Short notes on Lyapunov stability
by Sanand D

1 Introduction

All the discussion here is borrowed predominantly from Haddad, [2], Sastry and Khalil. We consider the following form for a general non linear dynamical system (continuous/discrete)

\[ \dot{x}(t) = f(t, x, u), x(t_0) = x_0, \ t \in [t_0, t_1]. \]  
\[ x(t + 1) = f(x(k), u(k)), \ k = 0, 1, \ldots \]  
where \( f \) for continuous time systems may assumed to be continuous in \( t \) and \( x, x \in \mathbb{R}^n, u \in \mathbb{R}^m, x \) being the state of the system and \( u \) being the input. In the absence of inputs or if inputs are fixed to some specified value, then one obtains

\[ \dot{x}(t) = f(t, x), x(t_0) = x_0, \ t \in [t_0, t_1]. \]  

1.1 Basic definitions for stability and invariance

In the following definitions, properties are said to be true

- **locally** if they are true for all \( x \in B_\epsilon(0) \) for some \( \epsilon \).
- **globally** if they are true for all \( x \in \mathbb{R}^n \).
- **uniformly** if they are true for all \( t_0 \geq 0 \).
- **semi globally** if they are true for all \( x \in B_\epsilon(0) \) for an arbitrary \( \epsilon \).

We may replace \( s(t, t_0, x_0) \) with \( x(t) \) in all the following definitions.

**Local stability definitions:**

**Definition 1.1** (stability). The equilibrium point \( 0 \) is said to be stable if, for every \( \epsilon > 0 \), there exists a \( \delta = \delta(\epsilon, t_0) \) such that \( \|x_0\| < \delta(\epsilon, t_0) \Rightarrow \|s(t, t_0, x_0)\| < \epsilon \) for all \( t \geq t_0 \).

**Definition 1.2** (uniform stability). The equilibrium point \( 0 \) is said to be uniformly stable if, for every \( \epsilon > 0 \), there exists a \( \delta = \delta(\epsilon) \) such that \( \|x_0\| < \delta(\epsilon) \Rightarrow \|s(t, t_0, x_0)\| < \epsilon \) for all \( t \geq t_0 \).

An equilibrium point is unstable if it is not stable. For autonomous systems stability and uniformly stability is the same.

**Definition 1.3** (attractivity). The equilibrium point \( 0 \) is attractive, if for each \( t_0 \in \mathbb{R}^+ \), there exists \( \eta(t_0) > 0 \) such that \( \|x_0\| < \eta(t_0) \Rightarrow s(t + t_0, t_0, x_0) \to 0 \) as \( t \to \infty \). It is said to be uniformly attractive if there exists \( \eta > 0 \) such that \( \|x_0\| < \eta \Rightarrow s(t + t_0, t_0, x_0) \to 0 \) as \( t \to \infty \) uniformly in \( t_0, x_0 \).

**Definition 1.4** (asymptotic stability). The equilibrium point \( 0 \) is asymptotically stable if it is stable and attractive. It is uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

**Definition 1.5** (exponential stability). The equilibrium point \( 0 \) is exponentially stable there exists constants \( c, \gamma, \epsilon \) such that \( \|s(t_0 + t, t_0, x_0)\| \leq c\|x_0\|e^{-\gamma t}, \forall t, t_0 \geq 0, \forall x_0 \in B_\epsilon(0) \). The constant \( \gamma \) is called an estimate of the rate of convergence.
All the definitions above are local since they are concerned with neighborhoods of the equilibrium point.

Global stability definitions:

Definition 1.6 (Global asymptotic stability). The equilibrium point 0 is globally asymptotically stable if it is stable and \( \lim_{t \to \infty} s(t, t_0, x_0) = 0 \) for all \( x_0 \in \mathbb{R}^n \).

Definition 1.7. The equilibrium point 0 is said to be globally uniformly asymptotically stable if it is uniformly stable and for each pair of positive numbers \( M, \epsilon \) with \( M \) arbitrarily large and \( \epsilon \) arbitrarily small, there exists a finite number \( T = T(M, \epsilon) \) such that \( \|x_0\| < M, t_0 \geq 0 \Rightarrow \|s(t + t_0, t_0, x_0)\| < \epsilon, \forall t \geq T(M, \epsilon) \).

Definition 1.8. The equilibrium point 0 is said to be globally exponentially stable if there exists constants \( \epsilon, \gamma \) such that \( \|s(t_0 + t, t_0, x_0)\| \leq \epsilon \|x_0\| e^{-\gamma t}, \forall t, t_0 \geq 0, \forall x_0 \in \mathbb{R}^n \).

For an equilibrium point to be either globally uniformly asymptotically stable or globally exponentially stable, a necessary condition is that it is the only equilibrium point.

Energy-like functions:

Definition 1.9 (class \( K, K^r, L, KL, KRL \) functions). A function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is of class \( K \) if it is continuous, strictly increasing, and \( \phi(0) = 0 \). It is said to belong to class \( K^r \) if it is in class \( K \) and in addition, \( \phi(x) \to \infty \) as \( x \to \infty \).

It is of class \( L \) if it is continuous on \([0, \infty]\), strictly decreasing, \( \phi(0) < \infty \) and \( \phi(x) \to 0 \) as \( x \to \infty \). A continuous function \( \beta : [0, \infty) \times [0, \infty) \to [0, \infty) \) is said to belong to class \( KL \) if, for a fixed \( s, \beta(r,s) \in KL \) with respect to \( r \), and for each fixed \( r, \beta(r,s) \in L \) with respect to \( s \). Similarly, we define class \( KRL \) functions.

In the definition of class \( K \) functions, if we change the domain from \([0, \infty)\) to \([0, a)\), we get class \( K_a \) functions. Then we can define class \( K_a^r \), \( K_a L \), \( K_a KL \) analogously.

Definition 1.10 (locally positive definite function (lpdf)). A continuous function \( V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+ \) is called locally positive definite if for some \( \epsilon > 0 \) and some \( \alpha() \) of class \( K \) functions,

\[
V(0, t) = 0 \text{ and } V(x, t) \geq \alpha(\|x\|) \quad \forall x \in B_\epsilon, t \geq 0.
\] (4)

Definition 1.11 (positive definite function (pdf)). A continuous function \( V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+ \) is called positive definite if (4) holds \( \forall x \in \mathbb{R}^n \). It is said to be radially unbounded if for some \( \alpha() \) of class \( K \) functions,

\[
V(0, t) = 0 \text{ and } V(x, t) \geq \alpha(\|x\|) \quad \forall x \in \mathbb{R}^n, t \geq 0.
\] (5)

Definition 1.12 (decrescent functions). A continuous function \( V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+ \) is called decrescent if there exists some \( \beta() \) of class \( K \) functions and an \( \epsilon > 0 \), such that

\[
V(x, t) \leq \beta(\|x\|) \quad \forall x \in B_\epsilon(0), t \geq 0.
\] (6)

Example 1.13. \( V(t,x) = (t+1)\|x\|^2 \) is a pdf but not decrescent.

Derivative along a trajectory: Let \( V(x(t)) \) be a real valued function on the solution trajectories of (3). Then by chain rule, \( \dot{V} = \nabla V(x) \cdot \dot{x} = \nabla V(x) \cdot f \). This is also called the Lie derivative of \( V \) along \( f \).

Theorem 1.14 (Basic equivalence). The equilibrium point 0 of (3) is

1. stable \( \iff \) for each \( t_0 \in \mathbb{R}^+ \), there exists \( d(t_0) > 0 \) and \( \phi_{t_0} \in K \) such that

\[
\|s(t, t_0, x_0)\| \leq \phi_{t_0}(\|x_0\|), \forall t \geq t_0, \forall x_0 \in B_{d(t_0)}(0).
\]
2. uniformly stable ⇐ there exists \( d > 0 \) and \( \phi \in \mathcal{K} \) such that
\[
\|s(t, t_0, x_0)\| \leq \phi(\|x_0\|), \forall t \geq t_0, \forall x_0 \in B_d(0).
\]

3. attractive ⇐ for each \( t_0 \in \mathbb{R}^+ \), there exists \( r(t_0) \) > 0, and for each \( x_0 \in B_{r(t_0)}(0) \), there exists a function \( \sigma_{t_0, x_0} \in \mathcal{L} \) such that
\[
\|s(t, t_0, x_0)\| \leq \sigma_{t_0, x_0}(t), \forall t \geq 0, \forall x_0 \in B_{r(t_0)}(0).
\]

4. uniformly attractive ⇐ there exists \( r > 0 \) and a function \( \sigma_{t_0, x_0} \in \mathcal{L} \) such that
\[
\|s(t, t_0, x_0)\| \leq \sigma(t), \forall t, t_0 \geq 0, \forall x_0 \in B_{r(t_0)}(0).
\]

5. asymptotically stable ⇐ there exists a number \( r(t_0) > 0 \), a function \( \phi_{t_0} \in \mathcal{K} \) and for each \( x_0 \in B_{r(t_0)}(0) \), there exists a function \( \sigma_{t_0, x_0} \in \mathcal{L} \) such that
\[
\|s(t, t_0, x_0)\| \leq \phi_{t_0}(\|x_0\|)\sigma_{t_0, x_0}(t), \forall t, t_0 \geq 0, \forall x_0 \in B_{r(t_0)}(0).
\]

6. i. uniformly asymptotically stable ⇐ there exists a number \( r > 0 \), a function \( \phi \in \mathcal{K} \) and a function \( \sigma \in \mathcal{L} \) such that
\[
\|s(t, t_0, x_0)\| \leq \phi(\|x_0\|)\sigma(t), \forall t, t_0 \geq 0, \forall x_0 \in B_r(0).
\]

ii. uniformly asymptotically stable ⇐ there exists a class \( \mathcal{KL} \) function \( \beta \) and a positive constant \( c \) independent of \( t_0 \) such that
\[
\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| \leq c.
\]

Proof. ([2] and Khalil) 1. \( \Leftarrow \) Given \( \epsilon > 0, t_0 \in \mathbb{R}^+ \), choose \( \delta(\epsilon, t_0) = \min\{d(t_0), \phi_{t_0}^{-1}(\epsilon)\} \).

\( \Rightarrow \) Fix \( t_0 \) and \( \epsilon > 0 \). There exists \( \delta(t_0, \epsilon) > 0 \) such that
\[
\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq t_0.
\]

Let \( \delta(t_0, \epsilon) \) be the supremum of all applicable \( \delta \). The function \( \delta \) is positive and non decreasing, but need not be continuous. Choose \( \theta_{t_0}(\epsilon) \in \mathcal{K} \) such that \( \theta_{t_0}(\epsilon) \leq \delta(\epsilon) \) for all \( \epsilon > 0 \) (need justification for existence of such function to be rigorous). Now choose \( \phi_{t_0} = \theta_{t_0}^{-1} \).

2. For uniform stability, we apply similar arguments without any \( t_0 \) dependence.

The proof of remaining cases is based on similar arguments. Intuitively, the first case says that the trajectories remain bounded where the bound is given by a level surface of a class \( \mathcal{K} \) function. Same for the second case where the level surface is independent of the initial time. For the third case, the criteria says that the magnitude of the trajectories (“energy” inside the trajectory) decreases, in other words, it is bounded by a decreasing function which goes to zero at infinity. This means the magnitude of the trajectory goes to zero as \( t \to \infty \) i.e., it approaches the origin. This explains attractivity of the origin. For asymptotic stability, the trajectory must be bounded and the origin must be attractive too. This explains the fifth and the sixth (i) case. The case 6(ii) (Khalil) gives an alternate formulation of uniform asymptotic stability using class \( \mathcal{KL} \) functions. The trajectory is bounded depending up on the first argument of \( \beta \) and it decreases to zero since \( \beta \to 0 \) as the second argument in \( \beta \) goes to infinity. Refer Khalil/[2] for rigorous arguments of all cases. \( \Box \)

Definition 1.15 (Domain of attraction). Suppose that \( 0 \) is an asymptotically stable equilibrium point of \( \dot{x} = f(x) \). Then the domain of attraction \( \mathcal{D}_0 \) is given by
\[
\mathcal{D}_0 := \{x_0 \in \mathbb{S} \subset \mathbb{R}^n | \text{ if } x(0) = x_0, \text{ then } \lim_{t \to \infty} x(t) = 0\}.
\]
Table 1: Basic Lyapunov stability theorems for autonomous systems (Theorem 2.1)

<table>
<thead>
<tr>
<th>Conditions on $V(x)$</th>
<th>Conditions on $-\dot{V}(x)$</th>
<th>Conclusion</th>
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<tbody>
<tr>
<td>$V(x) &gt; 0$, $x \neq 0$ locally</td>
<td>$\geq 0$ locally</td>
<td>stable</td>
</tr>
<tr>
<td>$V(x) &gt; 0$, $x \neq 0$ locally</td>
<td>$&gt; 0$, $\forall x \neq 0$ locally</td>
<td>asymptotically stable</td>
</tr>
<tr>
<td>$V(x) &gt; 0$, $x \neq 0$, $V(x) \to \infty$ as $|x| \to \infty$</td>
<td>$&gt; 0$, $\forall x \neq 0$ locally</td>
<td>globally asymptotically stable</td>
</tr>
<tr>
<td>$\alpha |x|^p \leq V(x) \leq \beta |x|^p$ locally</td>
<td>$\geq \epsilon V(x)$ locally</td>
<td>locally exponentially stable</td>
</tr>
<tr>
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<td>$\geq \epsilon V(x)$ globally</td>
<td>globally exponentially stable</td>
</tr>
</tbody>
</table>

2 Stability theory for Autonomous/time invariant systems

stability, asymptotic stability, exponential stability, global asymptotic stability, global exponential stability Lyapunov stability, invariant set stability, constructing Lyapunov functions, Converse Lyapunov theorems, instability theorems, linear systems and Lyapunov’s linearization

We consider non linear dynamical systems of the form

$$\dot{x} = f(x(t)).$$

(7)

2.1 Basic stability theorems using Lyapunov’s direct method

Theorem 2.1. Consider an autonomous system $\dot{x} = f(x(t))$, $x(0) = x_0$ where $x(t) \in S \subset \mathbb{R}^n$. Let $V : S \to \mathbb{R}$. Let $\alpha, \beta, \epsilon > 0$ and $p \geq 1$. Then, Table 1 holds.

Proof. Let $\epsilon > 0$ such that $B_\epsilon(0) \subset S$. Observe that since $\partial B_\epsilon(0)$ is compact and $V(x)$ is continuous, $V(\partial B_\epsilon(0))$ is compact hence, $V(\partial B_\epsilon(0)) = \min_{x \in \partial B_\epsilon(0)} V(x)$ exists and $\alpha > 0$ since $0 \notin \partial B_\epsilon(0)$ and $V(x) > 0$ for $x \in S$, $x \neq 0$. (Draw a picture of $S$ and $B_\epsilon$ inside $S$.)

Consider $\beta \in (0, \alpha)$ where $\alpha$ is the minimum value of $V$ on $\partial B_\epsilon(0)$. Let $\Omega_\beta = \{x \in S \mid V(x) \leq \beta\}$. Then, $\Omega_\beta \subset B_\epsilon(0) \setminus \partial B_\epsilon(0)$. This is true because if there exists $y \in \Omega_\beta$ such that $y \in \partial B_\epsilon(0)$, then $V(y) \geq \alpha > \beta$ which is a contradiction. (Draw this “ellipsoid” inside $B_\epsilon$.)

Since $\dot{V} \leq 0$, $V$ is non increasing and $V(x(t)) \leq V(x(0)) \leq \beta$. Therefore, $x(t) \in \Omega_\beta$ for all $x(0) \in \Omega_\beta$. Consider $B_\beta(0) \subset \Omega_\beta$ for some $\delta(\epsilon) > 0$. Therefore, for all $x(0) \in B_\beta(0)$, $V(x(0)) < \beta$ and $V(x(t)) \leq V(x(0)) \Rightarrow x(t) \in \Omega_\beta \subset B_\beta(0)$. Thus, $x(t) \in B_\beta(0) \Rightarrow \|x(t)\| < \epsilon$. This proves Lyapunov stability.

For the second case, $\dot{V} < 0$. Therefore, $V$ is decreasing and it is bounded from below by 0. We need to show that $x(t) \to 0$ as $t \to \infty$ i.e., for every $\epsilon > 0$, there exists $T > 0$ such that $\|x(t)\| < \epsilon$ for all $t > T$. By previous arguments, for every $\epsilon > 0$, we can choose $\beta > 0$ such that $\Omega_\beta \subset B_\epsilon$. Therefore, it is enough to show that $V(x(t)) \to 0$ as $t \to \infty$.

Since $V$ is decreasing and it is bounded from below by 0, let $V(x(t)) \to c \geq 0$ as $t \to \infty$. To show $c = 0$, we use contradiction. Suppose $c > 0$. By continuity of $V$, there exists $d > 0$ such that $B_d \subset \Omega_c$. Since $V(x(t)) \to c \geq 0$ as $t \to \infty$, $x(t)$ lies outside $B_d$ for all $t \geq 0$. Let $-\gamma = \max_{d \leq \|x\| \leq \epsilon} V(x)$. Since $\dot{V} < 0$, $-\gamma < 0$. Moreover,

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0)) - \gamma t.$$

Observe that the rhs eventually becomes negative contracting the assumption that $c > 0$. Therefore, $c = 0$ which proves asymptotic stability.

The third case is exactly similar to the second case. For any $x \in \mathbb{R}^n$, let $a = V(x)$. Since $\|x\| \to \infty \Rightarrow V(x) \to \infty$, for any $a > 0$, there exists $\epsilon > 0$ such that $V(x) > a$ when $\|x\| > \epsilon$. Therefore, $\Omega_a \subset B_\epsilon(0)$ which implies that $\Omega_a$ is bounded. Remaining arguments are exactly similar as to the second case.
Since \( \dot{V} \leq -eV(x) \), \( V(x(t)) \leq V(x(0))e^{-et} \), \( t \geq 0 \). By assumption, \( V(x(0)) \leq \beta \| x(0) \|^p \) and \( \alpha \| x(t) \|^p \leq V(x(t)) \). Therefore,

\[
\alpha \| x(t) \|^p \leq \beta \| x(0) \|^p e^{-et}, \ t \geq 0 \Rightarrow \| x(t) \| \leq \left( \frac{\beta}{\alpha} \right)^\frac{1}{p} \| x(0) \| e\left( \frac{et}{p} \right), \ t \geq 0
\]

which proves local exponential stability.

The last case follows similarly.

**Example 2.2** (Simple pendulum). Consider a simple pendulum where the constants are chosen such that \( \dot{\theta} + \sin(\theta) = 0 \). Choosing \( x_1 = \theta \) and \( x_2 = \dot{\theta}, x_1 \dot{x}_1 + x_2 \dot{x}_2 = -\sin(x_1) \). The total energy is \( \frac{1}{2}\dot{\theta}^2 - \cos(\theta) = \frac{1}{2}x_2^2 - \cos(x_1) \). Consider \( V(x_1,x_2) = 1 + \frac{1}{2}x_2^2 - \cos(x_1)0 \). Since this is continuously differentiable and lpdf, it is a Lyapunov function candidate. Moreover, \( \dot{V}(x_1,x_2) = \sin(x_1)x_1 + x_2\dot{x}_2 = \sin(x_1)x_2 + x_2(-\sin x_1) = 0 \). Therefore, the equilibrium point is stable.

Consider a simple pendulum with damping i.e., \( \dot{\theta} + \sin(\theta) + b\dot{\theta} = 0 \) (Khalil). Hence, the state equations are \( \dot{x}_1 = x_2, \dot{x}_2 = -\sin(x_1) - bx_2 \). Let \( V(x_1,x_2) \) be the same function as above. Thus, \( \dot{V} = \sin(x_1)x_1 + x_2\dot{x}_2 = \sin(x_1)x_2 + x_2(-\sin x_1) - bx_2^2 = -bx_2^2 \leq 0 \). Thus, the equilibrium point is stable for the damped pendulum. Consider a different Lyapunov function \( V(x_1,x_2) = x_1^TPx + \cos(x_1) \) where \( P > 0 \). Need to choose \( P \) such that \( \dot{V} < 0 \). Choose \( p_{11} = bp_{12}, p_{22} = 1 \). Now, \( \dot{V} = -\frac{1}{2}\alpha bx_1\sin(x_1) - \frac{1}{2}bx_2^2 \) and \( x_1\sin(x_1) > 0 \) for \( -\pi < x_1 < \pi \). Thus, choosing \( \{ x \in \mathbb{R}^2 \mid -\pi < x_1 < \pi \} \) as an open set around \( 0 \), \( \dot{V} < 0 \) on this set \( \Rightarrow \) asymptotic stability.

**Example 2.3** (Non linear series RLC circuit/Non linear mass spring damper). Non linear series RLC circuit (Sastry)/Mass-spring-damper. Suppose the inductor is linear but the resistor and capacitor are non linear (or mass-spring- damper system with non linear spring and damping). Let \( x_1 \) be the charge on the capacitor and \( x_2 \) be the current through the inductor. Therefore,

\[
\dot{x}_1 = x_2, \dot{x}_2 = -f(x_2) - g(x_1)
\]

where \( f \) is a continuous function modelling resistor current-voltage characteristics and \( g \) models capacitor charge-voltage characteristics. Suppose both \( f, g \) model locally passive elements i.e., there exists \( \sigma_0 \) such that

\[
\sigma f(\sigma) \geq 0, \sigma g(\sigma) \geq 0, \forall \sigma \in [-\sigma_0, \sigma_0].
\]

A Lyapunov function candidate is the total energy of the system i.e.,

\[
V(x) = \frac{x_2^2}{2} + \int_0^{x_1} g(\sigma)d\sigma
\]

where the first term represents the energy stored in the inductor (kinetic energy) and the second term represents the energy stored in the capacitor (potential energy). By passivity of \( g, V \) is lpdf. Moreover,

\[
\dot{V}(x) = x_2(-f(x_2) - g(x_1)) + g(x_1)x_2 = -x_2f(x_2) \leq 0
\]

when \( |x_2| < \sigma_0 \). Thus, this shows local stability of the origin.

**Example 2.4** (Rigid body and rotational motion). Consider a rotational motion of a rigid body in \( 3-D \) space. Let \( \omega \) be the angular velocity and \( I \) be the Inertia matrix. Then, in the absence of external torques, the motion is described by

\[
I\ddot{\omega} + \omega \times I\omega = 0
\]

where \( \times \) denotes cross product in \( \mathbb{R}^3 \). Do a change of basis such that \( I \) is a diagonal matrix. Let \( \omega_x, \omega_y, \omega_z \) be the components of the angular velocity. Equation (10) reduces to

\[
I_x\ddot{\omega}_x = -(I_z - I_y)\omega_y\omega_z, I_y\ddot{\omega}_y = -(I_x - I_z)\omega_x\omega_z, I_z\ddot{\omega}_z = -(I_y - I_x)\omega_x\omega_y.
\]
Without loss of generality assume that $I_x \geq I_y \geq I_z > 0$ and replace $\omega_x, \omega_y, \omega_z$ by $x, y, z$. Define $a = \frac{I_y - I_z}{I_x}, b = \frac{I_z - I_x}{I_y}, c = \frac{I_x - I_y}{I_z}$. Therefore, (11) becomes

$$\dot{x} = ayz, \quad \dot{y} = -bzx, \quad \dot{z} = cxy.$$  

For simplicity, suppose $I_x > I_y > I_z \Rightarrow a, b, c > 0$. For equilibrium, at least two quantities among $x, y, z$ must be zero. Thus, the equilibria are union of the three axes. Therefore, none of the equilibria is isolated. Consider the origin as an equilibrium point. Define a Lyapunov function candidate $V(x, y, z) = px^2 + qy^2 + rz^2$ where $p, q, r > 0$. Then $V$ is an lpdf.

$$\dot{V} = 2(px\dot{x} + qy\dot{y} + rz\dot{z}) = 2xyz(ap - bq + cr).$$

Choose $p, q, r$ such that $ap - bq + cr = 0$. Hence, the origin is stable.

**Example 2.5** (Non linear RC circuit). non linear resistive and linear capacitive circuit. Consider a bank of capacitors (linear characteristics) with capacitances $C_1, \ldots, C_n$ connected to a bank of resistors (non linear). Let $x = [x_1, \ldots, x_n]$ denote the voltages accross each capacitor. Then the current through the capacitors is $Gx$ where $G = \text{diag}(C_1, \ldots, C_n)$. Let $i(x)$ denote the current vector in the resistive network when the voltage vector $x$ is applied accross its terminals such that $i(0) = 0$. Let $i(x) = G(x)x$ where $G(.)$ is the non linear version of the conductance matrix. Therefore,

$$C\dot{x} = G(x)x \Rightarrow \ddot{x} = C^{-1}G(x)x.$$  

Clearly, $0$ is an equilibrium point. Consider the total energy stored in the capacitors i.e., $V(x) = \frac{1}{2}x^TCx$. Therefore,

$$\dot{V}(x) = \frac{1}{2}(x^TC\dot{x} + x^TC\dot{x}) = -\frac{1}{2}x^T[G^T(x) + G(x)]x.$$  

Let $M(x) := G^T(x) + G(x)$. If $M(x)$ is lpdf, then $0$ is asymptotically stable. Since $M$ is continuous, $M$ is lpdf if $M(0) > 0$.

Suppose $M(0) > 0$. Therefore, by continuity, there exists $\epsilon > 0$ such that $M(x) > 0$ for all $x \in B_\epsilon(0)$. Let $d := \inf_{x \in B_\epsilon(0)} M(x)$ and choose $\epsilon$ small enough such that $d > 0$. Then, $\dot{V} = -x^TM(x)x \leq -d\|x\|^2, \forall x \in B_\epsilon(0)$. Since, $V(x) = \frac{1}{2}x^TCx$,

$$\lambda_{max}(C)\|x\|^2 \geq V(x) \Rightarrow -d\|x\|^2 \leq \frac{-d}{\lambda_{max}(C)}V(x).$$

Therefore, $\dot{V} \leq \frac{-d}{\lambda_{max}(C)}V(x)$ and from Theorem 2.1 case 4, $0$ is locally exponentially stable. If $d = \inf_{x \in \mathbb{R}^n} M(x) > 0$, then $0$ is globally exponentially stable. It is also globally asymptotically stable since, $\dot{V} < 0$.

**Example 2.6.** not radially unbounded and finite escape time: Ex. on p.174 of [2]. It shows that if $V$ is not radially unbounded and satisfies all the remaining properties, then trajectories starting sufficiently far from the origin may have a finite escape time and the origin is not globally exponentially stable.


### 2.2 Invariant sets and stability theorems

**Lemma 2.8** (limit sets). If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to $S \subset \mathbb{R}^n$ for $t \geq 0$, then its limit set $L$ is nonempty, compact and invariant. Moreover, $x(t) \to L$ as $t \to \infty$.

**Proof.** The proof is slightly technical and we refer the reader to Haddad Theorem 2.41. \qed
Relaxing strict negative definite condition on $\dot{V}$:

**Theorem 2.9** (LaSalle’s invariance principle (Barbashin-Krasovskii-LaSalle Theorem)). Let $\Omega \subset S$ be a compact invariant set of $\dot{x} = f(x)$. Let $V : S \rightarrow \mathbb{R}$ be a continuously differentiable function such that $\dot{V} \leq 0$ in $\Omega$. Let $E$ be the set of points where $\dot{V}(x) = 0$. Let $M$ be the largest invariant set in $E$. If $x(0) \in \Omega$, then $x(t) \rightarrow M$ as $t \rightarrow \infty$.

**Proof.** Let $x(t)$ be a solution of $\dot{x} = f(x)$ starting in $\Omega$. Since $\dot{V} \leq 0$ in $\Omega$, $V$ is a decreasing function of $t$. Furthermore, since $V$ is continuous on the compact set $\Omega$, it is bounded from below on $\Omega$. Therefore, $\lim_{t \rightarrow \infty} V(x(t)) = a$ exists. The limit set $L$ of the trajectory $x(t)$ lies in $\Omega$ since $\Omega$ is invariant and closed. For any $p \in L$, there exists a sequence $\{t_n\}$ such that $t_n \rightarrow \infty$ and $x(t_n) \rightarrow p$ as $n \rightarrow \infty$. By continuity of $V(x)$, $V(p) = \lim_{n \rightarrow \infty} V(x(t_n)) = a$. Hence, $V(x) = a$ on $L$. By the previous lemma, $L$ is an invariant set and since $V$ is a constant function on $L$, $\dot{V} = 0$ on $L$. Therefore, $L \subset M \subset E \subset \Omega$. Since $x(t)$ is bounded, $x(t)$ approaches $L$ as $t \rightarrow \infty$. Hence, $x(t) \rightarrow M$ as $t \rightarrow \infty$.

Observe that there is no positiveness assumption on $V$ in the above theorem. However, invariance of $\Omega$ is assumed and the initial condition is assumed to belong to this invariant set. One can construct invariant sets using level sets of $V$ containing $x = 0$ such that $\dot{V} \leq 0$ in $E$ other than the trivial solution $x = 0$. Then, the origin is asymptotically stable.

**Corollary 2.10** (asymptotic stability using LaSalle). Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : S \rightarrow \mathbb{R}$ be a continuously differentiable positive definite function on $S$ (i.e., $V$ is lpdf) containing $x = 0$ such that $\dot{V} \leq 0$ on $S$. Let $E$ be the set of points where $\dot{V}(x) = 0$ and suppose that no solution can stay identically in $E$ other than the trivial solution $x = 0$. Then, the origin is asymptotically stable.

**Proof.** Follows as a consequence of the previous theorem.

**Corollary 2.11** (global asymptotic stability using LaSalle). Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable radially unbounded positive definite function on $\mathbb{R}^n \setminus \{0\}$ such that $\dot{V} \leq 0$ on $\mathbb{R}^n$. Let $E$ be the set of points where $\dot{V}(x) = 0$ and suppose that no solution can stay identically in $E$ other than the trivial solution $x = 0$. Then, the origin is globally asymptotically stable.

**Proof.** Again follows from the theorem.

**Example 2.12.** Consider the case of damped simple pendulum in Example 2.2. Since $\dot{V} = -b\dot{x}^2$, $\dot{V} = 0$ on the line $x_2 = 0$. To maintain $\dot{V} = 0$, the trajectory must be confined to the line $x_2 = 0$. Now $\dot{x}_2 = 0$ implies $\dot{x}_1 = 0$ for $n = 0, \pm 1, \pm 2, \ldots$. Therefore, on the segment $-\pi < x_1 < \pi$, the largest invariant set is the origin and this implies asymptotic stability by LaSalle.

**Example 2.13.** Non linear RLC circuit (Example 2.3), asymptotic stability of the origin using LaSalle: Recall from Example 2.3 that $\dot{V}(x_1, x_2) = -x_2 f(x_2) \leq 0$ for $x_2 \in [-\sigma_0, \sigma_0]$. Let $c = \min\{V(-\sigma_0, 0), V(\sigma_0, 0)\}$. Then, $\dot{V} \leq 0$ for $x \in \Omega_c := \{(x_1, x_2) \mid V(x_1, x_2) \leq c\}$. By LaSalle’s invariance principle, the trajectory approaches the largest invariant set in $\Omega_c \cap \{(x_1, x_2), \mid \dot{V}(x_1, x_2) = 0\} = \Omega_c \cap \{x_1, 0\}$ (since $f(x_2) = 0$ only when $x_2 = 0$). Note that $x_2 = 0$ in the largest invariant set which implies that $\dot{x}_1 = 0 \Rightarrow x_1 = x_{10}$. Furthermore $x_2 = 0 \Rightarrow \dot{x}_2 = 0 = -f(\dot{x}_1) - g(x_1) = 0 \Rightarrow \dot{x}_1 = 0 \Rightarrow x_1 = x_{10}$ (since, $g(x_1) = 0$ only when $x_1 = 0$). Therefore, the origin is the largest invariant set in $\Omega_c \cap \{x_1, x_2\}$. Hence, it is locally asymptotically stable.


$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = u$$
Let $\mathbf{x} = \mathbf{q}$ and $\mathbf{y} = \dot{\mathbf{q}}$. Therefore, $\dot{\mathbf{x}} = \mathbf{y}$ and $\dot{\mathbf{y}} = [M(\mathbf{x})]^{-1}[\mathbf{u} - C(\mathbf{x}, \mathbf{y})\mathbf{y}]$. Let $\mathbf{q}_d$ represent the desired value of the generalized co-ordinates. Suppose

$$\mathbf{u} = -K_p(\mathbf{q} - \mathbf{q}_d) - K_d\dot{\mathbf{q}} = -K_p(\mathbf{x} - \mathbf{q}_d) - K_d\mathbf{y},$$

where $K_p, K_d > 0$. Now the system equations are

$$\dot{\mathbf{x}} = \mathbf{y}, \dot{\mathbf{y}} = -[M(\mathbf{x})]^{-1}[K_p(\mathbf{x} - \mathbf{q}_d) + K_d\mathbf{y} + C(\mathbf{x}, \mathbf{y})\mathbf{y}]$$

The equilibrium point is $(\mathbf{x}, \mathbf{y}) = (\mathbf{q}_d, \mathbf{0})$. Consider a Lyapunov function

$$V(\mathbf{x}, \mathbf{y}) = \frac{1}{2}[\mathbf{y}^T M(\mathbf{x})\mathbf{y} + (\mathbf{x} - \mathbf{q}_d)^T K_p(\mathbf{x} - \mathbf{q}_d)]$$

$$\Rightarrow \dot{V} = \mathbf{y}^T M(\mathbf{x})\dot{\mathbf{y}} + \frac{1}{2}\mathbf{y}^T M(\mathbf{x})\mathbf{y} + \dot{\mathbf{x}}^T K_p(\mathbf{x} - \mathbf{q}_d)$$

$$= -\mathbf{y}^T [K_p(\mathbf{x} - \mathbf{q}_d) + K_d \mathbf{y} + C(\mathbf{x}, \mathbf{y})\mathbf{y}] + \frac{1}{2}\mathbf{y}^T M(\mathbf{x})\mathbf{y} + \dot{\mathbf{x}}^T K_p(\mathbf{x} - \mathbf{q}_d)$$

$$= -\mathbf{y}^T K_d\mathbf{y} + \frac{1}{2}\mathbf{y}^T [M(\mathbf{x}, \mathbf{y}) - 2C(\mathbf{x}, \mathbf{y})]$$

It turns out that $M - 2C$ is skew symmetric. Therefore, $\dot{V} \leq 0$ which implies stability. Applying Krasovskii-LaSalle principle, for the set of points where $\dot{V} = 0$ i.e., to points $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times 0$, the largest invariant set is given by the set of points in $\mathbb{R}^n \times 0$ where $(\mathbf{x}, \dot{\mathbf{y}}) = (\mathbf{0}, \mathbf{0})$. Now $\dot{\mathbf{x}} = 0$ implies that $\mathbf{y} = 0$ and $\dot{\mathbf{y}} = 0$ implies $[M(\mathbf{x})]^{-1}[K_p(\mathbf{x} - \mathbf{q}_d) + K_d\mathbf{y} + C(\mathbf{x}, \mathbf{y})\mathbf{y}] = 0$. Since $\mathbf{y} = 0$ and $M$ is invertible, this implies that $K_p(\mathbf{x} - \mathbf{q}_d) = 0$ and since $K_p > 0$, $\mathbf{x} = \mathbf{q}_d$. Therefore, $(\mathbf{q}_d, \mathbf{0})$ forms the largest invariant set and by LaSalle (Corollary 2.11), it is globally asymptotically stable.

Example 2.15 (Stability of a limit cycle using the invariance principle). Consider a system

$$\dot{x}_1 = x_2 + x_1(r^2 - x_1^2 - x_2^2)$$

$$\dot{x}_2 = -x_1 + x_2(r^2 - x_1^2 - x_2^2), r > 0$$

which has an equilibrium point at the origin. Observe that if $x_1^2 + x_2^2 = r^2$, then $\dot{x}_1 = x_2$ and $\dot{x}_2 = -x_1$ and the evolution remains on the circle $x_1^2 + x_2^2 = r^2$ and it forms an invariant set. Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2 - r^2)^2$. Thus, $V(x) \geq 0$ on $\mathbb{R}^2$. Note that $\dot{V}(x) = \nabla V.(f) = -(x_1^2 + x_2^2)(x_1 + x_2)^2 \leq 0$. Note that $V = 0$ either at the origin or on the circle $x_1^2 + x_2^2 = r^2$. Let $c > r$ and $M_c := \{x \in \mathbb{R}^2 | V(x) \leq c\}$. Since $\dot{V} \leq 0$ on $M_c$ and $V \geq 0$, $M_c$ is an invariant set containing the origin and the circle $x_1^2 + x_2^2 = r^2$. On both these sets, $\dot{V} = 0$ and the largest invariant set in $M_c$ is the union of the origin with the circle (they are not comparable as neither is a subset of the other). Therefore, all trajectories converge to either one of them by the invariance principle.

Observe that at the origin, $\dot{V}(\mathbf{0}) = \frac{r^4}{4}$. Suppose $r < \frac{r^4}{4}$ and let $r < c < \frac{r^4}{4}$. Let $M_c$ be as defined before. Note that $M_c$ excludes the origin in this case but contains the circle $x_1^2 + x_2^2 = r^2$. Now applying the invariance principle, all trajectories converge to the circle implying a stable limit cycle.

2.3 Constructing Lyapunov functions

Observe that by chain rule, $\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x)$. Let $g(x) = \frac{\partial V}{\partial x}$. Therefore, $\dot{V} = g(x).f(x)$. We need to construct $g$ such that it is a gradient of a positive definite function and $V(x) = g(x).f(x) < 0$. Note that

$$V(x) = \int_{0}^{x} g(s)ds = \int_{0}^{x} \sum_{i=1}^{n} g_i(s)ds_i. \quad (13)$$
Proposition 2.16. A function \( g : \mathbb{R}^n \to \mathbb{R} \) is the gradient vector of a scalar valued function \( V : \mathbb{R}^n \to \mathbb{R} \) if and only if \( \frac{\partial g}{\partial x_i} = \frac{\partial g_i}{\partial x_i}, \ i, j = 1, \ldots, n. \)

Proof. (\( \Rightarrow \)) trivial. (\( \Leftarrow \)) Suppose \( \frac{\partial g_i}{\partial x_i} = \frac{\partial g_j}{\partial x_j}, \ i, j = 1, \ldots, n. \) Let

\[
V(x) = \int_0^1 g(\sigma x) \, d\sigma = \int_0^1 \sum_{i=1}^n g_i(\sigma x) x_i \, d\sigma.
\]

Let \( y = \sigma x, \) therefore,

\[
\frac{\partial V}{\partial x_i} = \int_0^1 \sum_{j=1}^n \frac{\partial g_j}{\partial x_i}(\sigma x) x_j \, d\sigma + \int_0^1 g_i(\sigma x) \, d\sigma
\]

\[
= \int_0^1 \sum_{j=1}^n \frac{\partial g_i}{\partial x_j}(\sigma x) x_j \, d\sigma + \int_0^1 g_i(\sigma x) \, d\sigma
\]

\[
= \int_0^1 \sum_{j=1}^n \frac{\partial g_i}{\partial y_j}(y) x_j \, d\sigma + \int_0^1 g_i(\sigma x) \, d\sigma
\]

\[
= \int_0^1 \frac{d(g_i(\sigma x))}{d\sigma} \, d\sigma
\]

\[
= g_i(x), \ i = 1, \ldots, n.
\]

\( \square \)

Proposition 2.17. Let \( f, g : \mathbb{R}^n \to \mathbb{R}^n \) be continuously differentiable functions such that \( f(0) = 0. \) Then for every \( x \in \mathbb{R}^n, \) there exists \( \alpha \in [0, 1] \) such that

\[
g(x) \cdot f(x) = g^T(x) \frac{\partial f}{\partial x}(\alpha x) x.
\]

Proof. (MVT for vector valued functions (Theorem 2.16 of Haddad): Let \( D \subset \mathbb{R}^m \) and \( f : D \to \mathbb{R}^n \) which is continuously differentiable. Let \( x, y \in D \) such that \( \mathcal{L} := \{z \mid z = \mu x + (1 - \mu)y, \ \mu \in (0, 1)\} \subset D. \) Then, for every \( v \in \mathbb{R}^n, \) there exists \( z \in \mathcal{L} \) such that \( v^T[f(y) - f(x)] = v^T[f'(z)(y - x)] \). Let \( p \in \mathbb{R}^n. \) Then by mean value theorem, for every \( x \in \mathbb{R}^n, \) there exists \( \alpha \in [0, 1] \) such that

\[
g^T(p)f(x) = g^T(p)[f(x) - f(0)]
\]

\[
= g^T(p)\left[\frac{\partial f}{\partial x}(\alpha x) x\right].
\]

Hence, there exists \( \alpha \in [0, 1] \) such that \( g(p) \cdot f(p) = g^T(p)\frac{\partial f}{\partial p}(\alpha p)p. \) Since \( p \) is arbitrary, the result follows. \( \square \)
**Theorem 2.18** (Krasovskii’s theorem). Let $0$ be an equilibrium point of $\dot{x} = f(x(t))$, $x(0) = x_0$, $t \geq 0$ where $f : S \to \mathbb{R}^n$ is continuously differentiable and $S$ is an open set containing $0$. Let $P, R \in \mathbb{R}^{n \times n}$, $P, R > 0$ such that

$$
\left[\frac{\partial f(x)}{\partial x}\right]^{T} P + P \left[\frac{\partial f(x)}{\partial x}\right] \leq -R, \ x \in S, \ x \neq 0.
$$

(19)

Then the zero solution is a unique asymptotically stable equilibrium point with Lyapunov function $V(x) = f^{T}(x) P f(x)$. If $S = \mathbb{R}^n$, then the zero solution is a unique globally asymptotically stable equilibrium.

**Proof.** Suppose there exists $x_e \in S$, $x_e \neq 0$ and $f(x_e) = 0$. By the previous proposition, for every $x_e \in S$, there exists $\alpha \in [0,1]$ such that $0 = x_e^{T} P f(x_e) = x_e^{T} P \frac{\partial f(\alpha x_e)}{\partial x} x_e$. Hence,

$$
x_e \left[\frac{\partial f(\alpha x_e)}{\partial x}\right]^{T} P + P \left[\frac{\partial f(\alpha x_e)}{\partial x}\right] x_e = 0
$$

which contradicts (19). Therefore, $S$ does not contain any other equilibrium point of $f$. Note that $V(x) = f^{T}(x)P f(x) \geq \lambda_{\min}(P) \| f(x) \|_2^2 \geq 0$, $x \in S$. This implies that $V(x) = 0$ if and only if $f(x) = 0$ i.e., $x = 0$. Therefore, $V > 0$ on $S \setminus \{0\}$. Moreover,

$$
\dot{V} = V'(x) f(x) = 2 f^{T}(x) P \frac{\partial f(x)}{\partial x} - f(x) = f^{T}(x) \left[\frac{\partial f(x)}{\partial x}\right]^{T} P + P \left[\frac{\partial f(x)}{\partial x}\right] f(x)
$$

$$
\leq -f^{T}(x) R f(x) \leq -\lambda_{\min}(R) \| f(x) \|_2^2 \leq 0, \ x \in S.
$$

Since $f(x) = 0$ iff $x = 0$ in $S$, $\dot{V} < 0$ is $S \setminus \{0\}$ which proves that $0$ is asymptotically stable equilibrium point with Lyapunov function $V(x) = f^{T}(x) P f(x)$.

Now suppose $S = \mathbb{R}^n$. We need to show that $V(x) \to \infty$ as $\|x\| \to \infty$. By the previous proposition, for every $x \in \mathbb{R}^n$ and some $\alpha \in (0,1)$,

$$
x^{T} P f(x) = x^{T} P \frac{\partial f(x)}{\partial x} (\alpha x) = x^{T} \left[\frac{\partial f(\alpha x)}{\partial x}\right]^{T} P + P \left[\frac{\partial f(\alpha x)}{\partial x}\right] x
$$

$$
\leq -\frac{1}{2} x^{T} R x \leq -\frac{1}{2} \lambda_{\min}(R) \|x\|_2^2.
$$

(20)

This implies that $x^{T} P f(x) < 0$ since rhs above is negative. Therefore, $|x^{T} P f(x)| \geq \frac{1}{2} \lambda_{\min}(R) \|x\|_2^2$. Moreover, $|x^{T} P f(x)| \leq \lambda_{\max}(P) \|x\| \|f(x)\| \Rightarrow \|f(x)\| \geq \frac{|x^{T} P f(x)|}{\lambda_{\max}(P) \|x\|} \geq \frac{\lambda_{\min}(R)}{2 \lambda_{\max}(P)} \|x\|$. This implies by definition of $V$ that $V(x) \to \infty$ as $\|x\| \to \infty$. 

**Example 2.19.** Example 3.9 Haddad.

The following theorem allows us to find the domain of attraction as well.

**Theorem 2.20** (Zubov’s theorem). Let $0$ be an equilibrium point of $\dot{x} = f(x(t))$. Let $S \subset \mathbb{R}^n$ be bounded and suppose there exists a continuously differentiable function $V : S \to \mathbb{R}$ and a continuous function $h : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $h(0) = 0$ and

$$
0 < V(x) < 1, \ x \in S, \ x \neq 0
$$

(21)

$$
V(x) \to 1 \ \text{as} \ x \to \partial S
$$

(22)

$$
h(x) > 0, \ x \in \mathbb{R}^n, \ x \neq 0
$$

(23)

$$
V'(x) f(x) = -h(x) [1 - V(x)].
$$

(24)

Then, $0$ is asymptotically stable with domain of attraction $S$. 

Proof. It follows from (21), (23) and (24) that in the neighborhood \( B_{\epsilon}(0) \) of the origin, \( V(x) > 0 \) and \( \dot{V} < 0 \). Hence, \( 0 \) is locally asymptotically stable. To show that \( S \) is the domain of attraction, we need to show that \( x(0) \in S \) implies that \( x(t) \to 0 \) as \( t \to \infty \) and \( x(0) \notin S \) implies that \( x(t) \not\to 0 \) as \( t \to \infty \).

Let \( x(0) \in S \). Then from (21), \( V(x(0)) < 1 \). Let \( \beta > 0 \) be such that \( V(x(0)) \leq \beta < 1 \) and define \( \Omega_\beta := \{ x \in S \mid V(x) \leq \beta \} \). Since \( \Omega_\beta \subset S \), \( \Omega_\beta \) is bounded. Since \( \dot{V} < 0 \) on \( \Omega_\beta \), it is clear that \( \Omega_\beta \) forms an invariant set. If \( \dot{V} = 0 \), then \( h(x) = 0 \) which further implies that \( x = 0 \). Now by Theorem 2.9, \( x(t) \to 0 \) as \( t \to \infty \).

Let \( x(0) \notin S \) and suppose \( x(t) \to 0 \) as \( t \to \infty \). Thus, \( x(t) \to S \) for some \( t \geq 0 \). Therefore, there exists \( t_1, t_2 \) such that \( x(t_1) \in \partial S \) and \( x(t) \in S \) for all \( t \in (t_1, t_2) \). Let \( W(x) = 1 - V(x) \). Hence, \( W = h(x)W(x) \). Therefore,

\[
\int_{W(x(t_0))}^{W(x(t))} \frac{dW}{W} = \int_{t_0}^{t} h(s) \, ds = 1 - V(x(t)) = [1 - V(x(t))]e^{-\int_{t_0}^{t} h(x(s)) \, ds}.
\]

Let \( t = t_2 \) and \( t_0 \to t_1 \). Using (22), it follows that \( \lim_{t_0 \to t_1} [1 - V(x(t_0))] = 0 \) and \( \lim_{t_0 \to t_1} [1 - V(x(t))]e^{-\int_{t_0}^{t} h(x(s)) \, ds} > 0 \) which is a contradiction. Therefore, for \( x(0) \notin S \), \( x(t) \not\to 0 \) as \( t \to \infty \). \( \square \)

Example 2.21. Haddad: Example 3.10

Constants of motion or first integrals or integrals of motion or dynamic invariants or Casimir functions. A function \( C: S \to \mathbb{R} \) is an integral of motion if it is conserved along the flow of the dynamical system i.e., \( C'(x).f(x) = 0 \). Let \( C_i: S \to \mathbb{R} \) \( i = 1,\ldots,r \) be twice differentiable Casimir functions. Define

\[
E(x) := \sum_{i=1}^{r} \mu_i C_i(x)
\]

for \( \mu_i \in \mathbb{R}, \, i = 1,\ldots,r \).

Theorem 2.22 (Energy-Casimir theorem). Consider a non linear dynamical system \( \dot{x} = f(x) \) where \( f: S \to \mathbb{R}^n \) is Lipschitz. Let \( x_e \) be an equilibrium point and let \( C_i: S \to \mathbb{R}, \, i = 1,\ldots,r \) be Casimir functions of the dynamical system. Suppose \( C_i'(x_e) \) \( i = 2,\ldots,r \) are linearly independent and suppose there exists \( \mu = [\mu_1, \ldots, \mu_r]^T \in \mathbb{R}^r \) such that \( \mu_1 \neq 0 \), \( E'(x_e) = 0 \) and \( x^T E''(x_e)x > 0 \) for \( x \in \mathcal{M} \) where \( \mathcal{M} = \{ x \in S \mid C_i'(x)\,C_i(x_e) = 0, \, i = 2,\ldots,r \} \). Then, there exists \( \alpha \geq 0 \) such that

\[
E''(x_e) + \alpha \sum_{i=2}^{r} \left( \frac{\partial C_i(x_e)}{\partial x} \right)^T \left( \frac{\partial C_i(x_e)}{\partial x} \right) > 0.
\]

Furthermore, the equilibrium point \( x_e \) is Lyapunov stable with Lyapunov function

\[
V(x) = E(x) - E(x_e) + \frac{\alpha}{2} \sum_{i=2}^{r} \left[ C_i(x) - C_i(x_e) \right]^2.
\]

Proof. Note that since \( C_i \) are Casimir functions, \( C_i'(x).f(x) = 0 \). Moreover,

\[
\dot{V}(x) = V'(x).f(x)
\]

\[
= E'(x).f(x) + \alpha \sum_{i=2}^{r} \left[ C_i(x) - C_i(x_e) \right] C_i'(x).f(x)
\]

\[
= \sum_{i=1}^{r} \mu_i C_i'(x).f(x) + \alpha \sum_{i=2}^{r} \left[ C_i(x) - C_i(x_e) \right] C_i'(x).f(x) = 0
\]
for \( x \in S \). We need to show that \( V(x_e) = 0 \) and \( V(x) > 0 \), \( x \in S \setminus x_e \). Clearly, \( V(x_e) = 0 \) and

\[
V'(x) = E'(x) + \alpha \sum_{i=2}^{r} [C_i(x) - C_i(x_e)]C'_i(x) \Rightarrow V'(x_e) = 0.
\]

\[
V''(x) = E''(x) + \alpha \sum_{i=2}^{r} \{[C_i'(x)]^T C_i'(x) + [C_i(x) - C_i(x_e)]C''_i(x)\}
\]

\[
\Rightarrow V''(x_e) = E''(x_e) + \alpha \sum_{i=2}^{r} (\frac{\partial C_i(x_e)}{\partial x})^T (\frac{\partial C_i(x_e)}{\partial x}). \quad (30)
\]

Recall that by hypothesis, if \( x \in M \), then \( C_i'(x_e)x = 0 \), \( i = 2, \ldots, r \). Consider a subspace \( V_1 \) formed by linearly independent elements of \( M \) which is a \( r - 1 \) dimensional subspace and extend the basis of this subspace to a basis of \( \mathbb{R}^n \) by taking linearly independent vectors in the orthogonal complement \( V_1^\perp \). With respect to the new basis, elements of \( V_1 \) can be written as \( [\ast \ 0]^T \) and elements of \( V_1^\perp \) can be written as \( [0 \ \ast]^T \). Therefore, there exists an invertible matrix \( T \in \mathbb{R}^{n \times n} \) such that \( [0 \ I_{r-1}]Tx = 0 \) when \( x \in M \) and \( [I_{n-r-1} \ 0]Tx = 0 \) when \( x \in M^\perp \). Hence, for \( x_e \in S \),

\[
E''(x_e) = T^T \begin{bmatrix} E_1 & E_{12} \\ E_{12}^T & E_2 \end{bmatrix} T \quad (31)
\]

where \( E_1 > 0 \) and

\[
\sum_{i=2}^{r} (\frac{\partial C_i(x_e)}{\partial x})^T (\frac{\partial C_i(x_e)}{\partial x}) = T^T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T \quad (32)
\]

where \( N > 0 \). Substituting (31) and (32) in (30),

\[
V''(x) = T^T \begin{bmatrix} E_1 & E_{12} \\ E_{12} & E_2 + \alpha N \end{bmatrix} T.
\]

By choosing appropriate \( \alpha \), \( V''(x_e) > 0 \). Since \( V \) is twice differentiable and \( V(x_e) = V'(x_e) = 0 \), it follows from \( V''(x_e) > 0 \) that \( V > 0 \) in the neighborhood of \( x_e \). \( \square \)

Thus, using integrals of motions, one can construct a Lyapunov function.

**Example 2.23.** Example 3.11 Haddad.

### 2.4 Instability theorems

**Theorem 2.24** (instability theorem). *Let \( 0 \) be an equilibrium point of \( \dot{x} = f(x) \) and let \( V : S \to \mathbb{R} \) be a continuously differentiable function such that \( V(0) = 0 \) and \( V(x_0) > 0 \) for an \( x_0 \) with arbitrarily small \( ||x_0|| \). Let \( r > 0 \) and \( U = \{ x \in B_r(0) \mid V(x) > 0 \} \) such that \( \dot{V} > 0 \) on \( U \). Then, \( x = 0 \) is unstable.*

**Proof.** By continuity of \( V \) if \( V(x_0) > 0 \), then \( V > 0 \) in a neighborhood of \( x_0 \) hence, \( x_0 \) lies in the interior of \( U \). Let \( V(x_0) = a > 0 \). The trajectory starting at \( x_0 \) must eventually leave \( U \). One justifies this claim as follows. As long as \( x(t) \) is inside \( U \), \( V(x(t)) \geq a \) since, \( \dot{V} > 0 \) in \( U \). Let

\[
\gamma = \min\{\dot{V}(x) \mid x \in U, V(x) \geq a\}
\]

which exists since, the continuous function \( \dot{V} \) has a minimum over the compact set \( \{x \in U \mid V(x) \geq a\} \). Clearly, \( \gamma > 0 \). Note that

\[
V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))ds \geq a + \gamma t.
\]
This inequality shows that $x(t)$ can not stay in $U$ forever because $V(x)$ is bounded on $U$. Moreover, $x(t)$ can not leave through the surface $\{x \mid V(x) = 0\}$ since $V(x(t)) \geq 0$. Hence, it must leave $U$ through the sphere $\|x\| = r$. Because this can happen for arbitrarily small $\|x_0\|$, the origin is unstable.

**Theorem 2.25** (second instability theorem). Let $0$ be an equilibrium point of $\dot{x} = f(x)$ and let $V : S \to \mathbb{R}$ be a continuously differentiable function, $W : S \to \mathbb{R}$ and $\lambda, \epsilon > 0$ such that $V(0) = 0$ and $W(x) \geq 0$ for all $x \in B_\epsilon(0)$ and $\dot{V} = V'(x).f(x) = \lambda V(x) + W(x)$. Furthermore, assume that for every sufficiently small $\delta > 0$, there exists $x_0 \in S$ such that $\|x_0\| < \delta$ and $V(x_0) > 0$. Then, $x = 0$ is unstable.

**Proof.** Suppose there exists $\delta > 0$ such that if $x_0 \in B_\delta(0)$, then $x(t) \in B_\epsilon(0), t \geq 0$. Note that $V(x_0) > 0$. From the hypothesis, $\dot{V} = \lambda V(x) + W(x)$ and $W(x) \geq 0$. This implies that $\dot{V} \geq \lambda V(x)$ for $x \in B_\epsilon(0)$ i.e., $\dot{V} - \lambda V(x) \geq 0$ for $x \in B_\epsilon(0)$. Therefore,

$$e^{-\lambda t}V'(x(t)).f(x(t)) - \lambda e^{-\lambda t}V(x(t)) \geq 0, \quad t \geq 0$$

$$\Rightarrow \frac{d}{dt}[e^{-\lambda t}V(x(t))] \geq 0, \quad t \geq 0.$$ 

Integrating both sides, $e^{-\lambda t}V(x(t)) - V(x(0)) \geq 0 \Rightarrow V(x(t)) \geq e^{\lambda t}V(x(0))$, $t \geq 0$. Since $V(x_0) > 0$, $x(t) \notin B_\epsilon(0)$ as $t \to \infty$ which is a contradiction. Hence, the zero solution is unstable. 

**Theorem 2.26** (Chetaev’s instability theorem). Let $0$ be an equilibrium point of $\dot{x} = f(x)$ and let $V : S \to \mathbb{R}$ be a continuously differentiable function. Let $\epsilon > 0$ and an open set $\Theta \subset B_\epsilon(0)$ such that

$$V(x) > 0 \quad \forall x \in \Theta, \quad \sup_{x \in \Theta} V(x) < \infty, \quad 0 \in \partial \Theta \quad (33)$$

$$V(x) = 0 \quad \forall x \in \partial \Theta \cap B_\epsilon(0), \quad (34)$$

$$V'(x).f(x) > 0, \quad x \in \Theta. \quad (35)$$

Then, the zero solution is unstable.

**Proof.** Let $\mathcal{P} \subset \Theta$ be a closed set such that for $x_0 \in \Theta, x(t) \in \mathcal{P} \subset \Theta \subset B_\epsilon(0)$ for $t \geq 0$. Note that

$$V(x(t)) = V(x(0)) + \int_0^t V(x(s))ds$$

$$= V(x(0)) + \int_0^t V'(x(s)).f(x(s))ds \geq V(x(0)) + \alpha t \quad (36)$$

where $\alpha = \min_{x \in \mathcal{P}} V'(x).f(x) > 0$. This implies that $V(x(t)) \to \infty$ as $t \to \infty$ contradicting one of the hypothesis $\sup_{x \in \Theta} V(x) < \infty$. Therefore, there exists $T > 0$ such that $x(T) \in \partial \Theta$ (i.e., either $x(t)$ escapes $\Theta$ at some finite time) or $x(t) \to \partial \Theta$ as $t \to \infty$ (i.e., $x(t)$ escapes $\Theta$ at infinity). One of these two cases must be true since we have shown by contradiction that there is no closed set in $\Theta$ that contains $\lim_{t \to \infty} x(t)$.

Consider the first case i.e., suppose there exists $T > 0$ such that $x(T) \in \partial \Theta$. Since $V$ is strictly increasing on $\Theta$, $V(x(T)) > 0$. By hypothesis, $V(x) = 0$ on $\partial \Theta \cap B_\epsilon(0)$. This implies that $x(T) \notin \partial \Theta \cap B_\epsilon(0)$ hence, $x(T) \in \partial \Theta \setminus B_\epsilon(0)$. Observe that $\partial \Theta = \overline{\Theta} \cap \partial \Theta$ and $\overline{\Theta} \subset B_\epsilon(0)$. This implies that

$$\partial \Theta \setminus B_\epsilon(0) = (\overline{\Theta} \cap \partial \Theta) \setminus B_\epsilon(0) \subset (\overline{B_\epsilon(0) \cap \partial \Theta}) \setminus B_\epsilon(0) = \partial \Theta \cap \partial B_\epsilon(0).$$

Therefore, $x(T) \in \partial B_\epsilon(0)$. Similarly for the second case, if $x(t) \to \partial \Theta$ as $t \to \infty$, then $x(t) \to \partial B_\epsilon(0)$ as $t \to \infty$. Thus, there does not exists a $\delta > 0$ such that if $x_0 \in B_\delta(0)$, then $x(t) \in B_\epsilon(0)$ for $t \geq 0$. This implies that the zero solution is unstable. 

}$
Thus, it can be observed that Chetaev’s instability theorem requires milder conditions (milder in the sense that the properties need to be satisfied on a subset $\mathcal{O} \subset S$ and not on the whole of $S$) than Lyapunov instability theorems to conclude about instability of an equilibrium point.

Example 2.27.

2.5 Linear autonomous/LTI systems, linearization

Consider an autonomous linear system

$$\dot{x} = Ax.$$  \hfill (37)

**Theorem 2.28** (Stability for LTI). The zero solution of (37) is Lyapunov stable if and only if every eigenvalue of $A$ has a real part strictly less than zero or equal to zero and eigenvalues with zero real part have trivial Jordan structure (i.e., they are semi-simple).

**Proof.** Note that $e^{At}$ is bounded iff the condition on the eigenvalues of $A$ mentioned above is satisfied. \hfill $\square$

**Theorem 2.29** (Lyapunov asymptotic stability for LTI). Following are equivalent.

1. The system (37) is asymptotically stable.
2. The system (37) is exponentially stable.
3. All eigenvalues of $A$ have strictly negative real parts.
4. For every $Q > 0$, there exists a unique solution $P > 0$ to the following Lyapunov equation

$$A^T P + PA = -Q.$$  \hfill (38)

5. There exists $P > 0$ which satisfies the following Lyapunov matrix inequality

$$A^T P + PA < 0.$$  \hfill (39)

6. There exists $C \in \mathbb{R}^{m \times n}$ such that the pair $(C, A)$ is observable, and there exists $P > 0$ which satisfies

$$A^T P + PA + C^T C = 0.$$  \hfill (40)

7. For all $C \in \mathbb{R}^{m \times n}$ such that the pair $(C, A)$ is observable, and there exists $P > 0$ which satisfies (40).

**Proof.** The equivalence of first five statements is shown in Hespanha. Equivalence of the remaining two statements with the remaining ones is given in Sastry. (7) $\Rightarrow$ (6). To show (6) $\Rightarrow$ (1), let $V(x) = x^T P x$ and $-\dot{V} = x^T C^T C x$. Then, by Lasalle’s invariance principle, the trajectories starting from arbitrary initial conditions converge to the largest invariant set in the null space of $C$. Since $(C, A)$ is observable, largest invariant set in the null space of $C$ is the origin. Therefore, the origin is asymptotically stable.

(1) $\Rightarrow$ (7) Just as in (2) $\Rightarrow$ (4), let $P = \int_0^\infty e^{A^T t} C^T C e^{At} dt$. Clearly, $P \geq 0$. To show that $P > 0$, let $x^T P x = 0$. This implies that $Ce^{At} x = 0$. Differentiating $n$ times at the origin, we obtain $C x = CA x = \ldots = CA^{n-1} x = 0$. Since $(C, A)$ is observable, $x = 0 \Rightarrow P > 0$ and $P$ also satisfies (40). \hfill $\square$

**Theorem 2.30** (Lyapunov stability for LTI). The zero solution of (37) is Lyapunov stable $\iff$ there exists $P > 0$ and $Q \geq 0$ such that (38) holds.
Proof. ($\Leftrightarrow$) Follows by choosing $V(x) = x^T P x$. ($\Rightarrow$) Suppose the zero solution is stable. From Theorem 2.28, eigenvalues of $A$ lie in the left half complex plane with those having zero real parts being semi-simple. Without loss of generality, suppose $A$ is in Jordan canonical form with first $r$ Jordan blocks having eigenvalues strictly in the left half plane and the remaining ones with zero real parts and semi-simple. Let $A = \begin{bmatrix} J_I & 0 & J_0 \\ 0 & J_I & 0 \\ \end{bmatrix}$. It is clear that $J_0$ is skew symmetric. Since $J_I$ has eigenvalues in the LHP, there exists $P_1 > 0, Q_1 > 0$ such that $J_I^T P_1 + P_1 J_I = Q_1$. Let $P = \begin{bmatrix} P_1 & 0 \\ 0 & I \\ \end{bmatrix}$ and $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \\ \end{bmatrix}$ such that (38) holds. 

Remark 2.31. In the above theorem, if $C = \sqrt{Q}$ and $(C, A)$ is observable, then the origin is asymptotically stable.

2.6 Indirect method of Lyapunov and local linearization

These methods work only locally i.e., they tell about local stability of an equilibrium point. No conclusion can be made about global stability.

Theorem 2.32 (Lyapunov’s indirect theorem for autonomous systems using linearization). Let zero be an equilibrium point of (7) where $f : S \rightarrow \mathbb{R}^n$ is continuously differentiable and $S$ is open with $0 \in S$. Let $A = \frac{\partial f}{\partial x}|_{x=0}$. Then following holds:

1. (Local exponential stability of linearization) If $\text{Re}(\lambda) < 0$ where $\lambda \in \text{spec}(A)$, then the zero solution is exponentially stable.

2. (Unstability of linearization) If there exists $\lambda \in \text{spec}(A)$ such that $\text{Re}(\lambda) > 0$, then the zero solution is unstable.

Proof. Suppose $\text{Re}(\lambda) < 0, \lambda \in \text{spec}(A)$. Therefore, there exists a unique $P > 0$ which satisfies (38). Let $V(x) = x^T P x$. Note that using the Taylor expansion of $f$ around $0$,

\[
\dot{V} = V', f = 2x^T P [Ax + r(x)] \\
= x^T (A^T P + PA)x + 2x^T P r(x) \\
= -x^T Q x + 2x^T P r(x).
\]

Using Cauchy-Schwarz inequality and using $-x^T Q x \leq -\lambda_{\min}(Q) \|x\|_2^2$,

\[
\dot{V} \leq -\lambda_{\min}(Q) \|x\|_2^2 + 2\lambda_{\max}(P) \|x\|_2 \|r(x)\|_2.
\]

Since $\|r(x)\|_2$ goes to zero faster than $\|x\|_2$, for every $\gamma > 0$, there exists $\epsilon > 0$ such that $\|r(x)\|_2 < \gamma \|x\|_2$. Therefore, for all $x \in B_\epsilon(0)$,

\[
\dot{V} < -[\lambda_{\min}(Q) - 2\gamma \lambda_{\max}(P)] \|x\|_2^2. \tag{43}
\]

Choosing $\gamma \leq \lambda_{\min}(Q) / 2 \lambda_{\max}(P)$, it follows that $\dot{V}(x) < 0$ for all $x \in B_\epsilon(0) \setminus 0$. This shows local asymptotic stability. Local exponential stability follows from the inequality $\lambda_{\min}(P) \|x\|_2^2 \leq V(x) \leq \lambda_{\max}(P) \|x\|_2^2$.

For the proof of the second statement we refer the reader to Haddad p. 178.

For the converse of the first statement, refer Hespanha

Example 2.33. Consider damped simple pendulum from Example 2.2. Linearizing around the origin, it can be checked that the linearization matrix has eigenvalues in the left half plane which implies asymptotic stability. For a pendulum without damping, eigenvalues of the linearization matrix are purely imaginary. Therefore, we can not conclude about stability by linearization. But we have already checked stability for this case using Lyapunov function.

Consider another equilibrium point $x_1 = \pi, x_2 = 0$. It can be checked that the linearization matrix has an eigenvalue in the right half plane which implies unstable equilibrium point.
Example 2.34 (Van der Pol oscillator). Consider the Van der Pol oscillator ([2]) described by
\[ x_1 = x_2, \quad x_2 = -\mu(1-x_1^2)x_2 - x_1. \]
Linearize around 0. It turns out that if \( \mu > 0 \), then both eigenvalues of the linearized matrix have positive real parts which implies instability.

Theorem 2.35 (Lyapunov's indirect theorem for control systems using linearization). Consider a non linear control system \( \dot{x} = F(x, u) \) such that \( F(0,0) = 0 \) and \( F \) is continuously differentiable. Let \( A := \frac{\partial F}{\partial x}|_{(x,u)=(0,0)}, \) \( B := \frac{\partial F}{\partial u}|_{(x,u)=(0,0)} \) such that \( (A,B) \) is stabilizable i.e., there exists \( K \in \mathbb{R}^{m \times n} \) such that \( \text{spec}(A + BK) \) lies in the left half complex plane. With the linear control law \( u = Kx \), the zero solution of the closed loop non linear system \( \dot{x} = F(x, u) \) is locally exponentially stable.

Proof. The closed loop system \( \dot{x} = F(x, Kx) = f(x) \) and
\[ \frac{\partial f}{\partial x}|_{x=0} = [\frac{\partial F(x, Kx)}{\partial x} + \frac{\partial F(x, Kx)}{\partial u}K]|_{x=0} = A + BK. \]
Since \( A + BK \) is asymptotically stable, by the first statement of the previous theorem, the result follows.

One can locally linearize the system and use separation principle to construct a local controller which locally asymptotically stabilizes the non linear control system.

Applications in singularly perturbed systems ([2]).

2.7 Converse theorems

Does there exists a Lyapunov function for stable, asymptotically stable, exponentially stable dynamical systems? These are converse theorems. For time varying systems, such functions exist under some conditions. For time invariant or autonomous systems, Lyapunov stability does not guarantee existence of a continuously differentiable or continuous time-independent Lyapunov function. (Existence of lower semi-continuous Lyapunov function Haddad.) However, for asymptotically stable equilibrium points of autonomous systems, a continuously differentiable time-independent Lyapunov function exists.

Lemma 2.36 (Massera’s lemma). We refer the reader to Haddad p. 162.

Theorem 2.37 (asymptotic stability converse). Suppose the zero solution of (7) is asymptotically stable, \( f : S \rightarrow \mathbb{R}^n \) is continuously differentiable and let \( \delta > 0 \) be such that \( B_\delta(0) \subset S \) is contained in the domain of attraction of (7). Then, there exists a continuously differentiable function \( V : B_\delta(0) \rightarrow \mathbb{R} \) such that \( V(0) = 0, V(x) > 0 \) for all \( x \in B_\delta(0) \setminus \{0\} \) and \( V'(x).f(x) < 0 \) for all \( x \in B_\delta(0) \). 

Proof. Theorem 3.9 Haddad.

Theorem 2.38 (exponential stability converse). Suppose the zero solution of (7) is exponentially stable, \( f : S \rightarrow \mathbb{R}^n \) is continuously differentiable and let \( \delta > 0 \) be such that \( B_\delta(0) \subset S \) is contained in the domain of attraction of (7). Then, for every \( p > 1 \), there exists a continuously differentiable function \( V : S \rightarrow \mathbb{R} \) and scalars \( \alpha, \beta, \epsilon > 0 \) such that \( \alpha \|x\|^p \leq V(x) \leq \beta \|x\|^p \) for all \( x \in B_\delta(0) \) and \( V'(x).f(x) < -\epsilon V(x) \) for all \( x \in B_\delta(0) \).

Proof. Theorem 3.10 Haddad.

Corollary 2.39. Suppose the zero solution of (7) is exponentially stable, \( f : S \rightarrow \mathbb{R}^n \) is continuously differentiable and let \( \delta > 0 \) be such that \( B_\delta(0) \subset S \) is contained in the domain of attraction of (7). Then, there exists a continuously differentiable function \( V : S \rightarrow \mathbb{R} \) and scalars \( \alpha, \beta, \epsilon > 0 \) such that \( \alpha \|x\|^2 \leq V(x) \leq \beta \|x\|^2 \) for all \( x \in B_\delta(0) \) and \( V'(x).f(x) < -\epsilon V(x) \) for all \( x \in B_\delta(0) \).
Table 2: Basic Lyapunov stability theorems for non autonomous systems (Theorem 3.1)

<table>
<thead>
<tr>
<th>Conditions on $V(x,t)$</th>
<th>Conditions on $-\dot{V}(x,t)$</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>lpdf</td>
<td>$\geq 0$ locally</td>
<td>stable</td>
</tr>
<tr>
<td>lpdf, decrescent</td>
<td>$\geq 0$ locally</td>
<td>uniformly stable</td>
</tr>
<tr>
<td>lpdf</td>
<td>lpdf</td>
<td>asymptotically stable</td>
</tr>
<tr>
<td>lpdf, decrescent</td>
<td>lpdf</td>
<td>uniformly asymptotically stable</td>
</tr>
<tr>
<td>pdf, decrescent</td>
<td>pdf</td>
<td>globally uniformly asymptotically stable</td>
</tr>
<tr>
<td>$a|x|^p \leq V(t,x) \leq b|x|^p$ locally</td>
<td>$\geq c|x|^p$ locally</td>
<td>locally exponentially stable</td>
</tr>
<tr>
<td>$a|x|^p \leq V(t,x) \leq b|x|^p$ globally</td>
<td>$\geq c|x|^p$ globally</td>
<td>globally exponentially stable</td>
</tr>
</tbody>
</table>

Proof. Take $p = 2$ in the previous theorem.

**Theorem 2.40** (global exponential stability converse). Suppose the zero solution of (7) is globally exponentially stable, $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and globally Lipschitz. Then, for every $p > 1$, there exists a continuously differentiable function $V : S \to \mathbb{R}$ and scalars $\alpha, \beta, \epsilon > 0$ such that $\alpha\|x\|^2 \leq V(x) \leq \beta\|x\|^2$ for all $x \in B_0(0)$ and $V'(x).f(x) < -\epsilon V(x)$ for all $x \in B_0(0)$.

Proof. Theorem 3.11 of Haddad.

### 3 Stability theory for non autonomous/time varying systems

#### 3.1 Basic stability theorems for time varying systems

**Theorem 3.1.** Consider a non autonomous system $\dot{x} = f(x(t), t, x(0)) = x_0$ where $x(t) \in S \subset \mathbb{R}^n$. Let $V : S \times \mathbb{R} \to \mathbb{R}$. Let $a, b, c > 0$ and $p \geq 1$. Then, Table 2 holds.

Proof. Consider the first case. Since $V$ is an lpdf, there exists $\alpha \in \mathcal{K}$ such that $V(x,t) \geq \alpha(\|x\|)$, $\forall x \in B_0(0)$. Moreover, since $\dot{V} \leq 0$ locally, $V(x,t) \leq 0 \forall t \geq t_0$, $\forall x \in B_0(0)$. To show local stability of $0$, we need to show that for given $\epsilon > 0$, $t_0 \geq 0$, there exists $\delta = \delta(\epsilon, t_0)$ such that for all $\|x(t_0)\| < \delta$, $\|x(t)\| < \epsilon$, $\forall t \geq t_0$.

Define $\epsilon_1 := \min(\epsilon, s, r)$. Choose $\delta > 0$ such that

$$\beta(t_0, \delta) := \sup_{\|x\| \leq \delta} V(x, t_0) < \alpha(\epsilon_1).$$

Such $\delta$ always exists because, $\alpha(\epsilon_1) > 0$ and $\lim_{\delta \to 0} \beta(t_0, \delta) = \lim_{\delta \to 0} \sup_{\|x\| \leq \delta} V(x, t_0) = 0$. Note that

$$\alpha(\|x(t_0)\|) \leq V(x(t_0), t_0) < \alpha(\epsilon_1).$$

Since $\alpha$ is increasing, this implies that $\|x(t_0)\| < \epsilon_1$. We need to show that $\|x(t)\| < \epsilon_1$ for all $t \geq t_0$. Suppose this is not true. Let $t_1 > t_0$ be the first instant such that $\|x(t)\| \geq \epsilon_1$. Then

$$V(x(t_1), t_1) \geq \alpha(\epsilon_1) > V(x(t_0), t_0).$$

But this is a contradiction since $\dot{V}(x, t) \leq 0$ for all $\|x\| \leq \epsilon_1$. Thus, $\|x(t)\| < \epsilon_1$ for all $t \geq t_0$.

For the second case, since $V$ is decrescent, define

$$\beta(\delta) := \sup_{\|x\| \leq \delta} \sup_{t \geq t_0} V(x, t).$$

Note that $\beta$ is non decreasing and there exists $d > 0$ such that $\beta(\delta) < \infty$ for $0 \leq \delta \leq d$. Choosing $\delta$ such that $\beta(\delta) < \alpha(\epsilon_1)$, using similar arguments used in the previous case, the statement follows.

For the third case, it is clear from the first case that $0$ is stable. We need to show that $\lim_{t \to \infty} x(t) = 0$ i.e., $\lim_{t \to \infty} \|x(t)\| = 0$. To show this, we need to show that for every $\epsilon > 0$, 

there exists $\delta = \delta(\epsilon, t_0) > 0$ and $T(\epsilon, t_0) < \infty$ such that $\|x(t_0)\| < \delta \Rightarrow \|s(t, t_0, x(t_0))\| < \epsilon$ for all $t > T(\epsilon, t_0) \geq t_0$. For the sake of simplicity, we avoid giving further details here. Case 4 is proved in [2] and the proof of case 3 follows on similar lines. For cases 5, 6, 7, we refer the reader to [2]. □

Converses two first two cases are true ([2] p.160, Remark pt. 1).

Alternate geometric proof (Khalil):

Proof. 1. Since $V$ is an lpdf, there exists $\alpha \in X$ such that $\alpha(\|x\|) \leq V(t, x)$ for all $x \in B_\epsilon(0)$. Moreover, $\dot{V} = \frac{\partial V}{\partial x} + \frac{\partial f(t, x)}{\partial x} \leq 0$ for all $x \in B_\epsilon(0)$. Given $\epsilon < \min(r, s)$, $B_\epsilon(0) \subset S$. Choose $c > 0$ such that $c < \min(\min_{\|x\|=\epsilon} \alpha(\|x\|), r, s)$. Then, $\{x \in B_\epsilon(0) \mid \alpha(\|x\|) \leq c\}$ is in the interior of $B_\epsilon(0)$. Define

$$\Omega_{t,c} := \{x \in B_\epsilon(0) \mid V(t, x) \leq c\}.$$ 

Since $V(t, x) \leq c \Rightarrow \alpha(\|x\|) \leq c$, $\Omega_{t,c} \subset \{x \in B_\epsilon(0) \mid \alpha(\|x\|) \leq c\}$. Observe that $\dot{V} \leq 0$ on $\Omega_{t,c}$ for all $t \geq t_0$. Therefore, solution starting in $\Omega_{t_0,c}$ stays in $\Omega_{t_0,c}$ for all $t \geq t_0$. Now choose a $\delta$ ball around 0 which lies inside $\Omega_{t_0,c}$. For all $x_0$ lying in this $\delta$ ball, $x(t)$ remains inside $\Omega_{t_0,c} \subset B_\epsilon(0)$. This shows the stability of 0.

2. Now suppose $V$ is decrescent. Hence, $\alpha_1(\|x(t)\|) \leq V(t, x(t)) \leq \alpha_2(\|x(t)\|)$. Therefore,

$$\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(V(t_0, x(t_0))) \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$$

Therefore, given $\epsilon > 0$, let $\delta = \alpha_2^{-1}(\alpha_1(\epsilon))$. This shows uniform stability.

3. Since, $\dot{V} < 0$, $\Omega_{t,c}$ shrinks as $t \to \infty$. This implies asymptotic stability.

4. Follows from 2 and 3.

5. Since $V$ is a pef, follows from 4.

6, 7. Follows similarly using extension of arguments used for the time invariant case. Using the given inequalities, it follows that $\dot{V} \leq -\xi V = \epsilon V$. Now using comparison lemma and similar arguments used for time invariant case, the statements follow. □

Example 3.2.

For Lyapunov stability of periodic systems, refer [2]

Theorem 3.3 (Exponential stability and its converse). Sastry, Theorem 5.17.

Lemma 3.4 (Barbalat’s lemma). Suppose that $x(t)$ is bounded, $\dot{x}(t)$ is bounded, $w$ is uniformly continuous, and $\int_{t_0}^\infty w(x(t))dt < \infty$. Then, $w(x(t)) \to 0$ as $t \to \infty$.

Proof. Since $w$ is uniformly continuous, for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that $\|x - y\| < \delta_\epsilon \Rightarrow \|w(x) - w(y)\| < \frac{\epsilon}{2}$. Since $\dot{x}$ is bounded, $x(t)$ is uniformly continuous i.e., given any $\delta_\epsilon > 0$, there exists $\delta_t$ such that $|t_1 - t_2| < \delta_t \Rightarrow \|x - y\| < \delta_\epsilon \Rightarrow \|w(x) - w(y)\| < \frac{\epsilon}{2}$. Therefore, $w$ is uniformly continuous w.r.t. $t$.

Suppose $w(x(t)) \not\to 0$ as $t \to \infty$, i.e., there is an $\epsilon > 0$ and a sequence $\{t_k\} \to \infty$ such that $|w(x(t_k))| \geq \epsilon$ for all $k$. Let $t \in [t_k, t_k + \delta]$. Then,

$$|w(x(t))| = |w(x(t_k)) - w(x(t_k)) - w(x(t))| \geq |w(x(t_k))| - |w(x(t_k)) - w(x(t))| \geq \epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{2}.$$ 

Therefore,

$$\int_{t_k}^{t_k+\delta} w(x(t))dt = \int_{t_k}^{t_k+\delta} |w(x(t))|dt \geq \frac{\epsilon}{2} \delta > 0$$

(44)

where the equality holds since $w(x(t))$ retains its sign for $t_k \leq t \leq t_k + \delta$. Therefore, $\int_{t_0}^s w(x(t))dt$ cannot converge to a finite limit as $s \to \infty$, a contradiction. □
Theorem 3.5 (Generalization of LaSalle (LaSalle-Yoshizawa)). Suppose that the function \( f \) of (3) is Lipshitz continuous in \( x \), uniformly continuous in \( t \) in \( B_r(0) \). Let \( \alpha_1, \alpha_2 \in \mathbb{R} \) and \( V(t,x) \) such that
\[
\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|).
\]
Suppose there exists a continuous non negative function \( W(x) \) such that
\[
\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W(x) \leq 0. \tag{45}
\]
Then for all \( \|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(\rho)) \), the trajectories \( x(\cdot) \) are bounded and \( \lim_{t \to \infty} W(x(t)) = 0 \).

Proof. (Sastry, Theorem 5.27.) Consider \( \dot{V}(t,x) \leq -W(x) \), and integrate from \( t_0 \) to \( t \). Therefore,
\[
V(t,x) - V(t_0,x_0) \leq -\int_{t_0}^{t} W(x(s))ds \Rightarrow \int_{t_0}^{t} W(x(s))ds \leq V(t_0,x_0) - V(t,x) \leq V(t_0,x_0)
\]
where the last inequality holds because \( V > 0 \). Therefore, \( \int_{t_0}^{t} W(x(s))ds \) exists and is bounded.

Claim: \( \|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(\rho)) \Rightarrow \|x(t)\| \leq \rho \) for all \( t \geq t_0 \).

Let \( \rho < \min(\min_{\|x\| = \rho} \alpha_1(r)) \). Then, \( \{x \in B_r(0) \mid \alpha_1(\|x\| \leq \rho) \} \subset B_r(0) \). Let \( \mu := \alpha_2^{-1}(\alpha_1(\rho)) \).

Suppose \( x(t_0) \in B_r(0) \). Therefore, \( \alpha_1(\|x(t_0)\|) \leq V(t_0,x(t_0)) \leq \alpha_2(\|x(t_0)\|) \).

Let \( \Omega_0, \rho = \{x \in B_r(0) \mid V(t,x(t)) \leq \rho \} \). Therefore, since \( \dot{V} \) is decreasing, if \( x(t_0) \in \Omega_{t_0,\rho} \), \( x(t) \in \Omega_{t_0,\rho} \). Since \( \dot{V} \) is decreasing,
\[
V(t,x(t)) \leq V(t_0,x(t_0)).
\]
Moreover, \( \alpha_1(\|x(t)\|) \leq V(t,x(t)) \leq \alpha_2(\|x(t)\|) \). Therefore,
\[
\|x(t)\| \leq \alpha_1^{-1}(V(t,x(t))) \leq \alpha_1^{-1}(V(t_0,x(t_0))) \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))
\]
Therefore, if \( \|x(t_0)\| \leq \mu \), then \( \|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)) = \rho \). Thus, \( x(t) \) is bounded.

We need to show that \( W(x(s)) \to 0 \) as \( s \to \infty \). Since \( x(t) \) is bounded, it is contained in some compact set \( D \). Since \( W \) is continuous, it is uniformly continuous over \( D \) (continuous functions over compact sets are uniformly continuous) i.e., for every \( \epsilon > 0 \), there exists \( \delta_x > 0 \) such that \( \|x - y\| < \delta_x \Rightarrow \|W(x) - W(y)\| < \epsilon \).

Since \( f \) is Lipshitz, \( \|\dot{x}\| = \|f(t,x) - f(t,y)\| \leq L(\|x - y\|) \) (\( y \) in some neighborhood of \( x \), for all \( x \in D \) and for all \( t \). Thus, \( \dot{x} \) is bounded and now by Barbalat’s lemma, theorem follows. \( \Box \)

3.2 Instability theorems

Theorem 3.6 (non autonomous). Consider a system (1) such that there exists a continuously differentiable function \( V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R} \) and a time \( t_0 \geq 0 \) such that
- \( \dot{V} \) is lpdf,
- \( V(0,t) = 0, \forall t \geq t_0 \),
- there exists points \( x_0 \) in a neighborhood around \( 0 \) such that \( V(t_0,x_0) \geq 0 \).

Then \( 0 \) is an unstable equilibrium point.

Proof. [2] \( \Box \)

Theorem 3.7. Consider a system (1) such that there exists a continuously differentiable function \( V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R} \) and a constant \( r > 0 \) such that
Consider an LTV system

Theorem 3.8 (Chetaev’s instability theorem). Consider a system (1) such that there exists a continuously differentiable function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ an open ball $B_r(0)$, an open set $\Theta \subset B_r(0)$ and a function $\gamma \in \mathcal{K}$ such that

$$V(x,t) > 0, \forall t \geq 0, \forall x \in \Theta, \sup_{t \geq 0} \sup_{x \in \Theta} V(t,x) < \infty,$$

$$V(t,x) = 0, \forall t \geq 0, \forall x \in \partial \Theta \cap B_r(0),$$

$$V'(x) \cdot f(x) > \gamma(\|x\|), \forall t \geq 0, \forall x \in \Theta.$$

Then, the zero solution is unstable.

Proof. [2]

3.3 Linear autonomous/LTV systems, linearization

Consider an LTV system

$$\dot{x}(t) = A(t)x(t).$$

Let $x(t_0) = x_0$ and let $\Phi(t,t_0)$ be the state transition matrix (refer Hespanha or short notes on linear systems). Then $x(t) = \Phi(t,t_0)x(t_0)$. Note that for linear systems, local and global stability is the same.

Theorem 3.9. Consider an LTV system (49). Then for the equilibrium point $0$, conditions in Table 3 hold. Moreover, the last two cases are equivalent.

Proof. Suppose $\sup_{t \geq t_0} \Phi(t,t_0) = m(t_0) < \infty$. For an $\epsilon > 0$ and $t_0 \geq 0$, define $\delta(t_0) := \frac{\epsilon}{m(t_0)}$. Then,

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| = \|\Phi(t,t_0)x(t_0)\| \leq \|\Phi(t,t_0)\|\|x(t_0)\| < m(t_0)\delta = \epsilon.$$

This shows that $0$ is stable.

Suppose $\|\Phi(t,t_0)\|$ is unbounded function of $t$ for some $t_0 \geq 0$. We need to show that one can choose $x(t_0) \in B_\delta(0)$ such that $\|x(t)\| \geq \epsilon$ for some $t \geq t_0$. Select $\delta_1 \in (0,\delta)$. Since $\|\Phi(t,t_0)\|$ is

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<td>$\sup_{t \geq 0} \sup_{t \geq t_0} \Phi(t, t_0) = m &lt; \infty$</td>
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<td>$\lim_{t \to \infty} |\Phi(t, t_0)| = 0$</td>
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<tr>
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</tr>
<tr>
<td>$|\Phi(t, t_0)| \leq me^{-\lambda(t-t_0)}$</td>
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unbounded function of $t$, there exists $t \geq t_0$ such that $\|\Phi(t, t_0)\| > \frac{\delta}{3}$. Choose $v$ such that $\|v\| = 1$ and $\|\Phi(t, t_0)v\| = \|\Phi(t, t_0)\|$ and let $x(t_0) = \delta_1 v$. Thus, $x(t_0) \in B_5(0)$. Furthermore, 
\[ \|x(t)\| = \|\Phi(t, t_0)x(t_0)\| = \|\delta_1 \Phi(t, t_0)v\| = \|\Phi(t, t_0)\| > \epsilon. \]
Hence, $0$ is unstable.

We refer the reader to Sastry [2] for proofs of other statements.

**Quadratic Lyapunov functions for LTV systems:**

**Lemma 3.10.** Suppose $Q : \mathbb{R}^+ \to \mathbb{R}^{n \times n}$ is continuous and bounded, and that the equilibrium $0$ of (49) is uniformly asymptotically stable. Then, for each $t \geq 0$, the matrix 
\[ P(t) = \int_t^\infty \Phi^T(\tau, t)Q(t)\Phi(\tau, t)d\tau \]  
(50)
is well-defined; moreover, $P(t)$ is bounded as a function of $t$.

**Proof.** Uniform asymptotic stability is equivalent to exponential stability for LTV systems (Theorem 3.9). Therefore, $\|\Phi(\tau, t)\| \leq me^{-\lambda(\tau-t)}$, $\forall \tau \geq t \geq 0$. The boundedness of $Q$ and exponential boundedness of $\Phi$ implies that $P$ is bounded.

**Lemma 3.11.** Suppose that, in addition to the hypotheses of the previous lemma, the following conditions also hold:

1. $Q(t)$ is symmetric and positive definite for each $t \geq 0$; moreover, there exists a constant $\alpha > 0$ such that $\alpha \|x\|^2 \leq x^TQ(t)x, \forall t \geq 0$, $\forall x \in \mathbb{R}^n$.

2. The matrix $A(\cdot)$ is bounded i.e., $m_0 = \sup_{t \geq 0} \|A(t)\| < \infty$.

Under these conditions, the matrix $P(t)$ defined in (50) is positive definite for each $t \geq 0$; moreover, there exists a constant $\beta > 0$ such that $\beta \|x\|^2 \leq x^TP(t)x, \forall t \geq 0, \forall x \in \mathbb{R}^n$.

**Proof.** [2], Lemma 56, p.203.

**Theorem 3.12.** Suppose $Q(\cdot)$ and $A(\cdot)$ satisfy the hypotheses of Lemmas 3.10 and 3.11. Then, for each function $Q(\cdot)$ satisfying the hypotheses, the function $V(x, t) = x^TP(t)x$ is a Lyapunov function establishing the exponential stability of the equilibrium point $0$.


**3.4 Indirect method of Lyapunov and local linearization**

For a non autonomous system (3), define 
\[ A(t) = \frac{\partial f(x, t)}{\partial x}|_{x=0} \text{ and } f_1(x, t) = f(x, t) - A(t)x. \]  
(51)

Then it follows that $\lim_{x \to 0} \frac{\|f_1(x, t)\|}{\|x\|} = 0$. However, it may not be true that 
\[ \lim_{\|x\| \to 0} \sup_{t \geq 0} \frac{\|f_1(x, t)\|}{\|x\|} = 0. \]  
(52)

**Theorem 3.13.** Consider (3) where $f$ is continuously differentiable. Define $A(t), f_1(x, t)$ as in (51). Suppose (52) holds and $A(\cdot)$ is bounded. If $0$ is an exponentially stable equilibrium point of the linearized system $\dot{x}(t) = A(t)x(t)$, then it is also an exponentially stable equilibrium point of (3).

**Proof.** [2], Theorem 15, Section 5.5, p.211 or Sastry, Theorem 5.41, p.215.

**Example 3.14.** For LTV systems, exponential stability/uniform asymptotic stability is not characterized by the location of eigenvalues of $A(t)$ for all $t$, Example 4.22, Khalil.
3.5 Converse theorems

Converse theorems for uniform asymptotic stability, exponential stability and and global exponential stability are stated in [2] Section 5.7. The essence of these theorems is that they state that the sufficient conditions obtained before for stability are necessary as well.

3.5.1 Applications of converse theorems

**Theorem 3.15.** local exponential stability of autonomous systems (Section 5.8.1, [2]). This shows that the converse of the first statement of Theorem 2.32.

slowly varying systems (Section 5.8.2, [2]), Observer-Controller stabilization for time varying systems (separation principle): local stabilizing controllers and local separation principle for non linear systems via linearization (Section 5.8.3, [2]), stabilizing triangular/hierarchical systems (isolated subsystems) (Section 5.8.4, [2]).

4 Stability of periodic solutions and tracking trajectories

Let $\gamma(t)$ be a periodic solution of $\dot{x} = f(t,x)$. We want to investigate stability of this periodic solution for trajectories starting from nearby points. Let $y(t) = x(t) - \gamma(t)$. Thus, $y = 0$ is an equilibrium point of the time varying system $\dot{y} = f(t,y + \gamma(t)) - f(t,\gamma(t))$. Thus, one can characterize stability properties of $\gamma(t)$ (e.g., Lyapunov stability, asymptotic stability etc.) based on stability properties of $y = 0$.

**Example 4.1.** Suppose we are interested in tracking a trajectory $r(t)$. Again, let $y(t) = x(t) - r(t)$. Now, one needs to choose an input such that $y = 0$ is an equilibrium point of the time varying system $\dot{y} = f(t,y + r(t)) - \dot{r}(t)$. To achieve asymptotic tracking, one needs an input which makes $y = 0$ asymptotically stable.

**Example 4.2.** Recall the example of stabilization of a rigid robot. Suppose one wants to track a trajectory $r(t)$. Thus, one wants $(x,y) \rightarrow (r,\dot{r})$. Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} r \\ \dot{r} \end{bmatrix}$. One needs to find $u$ such that $0$ is an asymptotically stable equilibrium point of

$$
\dot{z} = \begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = \begin{bmatrix}
y - \dot{r} \\
M(x)^{-1}[u - C(x,y)y] - \dot{r}
\end{bmatrix}.
$$

Let $u = -K_p(x - r) - K_d(y - \dot{r}) + C(x,y)\dot{r} + M(x)\ddot{r}$ where $K_p, K_d > 0$. Therefore,

$$
\dot{z} = \begin{bmatrix}
y - \dot{r} \\
M(x)^{-1}[-K_p(x-r) - K_d(y-\dot{r})]
\end{bmatrix} = \begin{bmatrix}
z_2 \\
M(x)^{-1}[-K_pz_1 - K_dz_2 - C(x,y)z_2]
\end{bmatrix}
$$

and $\dot{z}$ has the required equilibrium point. Consider a Lyapunov function

$$
V(z_1, z_2) = \frac{1}{2}[z_2^T M(x) z_2 + z_1^T K_p z_1]
$$

$$
\Rightarrow \dot{V} = z_2^T M(x) z_2 + \frac{1}{2} z_2^T M(x) z_2 + z_1^T K_p z_1
$$

$$
= z_2^T (-K_p z_1 - K_d z_2 - C(x,y) z_2) + \frac{1}{2} z_2^T M(x) z_2 + z_1^T K_p z_2
$$

$$
= -z_1^T K_p z_2
$$

since $M - 2C$ is skew symmetric. Now we find the largest invariant set. Observe that $\dot{V}$ vanishes on $\mathbb{R}^n \times 0$. Note that $\dot{z}_1 = 0 \Rightarrow z_2 = 0$ and $\dot{z}_1 = 0 \Rightarrow M(x)^{-1}[-K_p z_1 - K_d z_2 - C(x,y) z_2] = 0$. Since $z_2 = 0$ and $M$ is invertible, $K_p z_1 = 0 \Rightarrow z_1 = 0$. Thus, $(0,0)$ forms the largest invariant set and global asymptotic stability follows from LaSalle.
5 Input to state stability

(Haddad Khalil) Boundedness of solutions

Consider an LTI system $\dot{x} = Ax + Bu$ where $A$ is Hurwitz. Recall that

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\dot{u}(\tau)d\tau.$$ 

Using $\|e^{A(t-t_0)}\| \leq ke^{-\lambda(t-t_0)}$,

$$\|x(t)\| \leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \int_{t_0}^{t} ke^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|d\tau$$

$$\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + k\|B\|\sup_{t_0 \leq \tau \leq t}\|u(\tau)\|e^{-\lambda t}\int_{t_0}^{t} e^{\lambda\tau}d\tau$$

$$= ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda}\sup_{t_0 \leq \tau \leq t}\|u(\tau)\| (1 - e^{-\lambda(t-t_0)})$$

$$\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k\|B\|}{\lambda}\sup_{t_0 \leq \tau \leq t}\|u(\tau)\|.$$ 

This shows that the zero input response decays to zero exponentially whereas; the zero state response is bounded for bounded inputs. Thus, $A$ Hurwitz implies that state remains bounded for an LTI system if the input is bounded.

We want to understand conditions under which $x(t)$ remains bounded starting from an initial condition.

**Definition 5.1** (Haddad/Khalil). *The solutions of (3) are*

1. uniformly bounded if there exists $\gamma > 0$, independent of $t_0 \geq 0$ such that for every $\delta \in (0, \gamma)$, there exists $\epsilon = \epsilon(\delta) > 0$, independent of $t_0$, such that

   $$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \ \forall t \geq t_0. \quad (53)$$

2. globally uniformly bounded if (53) holds for arbitrarily large $\delta$.

3. uniformly ultimately bounded with an ultimate bound $\epsilon$ if there exists $\gamma > 0$, independent of $t_0 \geq 0$, such that for every $\delta \in (0, \gamma)$, there exists $T = T(\delta, \epsilon)$, independent of $t_0$ such that

   $$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \ \forall t \geq t_0 + T. \quad (54)$$

4. globally uniformly ultimately bounded if (54) holds for arbitrarily large $\delta$.

For solutions of (7), we drop the word uniformly since the solution depends only on $t - t_0$.

We refer the reader to generalized LaSalle theorem (Theorem 3.5).

**Theorem 5.2** (boundedness). *Let $S \subset \mathbb{R}^n$ be an open set containing $0$ and $V : [0, \infty) \times S \rightarrow \mathbb{R}$ be a continuously differentiable function such that*

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t, x) \leq -W_3(x), \ \forall \|x\| \geq \mu > 0 \quad (55)$$

$$\forall t \geq 0 \text{ and } \forall x \in S \text{ where } \alpha_1, \alpha_2 \in \mathcal{K} \text{ and } W_3 \text{ is continuous positive definite function. Take } r > 0 \text{ such that } B_r \subset S \text{ and suppose that } \mu < \alpha_2^{-1}(\alpha_1(r)). \text{ Then, there exists a class } \mathcal{KL} \text{ function } \beta \text{ and }$$
for every initial state \( x(t_0) \), satisfying \( \|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r)) \), there exists \( T = T(x(t_0), \mu) \) such that the solution to (3) satisfies

\[
\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t_0 \leq t \leq t_0 + T
\]

(57)

\[
\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall t \geq t_0 + T.
\]

(58)

Moreover, if \( S = \mathbb{R}^n \), and \( \alpha_1 \in \mathbb{K}\mathbb{R} \), then (57) and (58) hold for any initial state \( x(t_0) \) with no restriction on how large \( \mu \) is.

**Proof.** (Sketch): Observe that if \( \mu = 0 \), then recalling conditions for uniform asymptotic stability of the origin (Theorem 1.14, Theorem 3.1) and comparing with the conditions of the present theorem, it is clear that one gets uniform asymptotic stability of the origin. The proof is based on extending similar ideas to the case where we can have uniform asymptotic stability w.r.t. a set rather than just a point. We refer the reader to Khalil for the complete proof.

Let \( \rho = \alpha_1(r) \), hence, \( \alpha_2(\mu) < \alpha_1(r) = \rho \). Since \( \|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r)) \), \( \alpha_2(\|x(t_0)\|) \leq \alpha_1(r) = \rho \). Let \( \eta = \alpha_2(\mu) \) and define \( \Omega_{\eta} = \{x \in B_r(0) \mid V(t,x) \leq \eta\} \) and \( \Omega_{t,\rho} := \{x \in B_r(0) \mid V(t,x) \leq \rho\} \). Note that if \( x(t_0) \) lies outside \( B_\mu(0) \) and inside \( B_r(0) \), then for \( t \geq t_0 \),

\[
\alpha_1(\|x(t)\|) \leq V(t,x(t)) \leq V(t,x(t_0)) \alpha_2(\|x(t_0)\|) \Rightarrow \|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|)).
\]

Note that \( B_\mu \subset \Omega_{t,\eta} \subset \Omega_{t,\rho} \subset B_r \). The fact that these sets are contained in \( B_r \) follows from definition. To show that \( B_\mu \subset \Omega_{t,\eta} \), note that \( \|x\| < \mu \Rightarrow \alpha_2(\|x\|) < \alpha_2(\mu) \). Moreover, \( V(t,x) \leq \alpha_2(\|x\|) < \alpha_2(\mu) = \eta \). Thus, \( B_\mu \subset \Omega_{t,\eta} \). Since \( \eta < \rho \), \( \Omega_{t,\eta} \subset \Omega_{t,\rho} \). Sets \( \Omega_{t,\eta}, \Omega_{t,\rho} \) have a property that a solution starting inside it can not leave it since \( V \) is negative on the boundary. A solution starting in \( \Omega_{t,\rho} \) must enter \( \Omega_{t,\eta} \) in finite time because in the set \( \Omega_{t,\rho} \setminus \Omega_{t,\eta} \), \( V \) satisfies \( V \leq -k < 0 \) where \( k = \min W(x) \) over the set \( B_\mu \leq \|x\| \leq B_r \). Therefore,

\[
V(t,x) \leq V(t_0,x(t_0)) - k(t-t_0) \leq \rho - k(t-t_0)
\]

which shows that \( V(t,x) \) reduces to \( \eta \) within the time interval \( [t_0, t_0 + (\rho - \eta)/k] \). For a solution starting in \( \Omega_{t,\eta} \), (58) is satisfied for all \( t \geq t_0 \). For a solution starting in \( \Omega_{t,\rho} \), it enters \( \Omega_{t,\eta} \) after time \( T \). Now (57) can be obtained by applying arguments used for checking uniform asymptotic stability (Theorem 1.14).

**Definition 5.3** (Input to state stability). The system (3) is said to be input-to-state stable if there exists a class \( \mathcal{KL} \) function \( \beta \) and a class \( \mathcal{K} \) function \( \gamma \) such that for any initial state \( x(t_0) \) and any bounded input \( u(t) \), the solution \( x(t) \) exists for all \( t \geq t_0 \) and satisfies

\[
\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|).
\]

(60)

**Theorem 5.4** (ISS for time independent systems). A system \( \dot{x} = F(x,u) \) is input-to-state stable \( \iff \) there exists a continuously differentiable radially unbounded pdf function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) and continuous functions \( \gamma_1, \gamma_2 \in \mathcal{K} \) such that for every \( u \in \mathbb{R}^m \),

\[
V'(x)F(x,u) \leq -\gamma_1(\|x\|), \quad \|x\| \geq \gamma_2(\|u\|)
\]

(61)

**Proof.** Sketch: (Haddad) \( \Rightarrow \) Let \( f(t,x) = F(x,u) \), \( V(t,x) = V(x) \) and \( W(x) = \gamma_1(\|x\|) \) in the previous theorem. One obtains that there exists \( t_1 \) such that \( \|x\| \leq \gamma(\|u\|) \), \( t \geq t_1 \) (this follows from the proof of (58)). And \( \|x\| \leq \eta(\|x_0\|, t) \), \( t \leq t_1 \) (this follows from the proof of (57)). Therefore, \( \|x\| \leq \eta(\|x_0\|, t) + \gamma(\|u\|) \), \( t \geq 0 \).

\( \Rightarrow \) involves technicalities.

\( \square \)
**Theorem 5.5** (ISS for time varying systems). Let \( V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function such that
\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -W(x), \forall \|x\| \geq \rho(\|u\|) > 0
\]
\( \forall (t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \), where \( \alpha_1, \alpha_2 \in \mathcal{K}, \rho \in \mathcal{K} \) and \( W > 0 \) on \( \mathbb{R}^n \). Then the system \( \dot{x} = F(t, x, u) \) is input-to-state stable with \( \gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho \).

*Proof. Sketch:* Again using \( f(t, x) = F(x, u) \), and using the same method used in the proof of Theorem 5.2, the proof can be constructed. \( \square \)

## 6 Center manifold theory

A manifold \( M \) is said to be invariant w.r.t. a dynamical system if \( x(0) \in M \Rightarrow x(t) \in M \) for all \( t \). As seen in the Introduction of nonlinear control, at an equilibrium \( x_0 \) of a nonlinear dynamical system, if one computes the spectrum of the linearization \( A \) and finds a number of eigenvalues in the left half-plane, then there is an invariant manifold (i.e., a manifold that is invariant under the flow and that is simply the graph of a mapping in this case) that is tangent to the corresponding (generalized) eigenspace; it is called the local stable manifold. All trajectories on this stable manifold are asymptotic to the point \( x_0 \) as \( t \to \infty \). Similarly, associated with the eigenvalues in the right half-plane is an unstable manifold.

If none of the eigenvalues associated with an equilibrium are on the imaginary axis, then the equilibrium is called hyperbolic. In this case, the tangent spaces to the stable and unstable manifolds span the whole of \( \mathbb{R}^n \). When there are zero eigenvalues or eigenvalues on the imaginary axis, one needs the notion of the center manifold as one cannot conclude about the local stability from linearization.

Consider a time invariant system (7). Let \( A = \frac{\partial f}{\partial x}|_{x=0} \) and suppose \( f \) is twice continuously differentiable. Then, (7) can be rewritten as
\[
\dot{x} = Ax + [f(x) - Ax] = Ax + \hat{f}(x)
\]
where \( \hat{f}(x) = f(x) - Ax \) such that \( \hat{f}(0) = 0 \) and \( \frac{\partial \hat{f}}{\partial x}(0) = 0 \). Since we are interested in the case when the linearization fails, suppose \( A \) has \( k \) eigenvalues with zero real parts and \( n-k \) eigenvalues with negative real parts. We can assume without loss of generality using a similarity transform \( T \) that \( A \) is in block diagonal form
\[
\begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix}
\]
where eigenvalues of \( A_1 \) has zero real parts and eigenvalues of \( A_2 \) have negative real parts.

Let \( Tx = \begin{bmatrix} y \\ z \end{bmatrix} \) which transforms (7) into the form
\[
\begin{align*}
\dot{y} &= A_1 y + g_1(y, z) \\
\dot{z} &= A_2 z + g_2(y, z)
\end{align*}
\]
(64)
(65)
where \( g_1 \) and \( g_2 \) inherit properties of \( \hat{f} \). They are twice continuously differentiable and
\[
g_i(0, 0) = 0, \quad \frac{\partial g_i}{\partial z}(0, 0) = 0
\]
(66)
for \( i = 1, 2 \).

**Definition 6.1.** If \( z = h(y) \) is an invariant manifold for (64) – (65) and \( h \) is smooth, then it is called a center manifold if
\[
h(0) = 0, \quad \frac{\partial h}{\partial y}(0) = 0.
\]
**Theorem 6.2** (Existence (Khalil, Theorem 8.1)). If $g_i$, $i = 1, 2$ are twice continuously differentiable and satisfy (66), all eigenvalues of $A_1$ have zero real parts, and all eigenvalues of $A_2$ have negative real parts, then there exists a constant $\delta > 0$ and a continuously differentiable function $h(y)$ defined for all $\|y\| < \delta$, such that $z = h(y)$ is a center manifold for (64) – (65).

Due to its invariance property, if the initial condition lies on the center manifold, then the solution remains on the center manifold. Since $z(t) = h(y(t))$, the evolution of the system in the center manifold is described by the $k$–th order differential equation

$$\dot{y} = A_1 y + g_1(y, h(y))$$

which is referred as reduced system.

Suppose $z(0) \neq h(y(0))$. Then $z(t) - h(y(t))$ represents the deviation of the trajectory from the center manifold at any time $t$. Let $w = z - h(y)$. This transforms (64) – (65) into

$$\dot{y} = A_1 y + g_1(y, w + h(y))$$

$$\dot{w} = A_2(w + h(y)) + g_2(y, w + h(y)) - \frac{\partial h}{\partial y}[A_1 y + g_1(y, w + h(y))].$$

In the new coordinates, $w = 0$ is the center manifold. The dynamics on the center manifold are characterized by $w(t) = 0 \Rightarrow \dot{w} = 0$. Substituting this in (69),

$$0 = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}[A_1 y + g_1(y, h(y))].$$

Since the above equation must be satisfied for any solution lying on the center manifold, we conclude that $h(y)$ must satisfy the pde (70). This provides a condition that the center manifold $z = h(y)$ must satisfy.

Adding and subtracting $g_1(y, h(y))$ to rhs of (68) and subtracting (70) from (69),

$$\dot{y} = A_1 y + g_1(y, h(y)) + N_1(y, w)$$

$$\dot{w} = A_2(w + h(y)) + N_2(y, w)$$

where $N_1(y, w) = g_1(y, w + h(y)) - g_1(y, h(y))$, $N_2(y, w) = g_2(y, w + h(y)) - g_2(y, h(y)) \frac{\partial h}{\partial y} N_1(y, w)$. One can check that $N_1, N_2$ are twice differentiable and $N_i(y, 0) = 0$, $\frac{\partial N_i}{\partial w}(0, 0) = 0$ for $i = 1, 2$.

**Theorem 6.3** (Stability condition). Under assumptions of Theorem 6.2, if the origin $y = 0$ of the reduced order system (67) is asymptotically stable (unstable), then the origin of the full system (64) – (65) is also asymptotically stable (unstable).

**Proof.** Theorem 8.2 of Khalil, the proof makes use of the converse Lyapunov theorem for asymptotic stability to construct a Lyapunov candidate for the reduced order system.

**Corollary 6.4.** Under assumptions of Theorem 6.2, if the origin $y = 0$ of the reduced order system (67) is stable and there is a continuously differentiable Lyapunov function $V(y)$ such that

$$\frac{\partial V}{\partial y}[A_1 y + g_1(y, h(y))] \leq 0$$

in some neighborhood of $y = 0$, then the origin of the full system (64) – (65) is stable.

**Proof.** Note that unlike asymptotic stability, converse theorem does not hold for Lyapunov stability. Therefore, if the origin of the reduced order system is known to be Lyapunov stable, we need an existence of Lyapunov candidate as well in our hypothesis to make the previous proof work.
Corollary 6.5. Under assumptions of Theorem 6.2, the origin $y = 0$ of the reduced order system (67) is asymptotically stable $\iff$ the origin of the full system (64) – (65) is also asymptotically stable.

Proof. $(\Rightarrow)$ Follows from the theorem above. $(\Leftarrow)$ Stability of the full system implies the stability of the reduced order system as well, so this implication is trivial.

To use Theorem 6.3, one needs to find the center manifold $z = h(y)$ for which one needs to solve the pde (70) with boundary conditions $h(0) = 0$, $\frac{\partial h}{\partial y}(0) = 0$. This is a difficult pde in general and one uses Taylor series approximation of the solution. We refer the reader to Khalil, Section 8.1 for more details.

7 Control Lyapunov functions and feedback stabilization

Consider non linear controlled dynamical system

$$\dot{x}(t) = \mathbf{F}(x(t), u(t)), \quad x(t_0) = x_0, \quad t \geq 0$$

(73)

where $x(t) \in S \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$ and $\mathbf{F} : S \times U \rightarrow \mathbb{R}^n$ satisfies $\mathbf{F}(0, 0) = 0$. Suppose $\mathbf{F}$ is Lipshitz continuous in the neighborhood of the origin in $S \times U$. We need to find a feedback control law $u(t) = \phi(x(t))$ such that the closed loop system is stable.

Definition 7.1 (Control Lyapunov function (Haddad)). Consider the controlled nonlinear dynamical system given by (73). A continuously differentiable positive-definite function $V : S \rightarrow \mathbb{R}$ satisfying

$$\inf_{u \in U} V'(x).\mathbf{F}(x, u) < 0, \quad x \in S, \quad x \neq 0$$

(74)

is called a control Lyapunov function.

If (74) holds, then there exists a feedback control law $\phi : S \rightarrow U$ such that $V'(x).\mathbf{F}(x, \phi(x)) < 0$, then by Theorem 2.1 case 2, the origin is asymptotically stable. Conversely, if there exists a feedback law such that the origin is asymptotically stable, then by Theorem 2.37, there exists a Lyapunov function $V$ such that $V'(x).\mathbf{F}(x, \phi(x)) < 0$ which implies the existence of a control Lyapunov function. Thus, (73) is feedback stabilizable if there exists a control Lyapunov function satisfying (74). The analogues of other cases of Theorem 2.1 also hold for control Lyapunov functions.

Consider an affine non linear control system

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \quad t \geq 0$$

(75)

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$.

Theorem 7.2 (Haddad). Consider the controlled nonlinear system given by (75). Then a continuously differentiable positive-definite, radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a control Lyapunov function of (75) if and only if

$$V'(x)f(x) < 0, \quad x \in \mathcal{R}$$

(76)

where $\mathcal{R} = \{x \in \mathbb{R}^n, x \neq 0 : V'(x)G(x) = 0\}$.

Proof. $(\Leftarrow)$ Obvious. $(\Rightarrow)$ If $x \notin \mathcal{R}$, then $\inf_{u \in U} V'(x).[f(x(t)) + G(x(t))u(t)] = -\infty$. When $x \in \mathcal{R}$, (76) must hold if $V$ is a control Lyapunov function.

We now construct an explicit feedback control law which is a function of the control Lyapunov function $V$ (Haddad). Let $\alpha(x) := V'(x)f(x)$ and $\beta(x) := G^T(x)(V'(x))^T$ and $c_0 \geq 0$. Let

$$\phi(x) = \begin{cases} -(c_0 + \frac{\alpha(x) + \sqrt{\alpha^2(x) + (\beta(x))^2}}{\beta(x)})\beta(x) & \text{if } \beta(x) \neq 0, \\ 0 & \text{if } \beta(x) = 0. \end{cases}$$

(77)
In this case, the control Lyapunov function turns out to be the Lyapunov function for the closed loop system with \( u = \phi(x) \). Observe that
\[
\dot{V}(x) = V'(x)[f(x) + G(x)\phi(x)] \\
= \alpha(x) + \beta^T(x)\phi(x) \\
= \begin{cases} 
-\alpha_0\beta^T(x)\beta(x) - \sqrt{\alpha^2(x) + (\beta^T(x)\beta(x))^2} & \text{if } \beta(x) \neq 0, \\
\alpha(x) & \text{if } \beta(x) = 0 
\end{cases} \\
< 0, \quad \forall x \in \mathbb{R}^n \setminus 0
\]
provided that Theorem 7.2 is satisfied. This implies that \( V \) is indeed a Lyapunov function for the closed loop system. Since \( f \) and \( G \) are smooth, the feedback law (77) is smooth everywhere except the origin.

**Theorem 7.3.** Consider the nonlinear dynamical system \( G \) given by (75) with a radially unbounded control Lyapunov function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \). Then the following statements hold:

1. The control law \( \phi(x) \) given by (77) is continuous at \( x = 0 \) if and only if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( 0 < \|x\| < \delta \), there exists \( u \in \mathbb{R}^n \) such that \( \|u\| < \epsilon \) and \( \alpha(x) + \beta^T(x)u < 0 \).

2. There exists a stabilizing control law \( \hat{\phi}(x) \) such that \( \alpha(x) + \beta^T(x)\hat{\phi}(x) < 0, x \in \mathbb{R}^n, x \neq 0 \) and \( \hat{\phi}(x) \) is Lipschitz continuous at \( x = 0 \) if and only if the control law \( \hat{\phi}(x) \) given by (77) is Lipschitz continuous at \( x = 0 \).

**Proof.** We refer the reader to Theorem 6.8 of Haddad. \( \square \)

### 8 Periodic systems

#### 8.1 Linear periodic systems

The following discussion is from [7]. Consider a linear periodic system \( \dot{x}(t) = A(t)x(t) \) such that \( A(t + T) = A(t) \), \( T > 0 \). Let \( \Phi(t, \tau) \) be the state transition matrix. Recall that for general linear systems,
\[
\Phi(t, t_0) := I + \int_{t_0}^{t} A(s_1)ds_1 + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)ds_2 ds_1 + \ldots.
\]
and \( \frac{d}{dt} \Phi(t, t_0) = A(t) \Phi(t, t_0) \). Hence, \( \frac{d}{dt} \Phi(t + T, t_0) = A(t + T) \Phi(t + T, t_0) \). Therefore, for periodic systems, \( \frac{d}{dt} \Phi(t + T, t_0) = A(t) \Phi(t + T, t_0) \) since \( A(t + T) = A(t) \). Furthermore, by the concatenation property of the state transition matrix, \( \Phi(t + T, t_0) = \Phi(t, t_0) \Phi(t + T, t_0) \). Let \( C(t) = \Phi^{-1}(t, t_0) \Phi(t + T, t_0) \). Therefore
\[
\dot{C} = -(\Phi^{-1}(t, t_0) A(t) \Phi(t, t_0) \Phi^{-1}(t, t_0)) \Phi(t + T, t_0) + \Phi^{-1}(t, t_0) A(t) \Phi(t + T, t_0) = 0.
\]
Hence, \( C \) is a constant matrix and \( C = C(t_0) = \Phi(t_0 + T, t_0) \). Therefore, \( \Phi(t + T, t_0) = \Phi(t_0) C \).

Define \( R := \frac{1}{T} \ln(C) \), hence, \( e^{RT} = C \). Let \( P(t) = \Phi(t, 0) e^{-Rt} \). Clearly \( P(t) \) is non-singular.
\[
P(t + T) = \Phi(t + T, 0) e^{-R(t + T)} = \Phi(t, 0) C e^{-R(t + T)} = \Phi(t, 0) C e^{-RT} e^{-Rt} = \Phi(t, 0) e^{-Rt} = P(t).
\]
Therefore, \( \Phi(t, 0) = P(t)e^{-Rt} \) where \( P(t) \) is non singular and periodic. This is called Floquet decomposition.

The state evolution is given by \( x(t) = \Phi(t, 0)x_0 = P(t)e^{-Rt}x_0 \). Since \( P(t) \) is periodic and continuous, it is bounded. Therefore the periodic linear system is globally exponentially stable \( \Leftrightarrow \)
\( R \) has eigenvalues in the strict LHP \( \iff \) \( C \) has eigenvalues strictly inside the unit circle. The matrix \( C = \Phi(t + T, t) \) is called monodromy matrix and eigenvalues of \( C \) are called Floquet multipliers.

Let \( y(t) := P^{-1}(t)x(t) \). Therefore,

\[
x = \dot{P}y + P\dot{y} \Rightarrow Ax = \dot{P}y + P\dot{y} \Rightarrow APy = \dot{P}y + P\dot{y} \Rightarrow y = P^{-1}(APy - \dot{P}y).
\]

Using \( P(t) = \Phi(t, 0)e^{-Rt} \) in the above equation,

\[
\dot{y} = P^{-1}(A\Phi(t, 0)e^{-Rt} - A\Phi(t, 0)e^{-Rt} + \Phi(t, 0)Re^{-Rt})y = e^{Rt}\Phi(t, 0)^{-1}(\Phi(t, 0)Re^{-Rt})y = e^{Rt}Re^{-Rt}y = Ry. \tag{79}
\]

Therefore the linear periodic system is globally exponentially stable iff the LTI system obtained above is exponentially stable.

### 8.2 Nonlinear periodic systems

Consider a nonlinear periodic system \( \dot{x} = f(t, x) \) such that \( f(t + T, x) = f(t, x) \) for all \( x \in \mathbb{R}^n, T > 0 \). Let \( s(t, 0, x_0) \) be the flow of this ode. Hence, \( \frac{d}{dt}s(t, 0, x_0) = f(t, x(t)) \). Consider \( s(t + T, T, x_0) \).

Because the time varying vector field \( f(t, x) \) is periodic with period \( T \), by existence and uniqueness, \( s(t + T, T, x_0) = s(t, 0, x_0) \).

Consider a fixed point or a periodic solution \( x^*(t) \) of this periodic system. Let \( z = x - x^* \) be the perturbation from this trajectory. Therefore,

\[
z = f(t, x) - t, x^*.
\]

Using linearization, one obtains a periodic LTV system. If the linear system is exponentially stable, then the nonlinear periodic system is locally exponentially stable at the solution \( x^* \).

It turns out that for periodic systems, \( 0 \) is stable/asymptotically stable iff is is stable/asymptotically stable ([2]). Lyapunov analysis for stability is applicable to periodic systems as well. One looks for a periodic lpdf Lyapunov function \( V \) with the same period as that of the system such that \( \dot{V} \leq 0 \) in some open neighborhood \( N \) of \( 0 \). Then, one can apply LaSalle-Krasovskii theorem to conclude convergence of trajectories towards the invariant set \( M \) of \( V \) where \( M \subset S \subset N \) such that \( S := \{ x \in N \mid \dot{V}(x) = 0 \} \) and \( M \) is the largest invariant set of \( S \). If \( S \) contains no invariant trajectory other than the zero trajectory, then \( 0 \) is uniformly asymptotically stable. If in addition, \( V \) is a pdf and radially unbounded, then \( 0 \) is globally uniformly asymptotically stable equilibrium point of the periodic system ([2]).

### 9 Discrete non linear systems

Consider an autonomous discrete time non linear system

\[
x(k + 1) = f(x(k)), \ x(0) = x_0, \ k \in \mathbb{Z}^+
\tag{80}
\]

where \( x \in S \subset \mathbb{R}^n \) (\( S \) open), \( 0 \in S \) and \( f(0) = 0 \). Assume that \( f \) is continuous on \( S \). The flow map is given by \( s: \mathbb{Z}^+ \times S \rightarrow S \) where \( s(0, x_0) = x_0, s(1, x_0) = f(s(0, x_0)) = f(x_0), s(2, x_0) = f(s(1, x_0)) \) and so on. There is an analogue of existence and uniqueness for discrete non linear system ([1], Theorem 13.1).

There are similar definitions on stability and asymptotic stability as we had in the continuous time case. Instead of exponential stability for continuous time systems, we define geometric stability for discrete systems where \( \|x(k)\| \leq \alpha\|x(0)\|\beta^{-k}, \alpha, \beta > 1 \). The discrete time varying systems are of the form

\[
x(k + 1) = f(k, x(k)), \ x(0) = x_0, \ k \in \mathbb{Z}^+.
\tag{81}
\]

There are natural discrete time counterparts of all results stated in the continuous time case. We refer the reader to Chapter 13 of Haddad ([1]) for complete details.
10 Advanced topics

10.1 Partial stability

Partial stability means stability with respect to part of the system's state. The state vector $x$ is divided into two components $x_1$ and $x_2$ and one asks for stability w.r.t. one of the two components. The sufficient stability conditions given in previous sections can be extended to obtain sufficient conditions for partial stability (Haddad [1], Section 4.2).

10.2 Finite time stability

In some applications, one needs to reach the equilibrium point in finite time rather than reaching it asymptotically.

10.3 Semistability

For systems with continuum of equilibria, one need an appropriate notion of stability instead of asymptotic stability.

10.4 Generalized Lyapunov theorems

Here the differentiability criteria is weakened and one only assumes lower semi-continuity of the Lyapunov function to obtain sufficient conditions for stability, asymptotic stability, invariance of sets and so on. Lyapunov function $V$ must be positive definite in the neighborhood of an equilibrium point but it should be decreasing i.e., $V(x(t))$ decreases as $t$ increases. These results allow us to patch different Lyapunov functions over different sets together without being worried about differentiability conditions to obtain sufficient conditions for stability. We refer the reader to Haddad ([1]), Section 4.8.

10.5 Lyapunov and asymptotic stability of sets and periodic orbits

We can define Lyapunov and asymptotic stability of sets by replacing the equilibrium point in the corresponding definitions with appropriate sets say $S_0$. One needs a Lyapunov function $V$ which is zero on the say $S_0$, positive on points in an open set containing $S_0$ which lie outside $S_0$ such that $V(x(t))$ is decreasing as $x(t)$ evolves in time. For extension of the results stated for equilibrium points, we refer the reader to Haddad ([1], 4.9).

Poincare’s theorem provides necessary and sufficient conditions for stability of periodic orbits (Section 4.10, [1]).

10.6 Vector Lyapunov functions

Here one uses vector Lyapunov function which requires less rigid requirements compared to a scalar Lyapunov function. One can obtain sufficient conditions for stability using the vector Lyapunov functions (Section 4.11, [1]).

10.7 Applications

Stability of switched systems, hybrid systems..

11 Appendix

11.1 lpdf, pdf and decrecent functions

The discussion here is from [2].
Lemma 11.1. Suppose $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, non-decreasing and $\phi(0) = 0$, $\phi(r) > 0 \ \forall r > 0$. Then, there is a class $\mathcal{K}$ function $\alpha$ such that $\alpha(r) \leq \phi(r)$, $\forall r$. Moreover, if $\phi(r) \to \infty$, as $r \to \infty$, then $\alpha$ can be chosen to be a class $\mathcal{K}$ function.

Proof. [2], Lemma 1, Section 5.2.

Lemma 11.2. A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is an lpdf $\iff$

- $V(0) = 0$.
- There exists $r > 0$ such that $V(x) > 0$, $\forall x \in B_r \setminus \{0\}$.

$V$ is a pdf $\iff$

- $V(0) = 0$.
- $V(x) > 0$, $\forall x \in \mathbb{R}^n \setminus \{0\}$.
- There exists $r > 0$ such that $\inf_{\|x\| \geq r} V(x) > 0$.

$V$ is radially unbounded $\iff V(x) \to \infty$ as $\|x\| \to \infty$ uniformly in $x$.

Proof. ($\Rightarrow$) Obvious. ($\Leftarrow$) Define $\phi(p) := \inf_{p\leq\|x\|\leq r} V(x)$. Then, $\phi(0) = 0$, $\phi$ is continuous and non-decreasing because as $p$ increases, the infimum is taken over a smaller region. Moreover, $\phi(p) > 0$ when $p > 0$. Now by the previous lemma, one can choose a class $\mathcal{K}$ function $\alpha$ such that $\alpha(\|x\|) \leq \phi(\|x\|) \leq V(x), \forall x \in B_r$.

Necessary conditions for pdf follow from the definition. The sufficient conditions follows from the above arguments (using the third condition). The last statement also follows.

A function $V(x) = \frac{x^2}{1+x^4} > 0$ on $\mathbb{R} \setminus \{0\}$ but it is not a pdf. Note that it violates the third condition in the above lemma. If $V$ is pointwise positive everywhere except the origin and radially unbounded, then $V$ is a pdf.

Lemma 11.3. A continuous function $V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ is an lpdf $\iff$

1. $V(t, 0) = 0$, $\forall t$.
2. There exists an lpdf $W : \mathbb{R}^n \to \mathbb{R}$ and a constant $r > 0$ such that $V(t, x) \geq W(x)$, $\forall t \geq 0, \forall x \in B_r(0)$.

$V$ is a pdf $\iff$

1. $V(t, 0) = 0$, $\forall t$.
2. There exists a pdf $W : \mathbb{R}^n \to \mathbb{R}$ such that $V(t, x) \geq W(x)$, $\forall t \geq 0, \forall x \in \mathbb{R}^n$.

$V$ is radially unbounded $\iff$ there exists a radially unbounded function $W : \mathbb{R}^n \to \mathbb{R}$ such that $V(t, x) \geq W(x)$, $\forall t \geq 0, \forall x \in \mathbb{R}^n$.

Proof. Consider lpdfs. If $V$ is an lpdf, then there exists a class $\mathcal{K}$ function $\alpha$ such that $V(t, x) \geq \alpha(\|x\|)$. Define $W(x) := \alpha(\|x\|)$. Conversely, suppose there exists $W$ satisfying given conditions. Then using the previous lemma, one can show that $V$ is an lpdf.

The necessary conditions for $V$ being pdf follow from the definition. For sufficiency, we can use the previous lemma. The last statement also follows similarly.

Example 11.4. Consider a quadratic function $V(x) = x^TPx$ where $P$ is a square real symmetric matrix. $V$ is pdf $\iff P$ is positive definite. Thus, the term positive definite matrix and positive definite function are consistent. Note that $V$ is radially unbounded.
Example 11.5. A function \( V_1(x_1, x_2) = x_1^2 + x_2^2 \) is an example of a radially unbounded pdf. Another function \( V_2(t, x_1, x_2) = (t + 1)(x_1^2 + x_2^2) \) is a pdf since it dominates time invariant \( V_1 \). A function \( V_3(t, x_1, x_2) = e^{-t}(x_1^2 + x_2^2) \) is not a pdf but it is decrescent.

A function \( W_1(x_1, x_2) = x_1^2 + \sin^2(x_2) \) is an lpdf but not a pdf. A function \( W_2(x_1, x_2) = x_1^2 + \tanh^2(x_2) \) is a pdf but not radially unbounded since, \( \tanh^2 x_2 \to 1 \) as \( |x_2| \to \infty \).

References


