed at higher
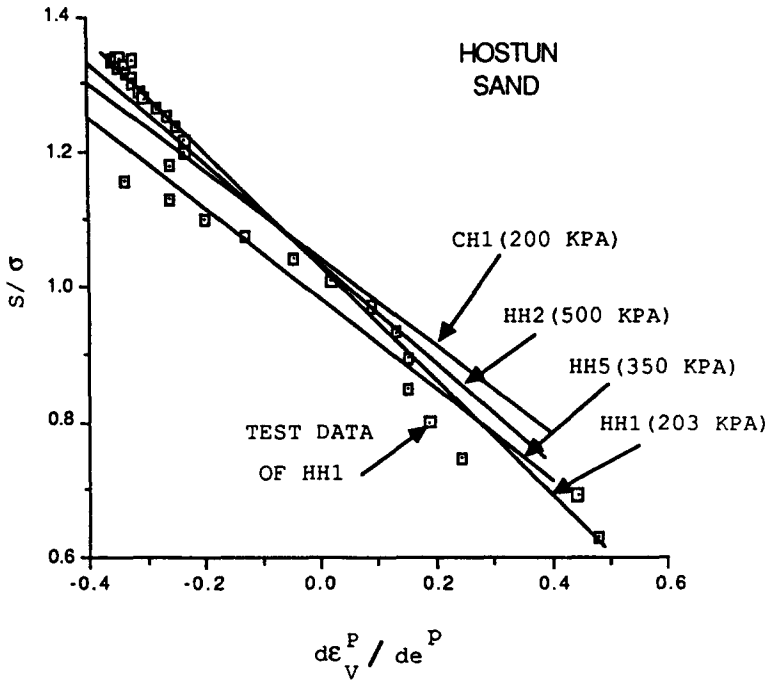
$$(S/\sigma) = 0.954557 - 0.974492(d\varepsilon_v^p / de^p)$$

Figure 3 Determination of Coupling Parameter

confining pressure (HH2 (σ_3 = 500 KPa)) show a more or less linear stress-dilatancy relationship which is to be expected for a dense sand. In the present study, however, the test results of HH1 has been used for the calibration purposes. It is to be noted from Figure 3 that the test results from the cubical device (CH1) is appreciably different from that obtained from the hollow device (HH1). This difference will get magnified further while predicting the test results for the cubical device, as will be seen in the next section. The coupling coefficient, $\Gamma 2o$, is defined as the ratio of σ/s at which $d\epsilon_v^p = 0$ and thus may be determined directly from Figure 3. The parameter Γo physically represents a stress level where the material starts to dilate. The other parameters k, ω_1 and ω_o may be obtained by solving the equation (69) for other sets of points lying on the stress-dilatancy curve.

For a frictional material (material following Mohr-Coulomb law), the functional for of the deviatoric hardening parameter is given by equation (63), which physically represents the failure envelope during shearing under constant hydrostatic stress, σ. The parameters δ_o and η may be physically interpreted as cohesion and tangent of friction angle as in Mohr-Coulomb theory. For pure sand δ_o = 0 and η is chosen in such a way that f_D = 1 at some convenient mean stress level. Thus as mentioned before f_D plays the role of a scaling factor that accounts for the mean stress level. Table 1.0 shows the value of the parameters $k, \Gamma_o, n, \delta_o, \omega_1$ and ω_o determined for Hostun sand. To have a yield surface type of theory ϕ_o (or B) in the equation (61) should be given a suitable large value so that

$$\phi_o = B = \sum_r B_r = \infty \tag{71}$$

Determination of the parameters A_r and α_r (for r number of mechanisms) requires determination of the incremental time scale dZ and the hardening parameter f_D. Knowing de^p, $d\epsilon^p$ and k it is possible to calculate dZ for every data point of a conventional triaxial test. Once dZ and f_D are known, it is possible to calculate dep/dZD at each point using the relation

$$\frac{d\epsilon^p}{dZ_D} = \frac{f_D}{\sqrt{1 + \left(k \frac{d\epsilon^p}{de^p}\right)^2}} = G(\omega) \tag{72}$$

where,

$$\omega = Z_D - Z_D^\circ$$

Thus, the expression for the deviatoric stresses may be specialized for the

conventional triaxial test by

$$s(\omega) = \int_0^\omega \rho(\omega - \omega') \, G(\omega') \, d\omega' \tag{73}$$

At each ω_n the integral in the above equation can be approximated as a summation over n terms; which leads to a system of linear simultaneous equations which may be easily solved by successive substitution to produce the following algorithm:

$$\rho(\omega_n) = \frac{dM(\omega_n)}{d\omega} \tag{74}$$

where,

$$M(\omega_n) = \frac{s(\omega_n) - \sum_{m=1}^{n-1} \Delta G_{n-m+1} \, M(\omega_m)}{\Delta G_1} \tag{75}$$

and

$$\Delta G_n = G(\omega_n) - G(\omega_{n-1}) \tag{76}$$

Both $s(\omega)$ and $G(\omega)$ are computed from the test data at equally spaced interval and then smoothened prior to determining the incremental values needed for equation (74) by fitting each with an exponential series. Figures 4 and 5 show the values of $s(\omega)$ and (ω) with increasing Z_D as calculated from the test results and the exponential series which are used to smooth the curves.

Determination of A_r and α_r amounts to fitting $p(\omega)$ with an exponential series subjected to the following constraints

$$A = \sum_{r=1}^N A_r = \infty \tag{77}$$

and

$$\sum_{r=1}^N \frac{A_r}{\alpha_r} = \sigma_Y \tag{78}$$

A Fortran program has been developed (Sengupta, 1989), based on equation (69) through (76), to compute $p(Z_D)$ at equal intervals of Z_D. A graphical procedure developed by Peters (1987) has been utilized to fit an exponential series to the computed values of $p(Z_D)$. The procedure is based on the observation that when the terms in the series are ordered such that $\alpha_1 < \alpha_2 < \ldots < \alpha_N$, the contribution of terms having the larger values of α_r to the summation becomes small as Z_D becomes a straight line having an intercept at $\omega = 0$ equal to A_1 and slope equal to α_1. By defining a new function ρ' such that

$$\rho' = \rho - A_1 \, e^{-\alpha_1 Z_D} \tag{79}$$

Table 1 Material Parameters for Sand

$$F_D = \delta_0 + \eta\,\sigma_m$$

$$F_H = (1 - \varphi_1)\,\varphi_0 + \varphi_1\,\sigma_m$$

K = 225000.0 KPA

G = 100000.0 KPA

k = 0.741 $\quad\gamma$ = 108 Pcf

Γ_0 = 1.026

η = 0.002

δ_0 = 0.0000001 $\quad\phi$ = 39 Deg.

φ_1 = 0.817

φ_0 = 1018.9

B = 1750000.0

A_1 = 16480.0 $\quad\alpha_1$ = 83.794

A_2 = 52000.0 $\quad\alpha_2$ = 403.2
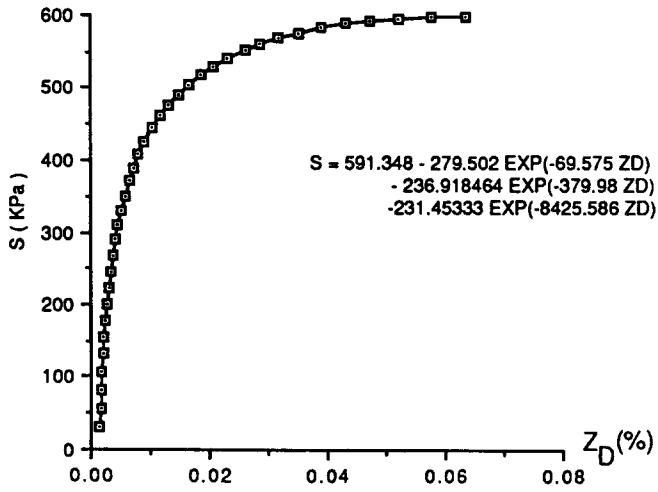
A_3 = 250000.0 $\quad\alpha_3$ = 773.55

$$S = 591.348 - 279.502 \, EXP(-69.575 \, ZD)$$
$$- 236.918464 \, EXP(-379.98 \, ZD)$$
$$-231.45333 \, EXP(-8425.586 \, ZD)$$

Figure 4   Functional Form of S

$$G = 0.855462 - 0.06145 \, EXP(-952.1628 \, ZD)$$
$$-0.199283 \, EXP(-369.996258 \, ZD)$$
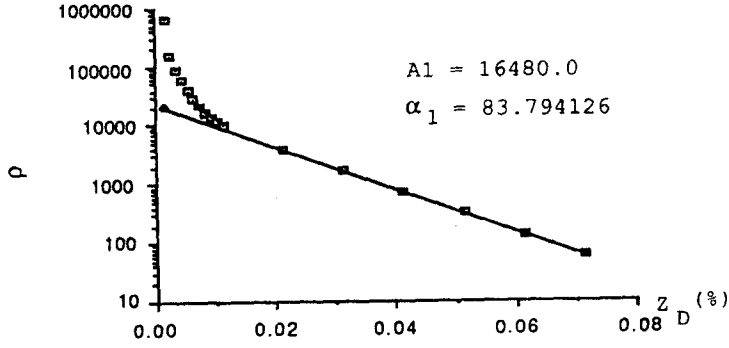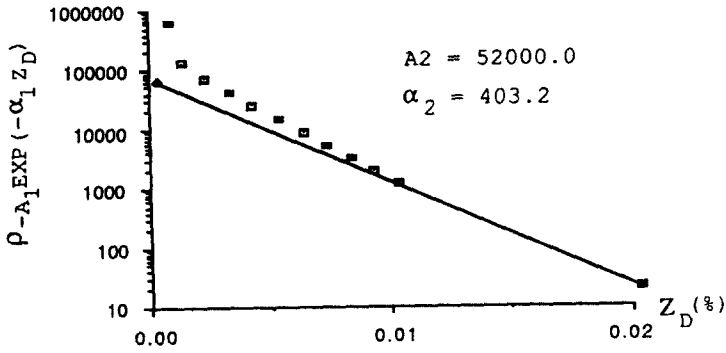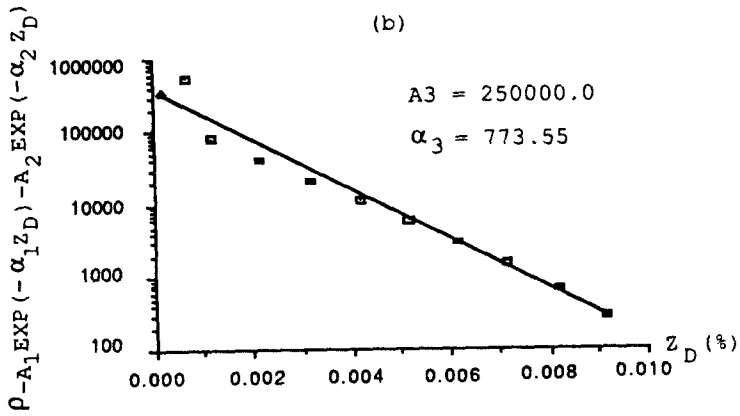$$-0.178564 \, EXP(-70.843904 \, ZD)$$
$$-0.295233 \, EXP(-9049.304192 \, ZD)$$
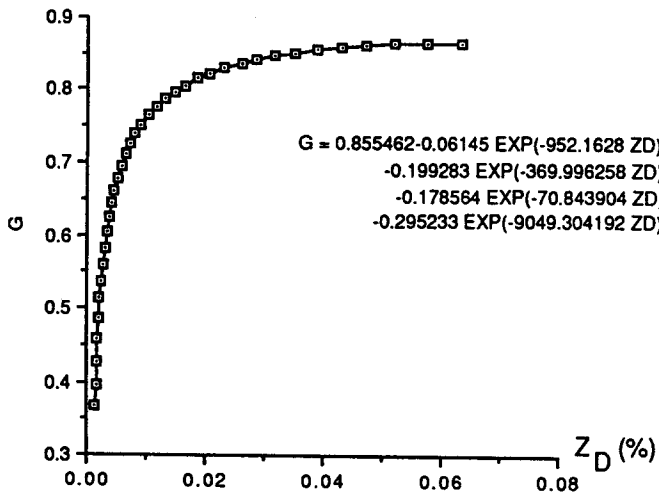
Figure 5   Functional Form of G

Figure 6 Determination of Internal Variables

The same procedure may be used to determine A_2 and α_2, etc. The process is repeated until the remainder ρ'_N can be approximated as a straight line or as a Dirac delta function. The exponential fitting procedure for $\rho(Z_D)$ has been illustrated in Figure 6. It has been found that three terms in the exponential series are sufficient to describe $\rho(Z_D)$ for the Hostun sand. Values of A_r and α_r are shown in Table 1. It may be necessary to adjust the values of A_3 and α_3 which represent the response of the initial part of $\rho(Z_D)$, to satisfy the constraints given by equations (77) and (78) and the volume change behavior of the material. The above discussion exemplifies the determination of material parameters from hollow triaxial tests. The parameters can also be determined from other tests as well, as shown in other cases.

The test results used in this study to show the determination of material parameters and calibration of the model for a cemented sand are taken directly from Avramidis (1985). Out of different types of cemented sands reported in that work, a cemented sand with 2% cement content; curing period of 15 days and with 80% relative density has been chosen specifically for the present study. A detailed description of the cemented sand and the testing procedure followed may be obtained from Avramidis (1985).

Figures 7 and 8 show the conventional triaxial test results at three different constant confining pressures. No independent hydrostatic tests were performed. So, in absence of any hydrostatic test results, the volumetric strains reported in the hydrostatic part at the very beginning of the triaxial tests are utilized to construct Figures 9 and 10. Bulk modulus for the cemented san, thus obtained, is shown in Figure 9. Young's modulus and shear modulus for the cemented sand at confining pressures of 49 KPa, 245 KPa and 490 KPa are shown in Figures 7 and 8. It was found that the bulk modulus obtained from Figure 9 does not represent the volumetric response of the material properly, since these hydrostatic results were representative of material behavior at very small magnitude of confining pressures. So it was decided to use the triaxial test data instead to obtain the bulk modulus at different confining pressures. For a conventional triaxial test in which the deviatoric stress $(\sigma_1 - \sigma_3)$ increases from zero while the confining pressure is held constant, bulk modulus may be defined as

$$K = \frac{(\sigma_1 - \sigma_3)}{3\,\varepsilon_v} \tag{84}$$

where ϵ_v is the volumetric strain corresponding to the deviatoric stress $(\sigma_1 - \sigma_3)$.
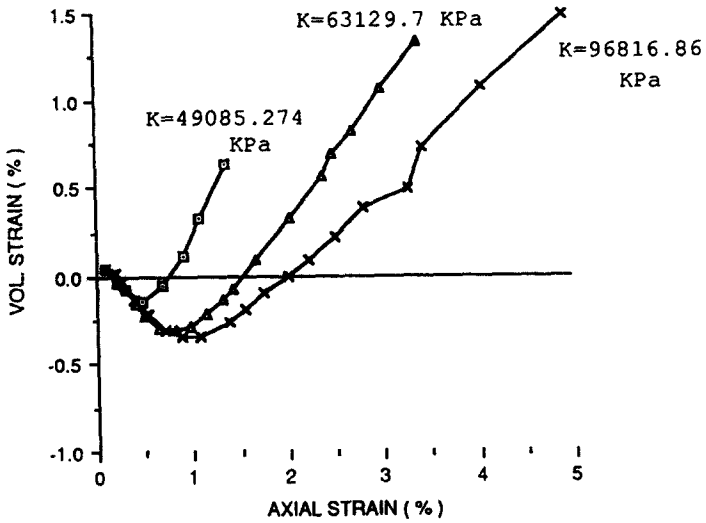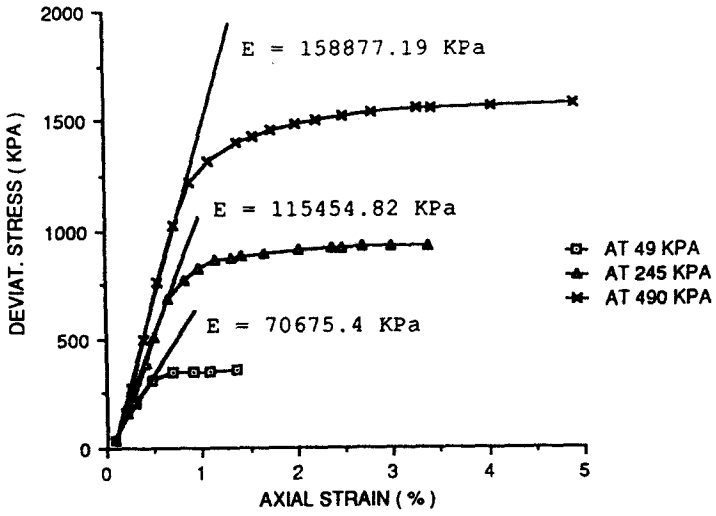
Figure 7 Stress-Strain Curve for Cemented Sand
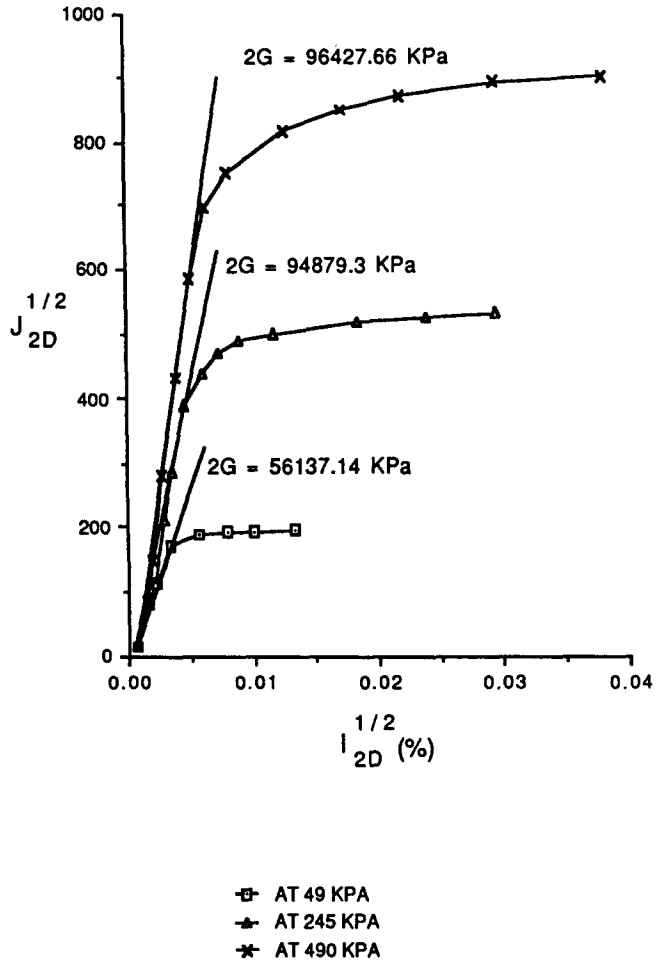(CC = 2 %; CP = 15 days; Dr. = 80 %)

Figure 8 Determination of Shear Modulus for
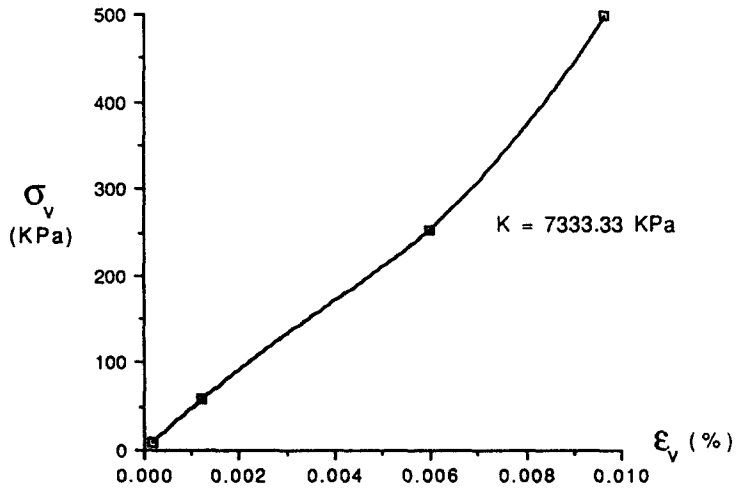Cemented Sand (CC = 2 %; Dr. = 80 %)
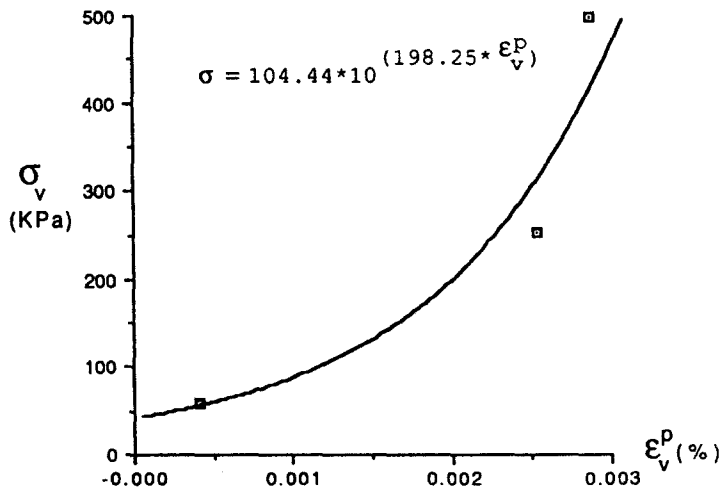
Figure 9 Hydrostatic Response of Cemented Sand

In the figure, the curve is labeled $K = 7333.33$ KPa, with axes σ_v (KPa) and ε_v (%).

Figure 10 Determination of Hydrostatic Parameters

In the figure, the curve is labeled $\sigma = 104.44 \times 10^{(198.25 \times \varepsilon_v^p)}$, with axes σ_v (KPa) and ε_v^p (%).
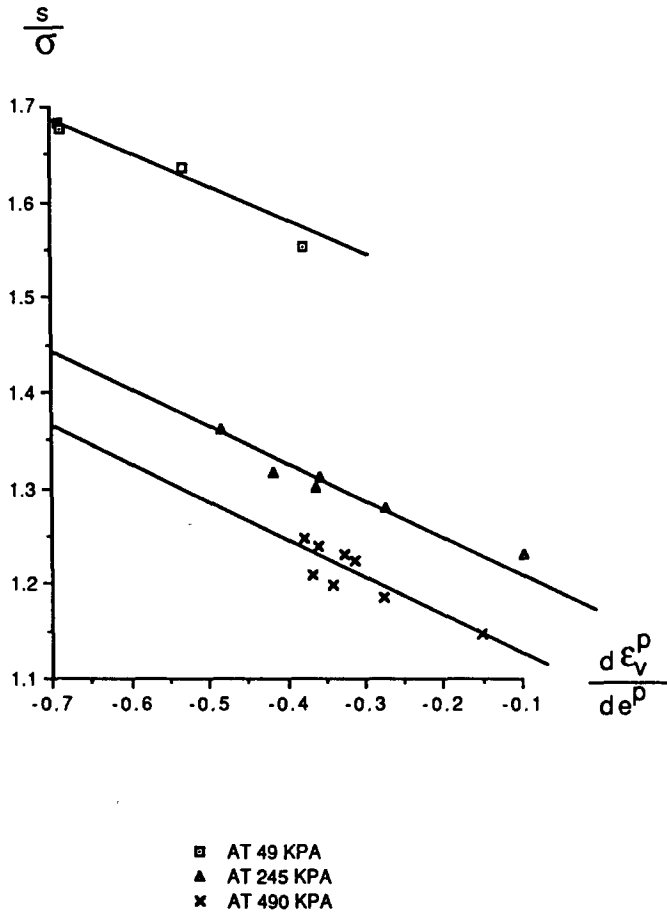
Figure 11 Stress-Dilatancy Relation for the Cemented Sand

$$\Gamma_o = 0.99 - 0.492 \cdot e^{-0.009423 \cdot \sigma_3}$$

Figure 12 Variation of Γ_o with σ

$$\varphi_o = 1200.0 - 1716.92 \cdot e^{-0.008891 \cdot \sigma_3}$$
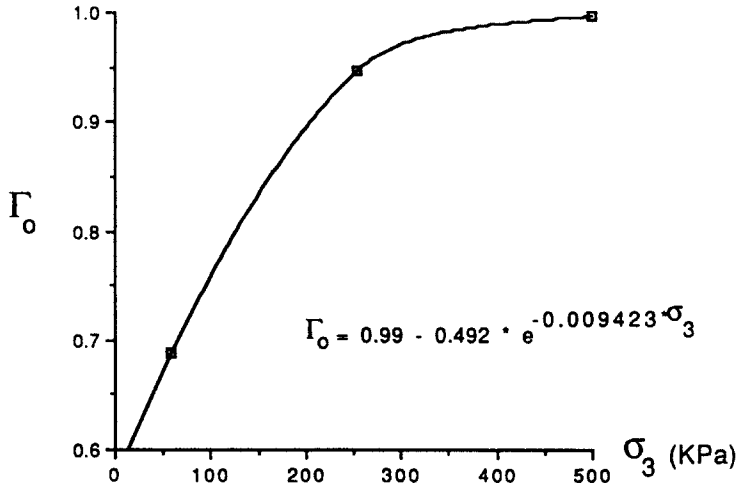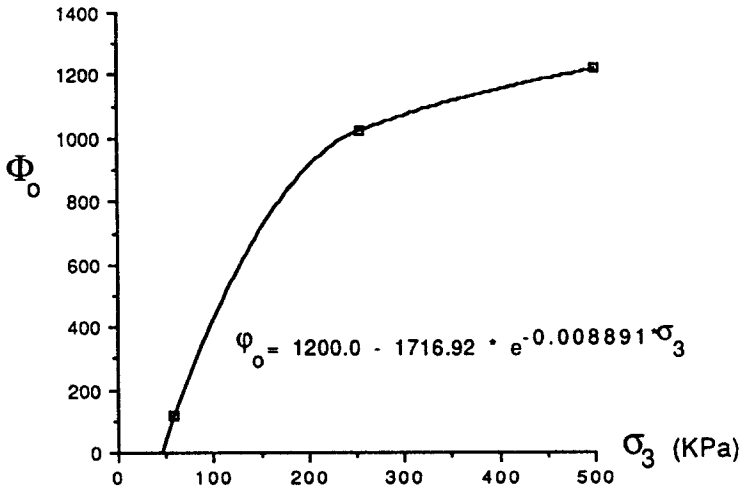
Figure 13 Variation of Φ_o with σ

The value of bulk modulus calculated from the above equation may be found to vary somewhat throughout the stress-strain curve. Study of the volume change behavior of a wide variety of soils has led to the following criteria for selecting which point to be used in calculating bulk modulus:

1. If the volume change curve does not reach a horizontal tangent prior to the stage at which 70% of the strength is mobilized, then the point on the stress-strain and volume change curves corresponding to a stress level of 70% will be used to calculate the bulk modulus.

2. If the volume change does reach a horizontal tangent prior to the stage at which 70% of the strength is mobilized, then the point on the volume change curve where the curve becomes horizontal and the corresponding point on the stress-strain curve should be used to calculate bulk modulus.

The bulk moduli obtained employing the above methodology are shown in Figure 7. For the given cemented sand it has been found that the following functional forms of Young's modulus and bulk modulus adequately represent the constitutive behavior of the material:

$$E = 62978.617 + 199.071\,\sigma_3 \qquad (85)$$

$$K = 44660.0 \times 10^{0.0006726\,\sigma_3} \qquad (86)$$

In the above relations σ_3 is the confining pressure. Knowing E and K, the shear modulus, G, may be calculated from the elastic theory. The values of shear moduli thus obtained are checked against those obtained directly from the triaxial test results and found to be close.

In the absence of any reliable hydrostatic test results, the ϕ_0 will be assumed to be a constant for a given confining pressure and thus may be obtained, as before, from equation (64). For the given cemented sand, ϕ_0 has been found to be a function of the confining pressure, σ_3 and well represented by the following equation:

$$\varphi_o = \varphi'_o - \varphi''_o\, e^{-\omega\,\sigma_3} \qquad (87)$$

where ϕ'_o, ϕ''_o and ω are material constants.
Figure 13 shows the variation of ϕ_0 with confining pressure and the assumed numerical values for the constants in the above equation.

Figure 11 shows the shear-volume relationship for the cemented sand at different confining pressure. The shear-volume relationship depicted by Figure 11 is significantly different from that obtained for Hostun sand. Unlike Hostun
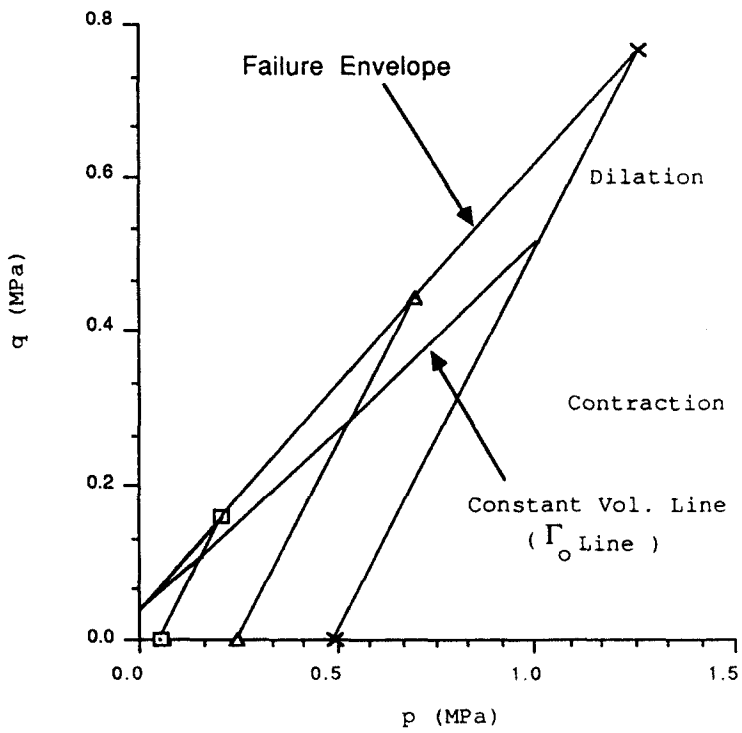
Figure 14 Q-P Plot and Determination of Hardening
    Function
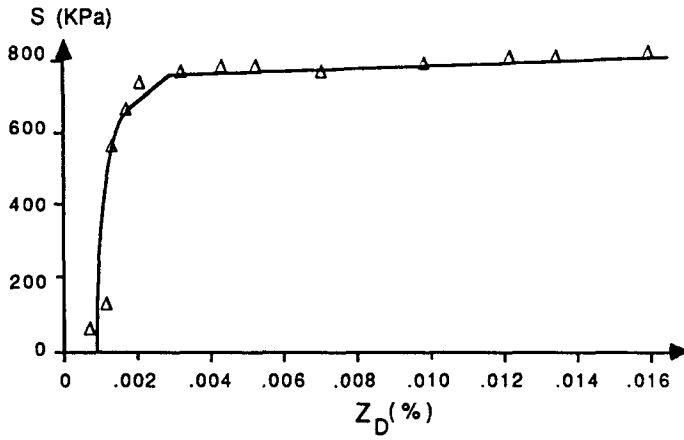
(a)
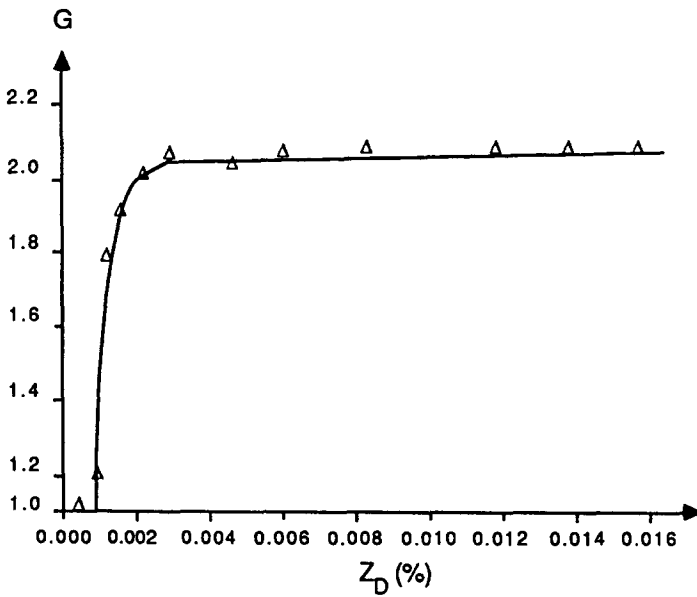
(b)

Figure 15 Variation of s and G with Z_D
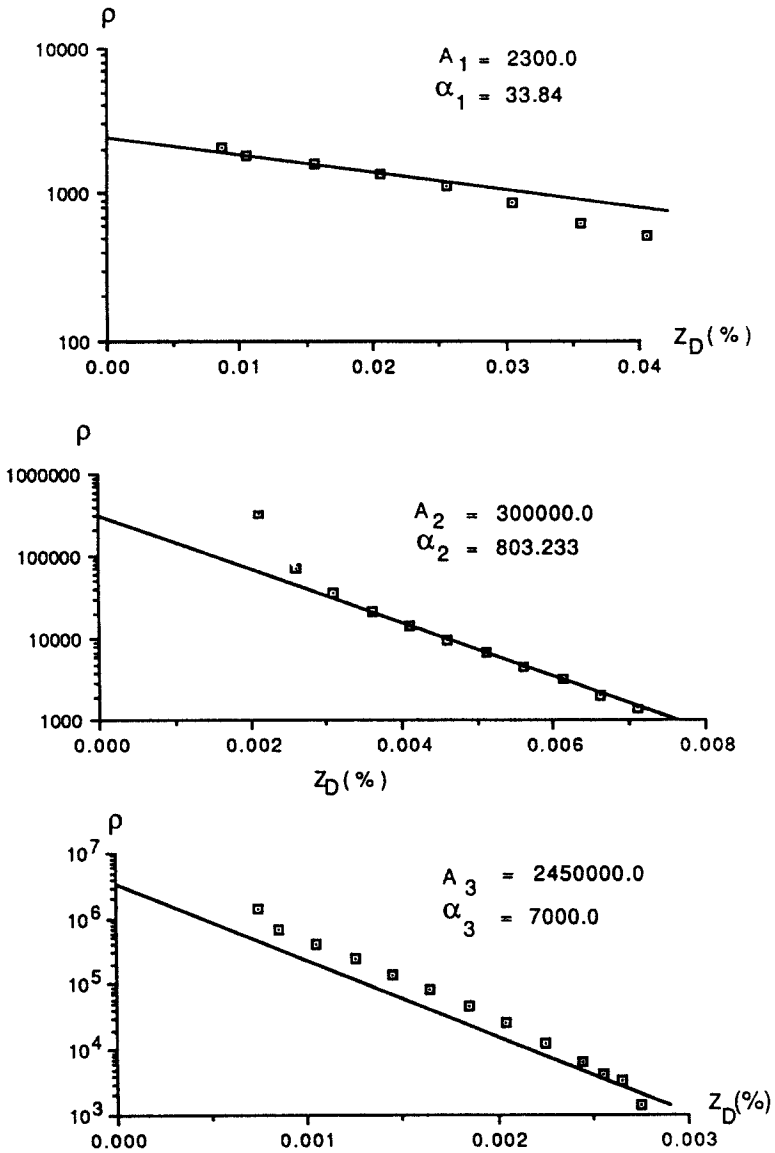
Figure 16 Determination of Kernel Functions for
Cemented Sand (AT 245 KPa)

sand, in cemented sand the shear-volume relationship is dependent on the initial confining pressure at which the test are conducted. As before a linear relationship is assumed for each individual case and the curves are fitted to each set of data in such a way that they are parallel to each other. Physically this means that the coupling of shear and volumetric parts remains indifferent to the confining pressure (k = constant), only the parameter Γ_o will vary depending on the confining pressure. Thus the stress level at which the material starts to dilate will be shifting depending on the confining pressure. The following functional form for Γ_o has been found to work well in the present case:

$$\Gamma_o = \Gamma'_o - \Gamma''_o e^{-\vartheta \sigma_3} \tag{88}$$

where Γ'_o, Γ''_o and θ are material constants.

The variation of Γ_o with the confining pressure and the values of the constants in the above equation are shown in Figure 12. Keeping with the prevailing ideas in the soil mechanics, the constant volume line (Γ_o - line) can be alternatively expressed by the following hyperbolic equation:

$$\frac{q}{p} = \frac{q_o}{p} + b \tag{89}$$

where q_o and b are constants and can be determined from the q-p plot (Figure 14) as the intercept on the q axis and the slope of the Γ_o line, respectively. For the cemented sand the values of q_o and b are found to be 0.0223 KPa and 0.493, respectively.

The other two parameters, k and ϕ_1 are obtained in the usual manner from equation 41 for two sets of data points lying on the shear-dilatancy curve for the case of σ_3 = 245 KPa.

Figure 14 shows the q-p plot and the failure envelope for the given cemented sand. The hardening parameters, δ_o and η are determined from the normalized equation of the failure envelope such that f_η = 1 at σ_3 = 245 KPa, where the kernel functions are evaluated.

It may be noted that for the determination of all the material parameters including the kernel functions, the result of the conventional triaxial test performed at a constant confining pressure of 245 KPa has been utilized.

Figure 15 shows the variation of $s(\omega)$ and $G(\omega)$ with endochronic time evaluated from the test results and the smoothened curves used in the determination of the kernel functions. It may be noted that in both cases the smoothened curves do not fit the initial part of the test results very well and

are a potential source of errors. Procedure followed for the determination of the kernel functions is described in Figure 16. It may be noted that in determining the third set of internal variables, A_3 and α_3, values obtained from the best fit are not used. Instead, they are obtained by trial and error such that the constraints given in equations (77) and (78) are satisfied and the volume change response of the material is well represented. The parameters, A_3 and α_3 represent the kernel function for the early part of the test. Thus, the above measures taken in the determination of both of them may be viewed as an attempt to rectify the errors that may have been incurred due to the misfit of the smoothening functions of $s(\omega)$ and $G(\omega)$ at the initial points of the test result.

CONCLUSIONS

The constitutive theory described above can model almost any material provided the parameters are chosen properly. This model, unlike the previously reported endochronic theory (Valanis, 1985), shows closure of hysteresis loop and follows Drucker's postulate when the model is simulating the classical plastic behavior. The present theory does predict the dilatant behavior of frictional materials, like sandy soil, very accurately. Being pseudo-static in nature, the theory is also capable of duplicating behavior of materials under dynamic conditions.

The material parameters for the model can be determined from ordinary consolidated drained triaxial test results. A number of ways are available, based on the type of data available and type of material behavior needs to model, for the determination of material parameters. The material parameters do possess some physical meaning and they can be explained from traditional soil mechanics theories.

NOTATIONS

α_r	Deviatoric Internal Variable Constants.
β_1	Material Constant.
β_2	Material Constant.
β_r	Hydrostatic Internal Variable Constants.
Γ_o	Material Constant Function.
Γ'_o	Material Constant.
Γ''_o	Material Constant.
γ	Unit Weight of a Material.
γ_{oct}	Octahedral Strain Invariant.

γ_o^p	Plastic Octahedral Strain Invariant.
δ_{ij}	Dirac-Delta Function.
δ_o	Material Constant.
ϵ_{ij}	Total Strains.
$\epsilon_{ij}^\bullet$	Elastic Strain.
ϵ^p	Hydrostatic Plastic Strain.
ϵ_{ij}^p	Plastic Strains.
ϵ_{ij}^p	Incremental Plastic Strains.
ϵ_v	Total Volumetric Strain.
ϵ_v^p	Plastic Volumetric Strain.
η	Entropy Density.
η_o	Material Constant.
v	Poisson's Ratio.
ξ	Intrinsic Time Measure.
ρ_o	$\Sigma_r A_r(-A)$
σ	Hydrostatic Stress ($-\sigma_{kk}/3$).
σ_{ij}	Total Stresses.
σ_{kk}	Volumetric Stress.
σ_v	Volumetric Stress ($-\sigma_{kk}$).
σ_y	Yield Stress of a Material.
τ	Shear Stress.
ϕ	Frictional Angle of a Material.
ϕ_o	$\Sigma_r B_r(-B)$.
φ_0	Material Constant.
φ_1	Material Constant.
$\varphi_\bullet$	A Porosity Dependent Function.
φ'_o	Material Constant.
φ''_o	Material Constant.
ψ	Helmholtz Free Energy Density Function.
A	$\Sigma_r A_r(-A)$.
A_r	Deviatoric Internal Variable Constants.
B	$\Sigma_r B_r$
B_r	Hydrostatic Internal Variable Constants.
b_D	Resistance Coefficient for Shear Part.
b_H	Resistance Coefficient for Hydrostatic Part.
$b_{ijkl}^{(r)}$	Resistance Tensors.
c	Cohesion of a Material.
E	Young's Modulus.
$\tilde{e}^\bullet$	Deviatoric Elastic Strains.

e^P	Deviatoric Plastic Strain Invariant.
$\tilde{e}^P$	Deviatoric Plastic Strains.
f_D	Deviatoric Hardening Function.
f_H	Hydrostatic Hardening Function.
G	Shear Modulus.
J_{2D}	Second Invariant of Stress.
K	Bulk Modulus.
k	Coupling Term.
$p(r)$	Hydrostatic Pseudo-Force Term.
$p_{kk}^{(r)}$	Internal Variables (Hydrostatic).
$\tilde{Q}$	Pseudo-Force Terms (Deviatoric).
Q_{ij}^r	Internal Forces.
$q_{ij}^{(r)}$	Internal Variables (Deviatoric).
$\tilde{s}$	Deviatoric Stress Tensor.
z	Intrinsic Time.
z_D	Deviatoric Intrinsic Time.
z_H	Hydrostatic Intrinsic Time.

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