1 Introduction to optimal control

Various optimization problems appear in open and closed loop control, deterministic and stochastic control and estimation theory. Optimal control is intersection of these areas. In general, the objective is to choose an optimal input w.r.t. some performance index which gives a cost function to be optimized subject to some constraints. Optimal control problems are some of the hardest optimization problems. These are infinite dimensional problems as we will see here.

Organization: We start with the topic: calculus of variations to study how to formulate infinite dimensional optimization problems. It turns out that continuous time optimal control problems are infinite dimensional optimization problems and calculus of variation forms a foundation for a mathematical formulation of these problems and gives tools to solve these problems using first and second order conditions. We consider both unconstrained and constrained problems (with equality constraints). In the third section, we list out different types of fixed/free time and endpoint problems for unconstrained case and different types of constraints for the constrained case and give first order necessary conditions. In the fourth section, we study variational approach to optimal control problems and list out certain strengths and limitations of this approach. To overcome the limitations, there is Pontryagin’s maximum principle (PMP) which gives necessary conditions for optimal control. We study its applications in time optimal, fuel optimal and energy optimal control problems. Then we consider dynamic programming and HJB theory which gives necessary and sufficient conditions for optimal control. We look at applications of HJB and dynamic programming in energy and time optimal control problem for continuous and discrete time systems. Finally, we also consider PMP on manifolds and some aspects of $H_{\infty}$ control.

2 Introduction to Calculus of variations

Calculus of variations form a backbone of optimal control theory, specifically, to derive Pontryagin’s maximum principle which gives necessary conditions to solve optimal control problems. For historic importance of this topic, we refer the reader to Liberzon. We mention some classical problems below (Liberzon). The technique of calculus of variations was invented to tackle these problems and is also of a foundational importance in classical and quantum mechanics.

2.1 Examples of variational problems

Control and estimation problems: Consider a single input system $\dot{x} = Ax + bu$ where $u : \mathbb{R} \rightarrow \mathbb{R}$. The space of all functions from $\mathbb{R}$ to $\mathbb{R}$ forms an infinite dimensional vector space. Suppose $x(0) = 0$ and we want to drive the state to $x_f$ at time $t$. Consider the following optimization problem.

$$\begin{align*}
\min_{u} & \quad J(u) = \int_{0}^{t} \|u(\tau)\|^2 d\tau \\
\text{subject to} & \quad \dot{x} = Ax + bu, \quad x_f = e^{At} \int_{0}^{t} e^{-A\tau} Bu(\tau) d\tau.
\end{align*}$$
The objective is to find \( u \) which minimizes the above cost functional subject to the constraint that the state is at \( x_f \) at time \( t \). Consider another problem as follows

\[
\min J(u) = \int_0^t d\tau
\]

subject to \( \dot{x} = A x + b u, \quad x_f = e^{A t} \int_0^t e^{-A \tau} B u(\tau) d\tau. \)

This is a minimum time problem. The objective is to find an input \( u \) which drives the state from the origin to \( x_f \) in minimum time. There are more general LQR and LQG problems as well which can be formulated as infinite dimensional optimization problems. Consider an estimation problem as follows

\[
\min J = \int_0^t \| y(\tau) - C \hat{x}(\tau) \|^2 d\tau
\]

subject to \( \dot{x} = A x + B u, \quad y = C x + n, \)

We want to find the best state estimate \( \hat{x} \) in this problem.

Problems from maths: Consider the shortest path problem. Given two points \( a \) and \( b \) in the given space, find the shortest path lying in the space which connects the two points. Let \( \gamma \) be a parameterized curve joining \( a \) and \( b \). Then \( \dot{\gamma} \) is the velocity vector. The distance between \( a \) and \( b \) along \( \gamma \) is given by \( \int_0^1 \sqrt{\dot{\gamma} \cdot \dot{\gamma}} dt \) where \( ds = \sqrt{\dot{\gamma} \cdot \dot{\gamma}} dt \) is the infinitesimal distance. The optimization problem is to find \( \gamma \) such that \( \int_0^1 \sqrt{\dot{\gamma} \cdot \dot{\gamma}} dt \) is minimized where \( \gamma(t_0) = a \) and \( \gamma(t_1) = b \) i.e.,

\[
\min_{\gamma} \int_0^1 \sqrt{\dot{\gamma} \cdot \dot{\gamma}} dt
\]

subject to \( \gamma(t_0) = a, \gamma(t_1) = b \)

Let \( \mathbb{R}^2 \) be the given space and let \( \gamma(t) = (x(t), y(t)) \) be a parameterized curve. Therefore, \( \dot{\gamma} = \left( \frac{dx}{dt}, \frac{dy}{dt} \right) \). Hence, \( \int_0^1 \sqrt{\dot{\gamma} \cdot \dot{\gamma}} dt = \int_0^1 \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt \). Suppose \( \frac{dx}{dt} \neq 0 \), thus eliminating the variable \( t \), \( \int_0^1 \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx \). Thus, we want to find a function \( y(x) \) such that \( \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx \) is minimum.

Isoperimetric problem involves finding a curve of a given length enclosing the largest area.

\[
\max_y \int_a^b y(x) dx
\]

subject to \( \int_a^b \sqrt{1 + y'(x)^2} dx = l \)

To solve these type of problems, one needs the theory of calculus of variations. The minimum energy problem and the minimum time problem are both constrained optimization problems whereas the shortest distance problem is an unconstrained optimization problem unless the underlying space is a manifold in \( \mathbb{R}^n \). To study calculus of variations, we need to understand function spaces and functionals on function spaces which is the next topic.

### 2.2 Functionals and function spaces

A functional assigns a value (real number) to functions belonging to some class. For example, it could be the length of a curve, surface area, time required to go from one point to another along a curve (defined by a function) joining the two points and so on.
Definition 2.1. Let $\mathbb{V}$ be an $\mathbb{F}$ vector space. A function space is a vector space over a field $\mathbb{F}$ whose elements are $\mathbb{F}$ valued functions on $\mathbb{V}$ or a subset $V_1$ of $\mathbb{V}$.

For example, $\mathbb{F}$ could be $\mathbb{R}$. The set of continuous functions $C^0 : [a, b] \to \mathbb{R}$ forms a function space. $C^0(a, b)$ denotes a space of continuous functions on an interval $[a, b]$. Note that $C^1(a, b)$ denotes a set of differentiable functions on $[a, b]$. Observe that these are infinite dimensional vector spaces. Vector spaces with a norm are called normed vector space (please refer short notes on linear algebra for norms on vector spaces). An example of a norm is as follows: let $\max_{a \leq x \leq b}|y(x)|$ be the maximum value of $y \in C^0(a, b)$. This defines a norm $\| \cdot \|_0$ on $C^0(a, b)$. A norm on $C^1(a, b)$ could be as follows. Let $y \in C^1(a, b)$. Then $\| y \|_1 = \max_{a \leq x \leq b}|y(x)| + \max_{a \leq x \leq b}|y'(x)|$.

We can also have function spaces where an element is a function of several variables e.g., $C^0(\mathbb{R}^n)$ i.e. a vector space of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Definition 2.2. A functional is a map $J$ from a set of function space $\mathcal{F}$ to $\mathbb{R}$. If $\mathcal{F}_1 \subset \mathcal{F}$ is a set. Then, we can also consider restriction of $J$ to the set $\mathcal{F}_1$. This restriction is also a functional on $\mathcal{F}_1$.

For example if $y \in C^0(a, b)$. Then $J(y) = \int_a^b y(x)dx$ is a functional. Suppose $y \in C^1(a, b)$. Then $J(y) = \int_a^b F(x, y(x), y'(x))dx$ is a functional where $y'$ is the derivative of $y$. Note that we can have functionals on a set of functions which is not a vector space for example, a set of curves passing through two fixed points is not a vector space. But the length of these curves defines a functional on this set of curves. If a map $J : \mathcal{F} \to \mathbb{R}$ is linear, then it is said to be a linear functional.

Definition 2.3. Let $\mathcal{F}$ be a normed function space and $J : \mathcal{F} \to \mathbb{R}$ be a functional on $\mathcal{F}$. Then $J$ is said to be continuous at $y \in \mathcal{F}$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\| y - \hat{y} \| < \delta$, then $|J(y) - J(\hat{y})| < \epsilon$.

We are interested in finding a functional from the set of functions which optimizes (maximizes or minimizes) the cost functional $J$ on the set of functions. We want to find the first order necessary conditions to find this optimum/extremum. From now on we will consider only minimization problems as maximizing $J$ is equivalent to minimizing $-J$. Thus, we want to consider the following optimization problem

$$\arg\min_{y \in \mathcal{F}} J(y), \text{ where } J : \mathcal{F} \to \mathbb{R}.$$ (1)

If $y^*$ is a local minimum of $J$ i.e., $y^* = \arg\min_{y \in \mathcal{F}} J(y)$, then there exists an $\epsilon > 0$ such that $J(y^*) \leq J(y)$ for all $\| y^* - y \| < \epsilon$.

2.2.1 Variation of Functionals

First order necessary condition: To find the minimum of a functional we need a notion of a derivative of a functional. For general functionals, the first order necessary condition for a local minimum at $y^*$ is $J(y^*) \leq J(y)$ for all $y$ in some neighbourhood of $y^*$.

Definition 2.4. (Liberzon) The first variation of $J : \mathcal{F}_1 \to \mathbb{R}$ (or the differential of $J$) where $\mathcal{F}_1 \subset \mathcal{F}$ is a linear functional $\delta J|_y : \mathcal{F} \to \mathbb{R}$ is a linear functional on $\mathcal{F}$ such that

$$J(y + \alpha \eta) = J(y) + \alpha \delta J|_y(\eta) + o(\alpha^2)$$ (2)

where $\alpha \in \mathbb{R}$, $\eta \in \mathcal{F}$, $y + \alpha \eta \in \mathcal{F}_1$ and $o(\alpha^2)$ represents higher order terms in $\alpha$ such that as $\alpha \to 0$, $o(\alpha^2) \to 0$ faster than $\alpha \to 0$.

$$\delta J|_y(\eta) = \lim_{\alpha \to 0} \frac{J(y + \alpha \eta) - J(y)}{\alpha}$$ (3)

Suppose $y \in \mathcal{F}_1 \subset \mathcal{F}$. We say that a perturbation $\eta \in \mathcal{F}$ is admissible if $y + \alpha \eta \in \mathcal{F}_1$ for all $\alpha$ close to 0.
This is called Gateaux derivative of $J$ which is the derivative of $J(y + \alpha \eta)$ w.r.t. $\alpha$ for fixed $y$ and $\eta$, evaluated at $\alpha = 0$. The first variation defined above is independent of the choice of norm on $\mathcal{F}$.

**Theorem 2.5.** A necessary condition for the differentiable functional $J(y)$ to have an extremum at $y^*$ is that $\delta J|_{y^*}(\eta) = 0$ for all admissible $\eta$.

**Proof.** Suppose $\delta J|_{y^*}(\eta) \neq 0$ for some admissible $\eta$. Thus,

$$J(y^* + \alpha \eta) - J(y^*) = \alpha \delta J|_{y^*}(\eta) + o(\alpha^2)$$

$$\Rightarrow J(y^* - \alpha \eta) - J(y^*) = -\alpha \delta J|_{y^*}(\eta) + o(\alpha^2).$$

But this contradicts the fact that since $y^*$ is a local minimum, $J(y^*) \leq J(y)$ for all $y$ in some neighbourhood of $y^*$.

**Definition 2.6.** (Gelfand) Note that we can also define the differential of $J$ w.r.t. a norm on $\mathcal{F}$. Let $\delta y$ be a variation (or a perturbation) in $y$. The corresponding variation in $J$ will be

$$\Delta J(\delta y) := J(y + \delta y) - J(y)$$

$$= \delta J|_{y}(\delta y) + o(\|\delta y\|^2)$$

where $\delta J|_{y}$ is the first variation of $J$ w.r.t. $\|\|$. This is called Fréchet derivative of $J$.

1 This is a stronger differentiability notion than the Gateaux derivative and allows more general perturbations.

**Weak and strong extremum:** Note that $C^1 \subset C$. Thus an element $y \in C^1$ can be regarded as an element of $C^1$ or an element of $C$. Therefore, we can have two types of extrema. If there exists an $\epsilon > 0$ such that for $\|y^* - \hat{y}\|_1 < \epsilon$, $J(y^*) \leq J(\hat{y})$, then we say $y^*$ is a weak minimum (extremum). On the other hand, if there exist an $\epsilon > 0$ such that for $\|y^* - \hat{y}\|_0 < \epsilon$, $J(y^*) \leq J(\hat{y})$, then we say $y^*$ is a strong minimum (extremum). It turns out that every strong minimum is a weak minimum. Observe that $\|y - \hat{y}\|_1 < \epsilon$ implies $\|y - \hat{y}\|_0 < \epsilon$. Therefore, if $J(y)$ is minimum w.r.t. all $\hat{y}$ such that $\|y - \hat{y}\|_0 < \epsilon$, then $J(y)$ is minimum w.r.t. all $\hat{y}$ such that $\|y - \hat{y}\|_1 < \epsilon$. Finding a weak minimum is easier than finding a strong minimum because the functionals usually considered in calculus of variations are continuous w.r.t. $\|\|_1$ norm on $C^1$ but not w.r.t. $\|\|_0$ norm on $C$ (Gelfand). Note that if we allow perturbations to lie in $C^1$, then the class of allowable perturbations are very restrictive as only differentiable perturbations are admissible. On the other hand if we allow perturbations to lie in $C$, then we have a richer class of admissible perturbations and minimum over this class gives a strong minimum. It is natural that first we look for tools to obtain weak minima and then proceed to find tools for obtaining strong minima.

**Remark 2.7.** One can also formulate Theorem 2.5 using the notion of the Fréchet derivative i.e., the first order necessary condition for an extremum is $\delta J|_{y}(\delta y) = 0$ for all admissible $\delta y$. Since we need differentiability of the functional w.r.t. the underlying norm, this gives a necessary condition for a weak extremum.

**Second order necessary condition (Gateaux derivative):** A bilinear form $B$ on a vector space $V$ is a function $B : V \times V \rightarrow F$ where $F$ is a field such that $B(v, w) \in F$ is linear in both variables $v$ and $w$. An example of a bilinear form is an inner product on a vector space (refer linear algebra notes). If we substitute $v = w$, then $Q(v) = B(v, v)$ gives a quadratic form on $V$. An example of a quadratic form is the Euclidean norm.

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1For finite dimensional vector spaces e.g., $\mathbb{R}^n$, all norms are equivalent. Therefore, the notion of derivative does not change according to the norm. This is not the case for function spaces as all norms are not equivalent in function spaces.
A quadratic form $\delta^2 J|_y : \mathcal{F} \to \mathbb{R}$ is called the second variation of $J$ at $y$ if for all $\eta \in \mathcal{F}$ and all $\alpha \in \mathbb{R}$,

$$J(y + \alpha \eta) = J(y) + \alpha \delta J|_y(\eta) + \alpha^2 \delta^2 J|_y(\eta) + o(\alpha^2).$$

The second order necessary condition for a local minimum $y^*$ states that $\delta^2 J|_{y^*}(\eta) \geq 0$.

### 2.3 Basic problems and Euler-Lagrange equation for weak extrema

Consider a function $L : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with continuous first and second partial derivatives w.r.t. all its arguments. Then among all functions $y$ of a single variable $x$ for $a \leq x \leq b$ and satisfy given boundary conditions $y(a) = y_0$ and $y(b) = y_1$, find the function $y$ for which 2

$$J(y) = \int_a^b L(x, y, y') dx$$

has a weak extremum. The function $L$ is called Lagrangian and it determines the running cost whereas $J$ is the total cost over an interval $[a, b]$. Note that in general we may have a Lagrangian function of the form $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ where $y : [a, b] \to \mathbb{R}^n$ are curves in $n$ dimensional space.

To derive Euler-Lagrange equations, we need some results which are described below.

**Lemma 2.8.** *(Gelfand)* If $\alpha$ is continuous in $[a, b]$ and

$$\int_a^b \alpha(x)h(x)dx = 0$$

for every $h \in C^0(a, b)$ such that $h(a) = h(b) = 0$, then $\alpha(x) = 0$ for all $x \in [a, b]$.

**Proof.** Suppose $\alpha \neq 0$ on $[a, b]$. By continuity of $\alpha$, we may assume w.l.o.g. that $\alpha$ is positive on $[x_1, x_2] \subset [a, b]$. Choose $h(x) = (x - x_1)(x_2 - x)$ and $h(x) = 0$ if $x$ lies outside $[x_1, x_2]$. But then, $\int_a^b \alpha(x)h(x)dx = \int_{x_1}^{x_2} \alpha(x)h(x)dx > 0$ a contradiction. Therefore, $h = 0$. \qed

**Lemma 2.9.** *(Gelfand)* If $\alpha$ is continuous in $[a, b]$ and

$$\int_a^b \alpha(x)h'(x)dx = 0$$

for every $h \in C^1(a, b)$ such that $h(a) = h(b) = 0$, then $\alpha(x) = c$ for all $x \in [a, b]$ where $c$ is a constant.

**Proof.** Let $c$ be a constant such that $\int_a^b [\alpha(x) - c]dx = 0$ and let $h(x) = \int_a^x [\alpha(\xi) - c]d\xi$. Thus, $h \in C^1(a, b)$ and $h(a) = h(b) = 0$. Note that

$$\int_a^b [\alpha(x) - c]h'(x)dx = \int_a^b \alpha(x)h'(x)dx - c[h(a) - h(b)] = 0.$$

But since $h'(x) = \alpha(x) - c$, $\int_a^b [\alpha(x) - c]h'(x)dx = \int_a^b [\alpha(x) - c]^2 dx$. Thus, $\alpha(x) = c$ for all $x \in [a, b]$. \qed

**Lemma 2.10.** *(Gelfand)* If $\alpha$ and $\beta$ are continuous in $[a, b]$ and

$$\int_a^b [\alpha(x)h(x) + \beta(x)h'(x)]dx = 0$$

for every $h \in C^1(a, b)$ such that $h(a) = h(b) = 0$, then $\beta$ is differentiable and $\beta'(x) = \alpha(x)$ for all $x \in [a, b]$.

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2Note that $L$ is a function on function spaces. However, while talking about its partial derivatives w.r.t. its arguments, the arguments can be considered as independent variables. Therefore, one can always use Taylor’s expansion when $L$ is differentiable in a neighbourhood of a point.
In this case the Euler-Lagrange equations are:

\[ \int_a^b \frac{d}{dx}(A(x)h(x))\,dx = \int_a^b A'(x)h(x)\,dx + \int_a^b A(x)h'(x)\,dx. \]

Note that \( \int_a^b \frac{d}{dx}(A(x)h(x))\,dx = A(b)h(b) - A(a)h(a) = 0 \) since \( h(a) = h(b) = 0 \). Thus, \( \int_a^b A'(x)h(x)\,dx = -\int_a^b A(x)h'(x)\,dx \). Since \( A'(x) = \alpha(x) \), \( \int_a^b \alpha(x)h(x)\,dx = -\int_a^b A(x)h'(x)\,dx \). Substituting this in the equation given in the statement of the lemma,

\[ \int_a^b [-A(x) + \beta(x)]h'(x)\,dx = 0. \]

Then, by the previous lemma, \( \beta(x) - A(x) = c. \) Therefore, by the definition of \( A(x) \), \( \beta'(x) = \alpha(x) \).

**Proof.** Let \( A(x) = \int_a^x \alpha(\xi)\,d\xi \). Then \( \int_a^b \frac{d}{dx}(A(x)h(x))\,dx = \int_a^b A'(x)h(x)\,dx + \int_a^b A(x)h'(x)\,dx \). Note that \( \int_a^b \frac{d}{dx}(A(x)h(x))\,dx = A(b)h(b) - A(a)h(a) = 0 \) since \( h(a) = h(b) = 0 \). Thus, \( \int_a^b A'(x)h(x)\,dx = -\int_a^b A(x)h'(x)\,dx \). Since \( A'(x) = \alpha(x) \), \( \int_a^b \alpha(x)h(x)\,dx = -\int_a^b A(x)h'(x)\,dx \). Substituting this in the equation given in the statement of the lemma,

\[ \int_a^b [-A(x) + \beta(x)]h'(x)\,dx = 0. \]

Then, by the previous lemma, \( \beta(x) - A(x) = c. \) Therefore, by the definition of \( A(x) \), \( \beta'(x) = \alpha(x) \).

**Theorem 2.11.** Let \( J(y) = \int_a^b L(x, y, y')\,dx \) be a functional defined on the set of functions \( y \in \mathcal{C}^1(a, b) \) such that \( y(a) = y_0 \) and \( y(b) = y_1 \). Then a necessary condition for \( J(y) \) to have an extremum for a given function \( y \) is that \( y \) satisfies the Euler-Lagrange equation

\[ \frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} = 0. \tag{7} \]

**Proof.** Suppose \( y \) is an extremum. Consider admissible perturbations \( y + \delta y \) such that \( \delta y(a) = \delta y(b) = 0 \). Therefore,

\[ \Delta J = J(y + \delta y) - J(y) = \int_a^b L(x, y + \delta y, y' + \delta y')\,dx - \int_a^b L(x, y, y')\,dx \]

\[ = \int_a^b \left[ \frac{\partial L(x, y, y')}{\partial y} \delta y + \frac{\partial L(x, y, y')}{\partial y'} \delta y' \right]\,dx + h.o.t. \]

Therefore, the first variation \( \delta J|_y = \int_a^b \frac{\partial L(x, y, y')}{\partial y} \delta y + \frac{\partial L(x, y, y')}{\partial y'} \delta y'|\,dx \). From Theorem 2.5, \( \delta J|_y = \int_a^b \frac{\partial L(x, y, y')}{\partial y} \delta y + \frac{\partial L(x, y, y')}{\partial y'} \delta y'|\,dx = 0 \). Now from Lemma 2.10, \( \frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} = 0 \).

**Note that we also denote a partial derivative of \( L \) w.r.t. \( x, y \) and \( y' \) as \( L_x, L_y \) and \( L_{y'} \) respectively.** The integral curve of Euler-Lagrange differential equation are called extremals. **Variable end point problem:** Consider the following problem. Among all curves whose end points lie on two vertical lines \( x = a \) and \( x = b \), find the curve for which the functional \( J(y) = \int_a^b L(x, y, y')\,dx \) has an extremum. By the same computation done in the proof of Theorem 2.11,

\[ \delta J|_y = \int_a^b (L_y \delta y + L_{y'} \delta y')\,dx = \int_a^b (L_y - \frac{d}{dx} L_{y'}) \delta y(x)\,dx + L_{y'} \delta y(x)|_{x=a}^{x=b}. \]

Consider a perturbation \( \delta y \) such that \( \delta y(a) = \delta y(b) = 0 \). Therefore, the first order necessary condition \( \delta J|_y = 0 \) would imply that \( L_y - \frac{d}{dx} L_{y'} = 0 \). But this along with \( \delta J|_y = 0 \) implies that \( L_{y'} \delta y(x)|_{x=a}^{x=b} = L_{y'}|_{x=b} \delta y(b) - L_{y'}|_{x=a} \delta y(a) = 0 \). Since \( \delta y \) is arbitrary, \( L_{y'}|_{x=b} = 0 \) and \( L_{y'}|_{x=a} = 0 \). These are sometimes called natural boundary conditions.

**Multiple degrees of freedom:** Suppose that a Lagrangian function \( L \) has several degrees of freedom i.e.

\[ J(y_1, \ldots, y_n) = \int_a^b L(x, y_1, \ldots, y_n, y'_1, \ldots, y'_n)\,dx, \text{ where } y_i(a) = y_{0i}, y_i(b) = y_{1i}. \tag{8} \]

In this case the Euler-Lagrange equations are: \( \frac{\partial L}{\partial y_i} - \frac{d}{dx} \frac{\partial L}{\partial y'_i} = 0 \) for \( 1 \leq i \leq n \) which give a first order necessary conditions for a weak extrema. For variable endpoint problems with several variables, we get \( L_{y'_i}|_{x=b} = 0 \) and \( L_{y'_i}|_{x=a} = 0 \) for \( 1 \leq i \leq n \).
Special types of Lagrangians: Note that using chain rule on the rhs of the Euler-Lagrange equation, \(^3\)

\[
L_y = \frac{d}{dx} L_{y'} = L_{y''} + L_{y' y'} y' + L_{y''} y''
\]

- Suppose \(L = L(x, y')\) i.e. there is no dependence on \(y\). Thus \(L_y = 0\). Hence, \(\frac{d}{dx} L_{y'} = 0\). This implies that \(L_{y'} = c\) for some \(c \in \mathbb{R}\). We will see soon that the quantity \(L_{y'}\) can be identified with the momentum when \(L\) is constructed for problems from mechanics. Furthermore, \(L_{y'}\) is a function in \(y'\) and \(x\). Integrating \(y' = f(x, c)\), we obtain curves \(y(x)\).

- Suppose \(L = L(y, y')\) i.e. there is no dependence on \(x\). Therefore, from Equation (9), \(L_y = L_y y' + L_{y'} y''\). Multiplying by \(y'\),

\[
-L_y y' + L_{y'} (y')^2 + L_{y''} y' y'' = 0
\]

Note that \(\frac{d}{dx}(L_y y' - L) = L_{y''} y' + L_{y'} (y')^2 + L_{y''} y' y'' - L_y y' - L_y y'' = L_{y''} (y')^2 + L_{y'} y'' - L_y y'\) since there is no dependence on \(x\). Therefore, from Equation (10), \(\frac{d}{dx}(L_y y' - L) = 0\). The quantity \(L_y y' - L\) is called the Hamiltonian and \(\frac{d}{dx}(L_y y' - L) = 0\) implies that it remains constant throughout. We will see later that this quantity is actually the energy for physical systems. Again, using \(L_y y' - L = c\), one can obtain \(y' = f(y, d)\). Using the method of separation of variables, one obtains \(y\) by integration.

- Suppose \(L = L(x, y)\) i.e. there is no dependence on \(y'\). Thus, \(L_{y'} = 0 \Rightarrow L_y = 0\). Observe that \(L_y = 0\) involves only algebraic equations in \(y\) and \(x\) and after solving them, one obtains \(y(x)\).

Example 2.12. Consider the third case. If \(L\) is independent of \(y'\), then the Euler-Lagrange equation gives a constraint \(L_y = 0\). Suppose \(L = y\) and we want to minimize \(\int_a^b y dx\). Observe that from Euler-Lagrange equations, we get a constraint \(L_y = \frac{d}{dx} L_y = 0\), which does not make sense. This is because the solutions become unbounded. Note that solutions \(y = -k (k \in \mathbb{R}^+)\) form an unbounded set and there is no unique minimizer. Therefore, Euler-Lagrange equation has no solution in this case.

Consider \(L(x, y) = y^2 x - y^2 x^2\). Thus, we obtain \(L_y = 4y^2 x - 2y x^2 = 0\) from Euler-Lagrange equation. Therefore, the solution is \(y^2 = \frac{1}{2} x\). Thus, in general, due to \(L\) being independent of \(y'\), we get an algebraic constraint that \(L_y\) must be equal to zero. It is given that the Lagrangian is independent of \(y'\), so it could be a general function of \(x\) and \(y\). Due to Euler-Lagrange, we get \(L_y = 0\) and it may or may not have a solution depending on the problem.

Applications: The shortest path problem, Brachistochrone problem, minimum surface area of revolution problem can be solved by finding solutions of Euler-Lagrange equations. For constrained optimization problems, we construct augmented Lagrangians involving constraints and apply Euler-Lagrange equation for the augmented Lagrangian.

Example 2.13. Shortest path problem: Applying Euler-Lagrange equation to the Lagrangian of the shortest path problem, one obtains

\[
\frac{d}{dx} \left( \frac{y'}{\sqrt{1 + (y')^2}} \right) = 0 \Rightarrow \frac{y'}{\sqrt{1 + (y')^2}} = c
\]

Solving for \(y'\), one obtains \(y' = m\) where \(m = \frac{c}{\sqrt{1 - c^2}}\). Integrating \(y' = m\), one obtains an equation of a straight line.

\(^3\)It would seem that we need \(L\) and \(y\) both to lie in \(\mathbb{R}^2\) to satisfy the second order differential equation. We refer the reader to [2], 2.3.3, page 40. Using integration by parts, one can obtain an integral form of Euler-Lagrange equation and there is no need to put an extra differentiability assumption on \(L\) or \(y\).
**Example 2.14.** Brachistochrome: Let $A$ denote the origin. Suppose an object of mass $m$ is allowed to fall along a curve joining points $A$ and $B$ under the effect of gravity. Find the curve along which the time required to go from $A$ to $B$ is minimum.

Let $y(x)$ be a curve joining $A$ and $B$. We want to minimize $\int_0^1 dt$. Note that $ds = v dt$ where $ds$ is the infinitesimal distance travelled in time $dt$ along the curve. At point $A$, both kinetic and potential energy are zero i.e., $T = U = 0$. By the law of conservation of energy, at any point along the curve, $T + U = 0$. Therefore, $\frac{1}{2}mv^2 + mgy(x) = 0 \Rightarrow v = \sqrt{-2gy(x)}$. Thus, the cost function becomes

$$\int_A^B \frac{ds}{\sqrt{-2gy(x)}} = \int_A^B \frac{\sqrt{1 + y'(x)^2}}{\sqrt{-2gy(x)}} dx$$

Now applying Euler-Lagrange equation to the Lagrangian and solving the corresponding equation, one obtains the required curve (cycloid). (If we take the downward direction of $y$ as the positive direction, then we obtain $\frac{1}{2}mv^2 - mgy(x) = 0 \Rightarrow v = \sqrt{2gy(x)}$.)

**Functions of several variables:** In this case, we have several independent variables $x_1, \ldots, x_n$ rather than just one independent variable $x$. The Euler-Lagrange equation for this case involves partial derivatives $\frac{\partial}{\partial x_i}$ in place of $\frac{d}{dx}$. We refer the reader to Gelfand and Fomin for details.

**Remark 2.15.** Note that if the functionals are convex (e.g., norm functionals, they are convex due to triangle inequality property of norms), then the first order conditions are also sufficient for a minima and it is a global minima.

### 2.3.1 Variational derivative and invariance of Euler-Lagrange equations under new co-ordinates

It turns out that the variational derivative of $J$ w.r.t. $y$ is given by $\delta J = L_y(x, y, y') - \frac{d}{dx}L'_y(x, y, y')$. Moreover, Euler-Lagrange equations remain invariant under change of co-ordinates. For details, we refer the reader to Gelfand.

### 2.4 Hamiltonian and the principle of least action

The origin of Lagrangian and Hamiltonian lie in classical mechanics. This was an abstract approach to obtain system of equations for a dynamical systems from which Newton’s laws of motion follow as a consequence. The advantage of this approach is that it is more general and it works for a much wider class of dynamical systems rather than just mechanical systems. The Lagrangian, Hamiltonian and the principle of least action lie at the root of understanding classical and quantum mechanics. It also has a natural extension to optimal control problems as we will see.

For mechanical systems, we define the Lagrangian as $L(q, \dot{q}) = T(\dot{q}) - U(q)$ where $q$ is the position and $\dot{q}$ is the velocity i.e. time derivative of $q$. For non conservative systems where there are losses due to friction, we also incorporate a dissipation term in the Lagrangian. $L$ may be time dependent or independent according to the underlying system. Using this Lagrangian, one can define a cost function $J = \int_a^b L(q, \dot{q}) dt$. It turns out that the trajectories of a system are extremals of the cost function $J$ hence, are given as solution trajectories of Euler-Lagrange equations $\frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$.

### 2.4.1 Hamilton’s equations

We define the momentum and the Hamiltonian as follows:

$$p := L_y'(x, y, y')$$

$$H(x, y, y', p) := p y' - L(x, y, y')$$
We treat \( y \) and \( p \) as canonical variables which depend only on \( x \). Note that since by definition, \( p = L_y'(x, y, y') \) and by the definition of partial derivatives, \( y \) is kept constant while taking a partial derivative w.r.t. \( y' \) (\( x \) too if there is an explicit dependence of \( L \) on \( x \)). Therefore, \( p \) is a function of \( y' \) and \( y' \) is a function of \( x \). Hence, \( p \) is also an implicit function of \( x \) but \( p \) is independent of \( y \).

Observe that \( H_p = y' \) and \( H_y = -L_y = -\frac{\partial}{\partial x} L_{y'} = -p' \) (since \( p \) is independent of \( y \), \( \frac{\partial f}{\partial y} = 0 \)).

2.4.2 The principle of least action

From now onwards, we will use \( t \) instead of \( x \) as our independent variable. Earlier, we used \( x \) as an independent variable because calculus of variations works for more general math problems which are not time dependent (e.g., least area and so on).

The principle of least action states that the trajectories of a mechanical system (or any dynamical system) are extremals of the functional \( \int_{t_0}^{t_1} (T - U) dt \) where \( T \) is the kinetic energy and \( U \) is the potential energy. The integral is called action integral. Note that \( q \) denotes position variable and \( \dot{q} \) denotes velocity. The Lagrangian \( L(t, q, \dot{q}) = T - U = \frac{1}{2} m (\ddot{q} \ddot{q}) - U(q) \). For multiple degrees of freedom \( q \) is a vector. Note that \( H = \dot{q} L - \frac{m (\dot{q} \dot{q})}{2} + U(q) = \frac{1}{2} m (\dot{q} \dot{q}) + U(q) = E \) which is the total energy.

Newton’s laws of motion, Euler-Lagrange equations and Hamilton’s equations: It turns out that Newton’s laws of motion follow as a consequence of Euler-Lagrange equations. From the principle of least action, one obtains \( \frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \). Note that \( \frac{\partial L}{\partial \dot{q}} = -\nabla U(q) = F \) where \( F \) is the force which is negative gradient of potential energy; and \( \frac{d}{dt} \frac{\partial L}{\partial q} = \frac{d}{dt} m \ddot{q} = m \dddot{q} \). Thus, we get \( F = ma \) which is Newton’s 2nd law of motion.

The principle of least action is more general than Newton’s laws of motion. It is applicable to all dynamical systems. For electrical systems, one can define kinetic energy using the rate of flow of charge and potential energy using charges in the system. Therefore, one can define a Lagrangian and obtain electrical laws by applying Euler-Lagrange equations and so on. Thus, for any mechanical, electrical, electro-mechanical or any dynamical system, equations of dynamics are obtained using Euler-Lagrange equations. For non-conservative or dissipative systems i.e., systems where there is friction or resistive elements or magnetic fields, one can include an appropriate dissipation function in the Lagrangian and obtain equations of motion. Action principle also works for quantum mechanical systems.

Note further that the Hamiltonian \( H = p^T \dot{q} - L(q, \dot{q}) \) where \( p = \frac{\partial L}{\partial \dot{q}} \). Therefore, the Hamilton’s equations become

\[
H_p = \dot{q}, \ H_q = -\frac{\partial L}{\partial \dot{q}} = -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = -\frac{d}{dt} p = -\dot{p}.
\]

Thus, we have the following first order equations of motion

\[
\dot{q} = H_p, \ \dot{p} = -H_q
\]

where \( q \) and \( p \) are position and momentum variables respectively. One can check that for mechanical systems, \( p = \frac{\partial L}{\partial \dot{q}} = m \ddot{q} \) which is indeed momentum. Observe that Hamilton’s equations also give Newton’s laws. Recall that the Hamiltonian \( H = T + U \) represents the total energy. The equation \( \dot{p} = -H_q \) indicates that the rate of change of momentum is equal to \(-H_q \) which is \(-\nabla_q U(q) = F \), the force. Thus, we obtain \( F = ma \).

Observe that \( p = \frac{\partial L}{\partial \dot{q}} = m \ddot{q} \), hence \( \dot{q} = \frac{p}{m} \). Writing \( H \) w.r.t. the new variables \( q, p \), we get \( H = \frac{1}{2} m \left( \frac{p}{m} \right)^2 + U(q) = \frac{1}{2} m p^2 + U(q) \). Now the first equation of Hamilton says that \( \dot{q} = H_p = \frac{p}{m} \) which is just a restatement of \( \dot{q} = m \ddot{q} \).

It is clear that Euler-Lagrange equations and Hamilton’s equations are be obtained from one another by change of variables and hence, are equivalent. Newton’s laws follow as a consequence.
Conservation of energy, momentum, angular momentum: Consider conservative systems where the Lagrangian $L$ does not depend explicitly on time. Therefore, the Hamiltonian also does not depend explicitly on time but there is an implicit dependence via $q$ and $p$. Therefore,

$$\frac{d}{dt}H = \frac{\partial H}{\partial q} \frac{dq}{dt} + \frac{\partial H}{\partial p} \frac{dp}{dt}.$$ 

Using Hamilton’s equations, we get $\dot{H} = H_q(H_p) + H_p(-H_q) = 0$ i.e., $H$ is constant along system trajectories which implies that energy is constant along system trajectories for conservative systems. This is also called time translational symmetry. (This is a consequence of a theorem of Noether that symmetry implies conservation laws which is one of the basic fundamental theorems of classical mechanics.)

Consider a mechanical system where there is no external force acting on the system. Therefore, $F = -\nabla_q U(q) = 0$ i.e., the potential energy is constant and independent of $q$. We may assume that the potential energy is zero as adding a constant does not affect the action. Now $\dot{p} = H_q = 0$ since $H$ is independent of $q$. This says that the rate of change of momentum is zero in the absence of an external force. Thus momentum is conserved if there is a translational symmetry w.r.t. space. Likewise for rotational motions, in the absence of external torques, angular momentum is conserved due to rotational symmetry.

2.5 Variational problems with constraints

We now assume that there are some constraints imposed on the admissible curves. We will see that we can use Lagrange multipliers just as in case of finite dimensional static optimization problems to solve variation problems with constraints.

**Integral constraints:** Consider a constraint on $y$ of the following form:

$$C(y) := \int_a^b M(x,y,y')dx = C_0.$$  \hfill (13)

**Non integral constraints:** These constraints could be holonomic $M(x,y) = 0$ or non-holonomic $M(x,y,y') = 0$ (Liberzon).

We give first order necessary conditions for both integral and non-integral constraints in Section 3.2. Just like we saw in the case of finite dimensional optimization problems, the first order conditions are obtained using Lagrange multipliers. We However, avoid the technical details regarding Lagrange multipliers for infinite dimensional optimization (e.g., geometric interpretations of finite dimensional optimization and so on) and take it for granted that it works in the same spirit as the finite dimensional case. Some discussion about this can be found in Liberzon (Chapter 2).

**An example with inequality constraints:** We give below an example of a problem with inequality constraints.

**Example 2.16** (Roughan). (Obstacle avoidance problem) Consider a problem of finding the shortest path between two points subject to inequality constraints. Consider two points $(-b,0)$ and $(b,0)$ in the $x-y$ plane ($b > 0$). Suppose there is a circular lake of radius $a$, $a < b$ centered at the origin. The problem is to find the shortest path from $(-b,0)$ to $(b,0)$ which avoids the lake completely.

Let $f(x,y,y')$ denote the running cost without explicitly writing down its expression (i.e., $f(x,y,y') = \sqrt{1 + (y')^2}$). Therefore, the problem is as follows:

$$\min \int_b^b f(x,y,y')dx$$

subject to $y(x) \geq g(x)$.  

$$,$$
Let \( z(x) \) be a slack function such that \( y(x) = g(x) + z(x)^2 \). Hence, \( y' = g' + 2zz' \). Substituting this constraint in the previous form, we get

\[
\min \int_{-b}^{b} f(x, z^2 + g, 2zz' + g')dx
\]

From Euler-Lagrange equations, we get \( \frac{d}{dx} \frac{\partial f}{\partial z'} - \frac{\partial f}{\partial y} = 0 \). By chain-rule,

\[
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial z} = \frac{\partial f}{\partial y} 2z + \frac{\partial f}{\partial y'} 2z',
\]

\[
\frac{\partial f}{\partial z'} = \frac{\partial f}{\partial y} \frac{\partial y}{\partial z'} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial z'} = \frac{\partial f}{\partial y'} 2z.
\]

Substituting above equations in Euler-Lagrange equation,

\[
\frac{d}{dx} \left( \frac{\partial f}{\partial y'} 2z \right) = \frac{\partial f}{\partial y} 2z + \frac{\partial f}{\partial y'} 2z' \Rightarrow 2z', \frac{\partial f}{\partial y'} + 2z \frac{d}{dx} \frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y} 2z + \frac{\partial f}{\partial y'} 2z'.
\]

\[
\Rightarrow z \left[ \frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} \right] = 0. \tag{14}
\]

Thus, extremals are given by \( z(x) = 0 \) and solutions of \( \frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \). But we know that solutions of \( \frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \) when \( f = \sqrt{1 + (y')^2} \) are straight lines and \( z(x) = 0 \) implies that we are on the constraint curve \( g(x) \). One needs to construct the shortest path by using these two curves.

Consider the following path. Draw a straight line from the point \((-b,0)\) to the curve \( g(x) \) (i.e., the circle) meeting the circle at point \( P_1 = (-x_1, y_1) \). Then traverse along the circle up to point \( P_2 = (x_2, y_2) \) and draw another straight line from \( P_2 \) to \( (b,0) \). Let \( (x_1, y_1) = (a \cos (\theta), a \sin (\theta)) \). The total distance \( d(\theta) = 2\sqrt{(b-a \cos (\theta))^2 + a \sin^2 (\theta)} + a (\pi - 2\theta) \). To find the minimum distance, use \( d'(\theta) = 0 \). From this, one obtains \( \cos (\theta) = \frac{a}{b} \). Thus, we can find the location of \( P_1 \) and \( P_2 \) on the circle and the straight lines joining \((-b,0) \) to \( P_1 \) and \((b,0) \) to \( P_2 \) along with the circular arc forms the shortest path avoiding the obstacle. One can check that at point \( P_1 \), the straight line joining the starting point and \( P_1 \) is tangential to the circle.

### 2.6 Second order conditions for a weak extrema

The necessary second order conditions for weak minima require that the first variation must vanish and the second variation is positive semi-definite. For sufficient conditions for weak extrema, we refer the reader to Gelfand and Fomin.

### 2.7 Necessary conditions for strong extrema

(Weierstrass-Erdmann corner conditions and Weierstrass excess function. Liberzon, Gelfand, Roughan) Suppose we have a weak minima. To check whether it is a strong minima, we need to allow perturbations which are continuous but may not be differentiable. Such class of perturbations are given by piecewise differentiable variations which are differentiable everywhere except finitely many points. It is clear that if a curve is a weak minima of some cost functional, then for differentiable variations of that curve, the cost increases. Suppose we consider piecewise differentiable variations i.e., variations with finitely many corner points. Observe that such piecewise differentiable variations can be approximated arbitrarily closely by differentiable variations. Therefore, if it is not a strong minima, i.e., there exists a piecewise differentiable variation for which the cost decreases; then there must a differentiable variation for which the cost decreases which is not possible. The same argument can be applied when there are finitely many jump discontinuities as functions with jump discontinuities can be approximated arbitrarily closely by differentiable functions. Thus, if a weak extrema exists, then it is also a strong extrema.
However, we may have a situation where the weak extrema does not exist but a strong extrema exists. Consider the following example where there is no weak minima.

**Example 2.17** (Liberzon). Let

\[ J(y) = \int_{-1}^{1} y^2 (y' - 1)^2 \, dx \quad \text{subject to boundary conditions} \quad y(-1) = 0, y(1) = 1. \]

Clearly, \( J(y) \geq 0 \). There are \( C^1 \) curves giving cost arbitrarily close to zero but not equal to zero. But the curve

\[
y(x) = 0, \quad -1 \leq x \leq 0 \\
y(x) = x, \quad 0 \leq x \leq 1
\]

gives \( J(y) = 0 \). Observe that this curve has a corner at the origin. It forms a strong minima.

Observe that if a strong extremum is \( C^1 \), then it is also a weak extremum. If a strong extremum is piecewise \( C^1 \), then it is not a weak extremum. But we can generalize the definition of weak norm to allow piecewise \( C^1 \) functions:

\[
\|y\|_1 := \max_{a \leq x \leq b} |y(x)| + \max_{a \leq x \leq b} \{|y'(x^-)|, |y'(x^+)|\}
\]

where \( |y'(x^-)| \) and \( |y'(x^+)| \) denote the left and right derivative respectively at \( x \). With this modified definition, the strong extrema is also a weak extrema. Therefore, for piecewise differentiable functions, weak and strong minima are the same. The necessary conditions for \( y \) to be a strong extremum are that \( L_y y' \) and \( y' L_y y' - L \) are both continuous at corner points. These are Weierstrass-Erdmann corner conditions for a strong extrema. To obtain a strong extrema, one needs to patch up \( C^1 \) extremal curves at corner points making sure that the first and the last curve in the patched up solution satisfy the boundary conditions.

Weierstrass excess function provides necessary conditions for a piecewise \( C^1 \) curve to be a strong minima (not just a strong extrema). (Liberzon, Gelfand).

### 2.8 Sufficient conditions for strong extrema

We refer the reader to Gelfand and Fomin for sufficient conditions for strong extrema. This requires Weierstrass excess function and the concept of fields.

### 2.9 Summary

- We saw some examples in geometry, physics and engineering where the infinite dimensional optimization problems naturally appear.

- We studied how to formulate an infinite dimensional optimization problem. The search space here is the space of functions. The objective is to find an optimum in this function space. To check if a candidate is an optimum, we need to evaluate the cost function at points in its neighbourhood. Thus, we need a notion of norm on function spaces. This leads to weak and strong minima depending upon the type of the function space one has and the norm on that function space as well.

- We studied variations in function spaces and calculus of variations to obtain first and second order conditions for optimality. The first order conditions lead to Euler-Lagrange equation which is a necessary condition for weak extrema.

- We studied Hamilton’s formalism and the principle of least action and their interrelationship with Euler-Lagrange equation and Newton’s laws.
• We studied how weak and strong minima are related and the necessary conditions for strong extrema (Weierstrass-Erdmann)/minima (Weierstrass excess function).

• There are separate second order necessary and second order sufficient conditions as well.

3 Variational problems with time as an independent variable and the first order necessary conditions for optimality

In here, we describe in short the types of problems with time as an independent variable. The discussion below is borrowed mostly from Kirk. First we mention the following four types where we obtain first order necessary conditions for a weak extrema. In the next section, we will encounter generalizations of these types as well as relations between them when studied as optimal control problems.

3.1 Functionals without constraints

The four types of problems are listed below.

1. Fixed time and fixed end points: This is the simplest type of problem where the initial time $t_0$ and final time $t_f$ are fixed along with the boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_f$. The cost functional is $J(x) = \int_{t_0}^{t_f} L(t, x, \dot{x}) dt$ and the problem is to find $x$ which is an extremum of $J$ with the specified boundary conditions. Observe that

$$\Delta J = J(x + \delta x) - J(x) = \int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} + h.o.t. \right] dt$$

$$\Rightarrow \delta J = \int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right] dt$$

Now from the first order necessary condition $\delta J = 0$ and Lemma 2.10, we get the Euler-Lagrange equation for an extrema $x^*$: $\frac{\partial L}{\partial x}|_{x^*} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}|_{x^*} = 0$.

2. Final time $t_f$ specified, endpoint $x_f$ free: In problem, we again obtain

$$\delta J = \int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right] dt$$

integrating by parts, $\frac{\partial L}{\partial x} \delta x \bigg|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left[ \frac{\partial L}{\partial x} \delta x - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \delta x \right] dt$

From the optimality conditions, the first variation $\delta J|_{x^*} = 0$ where $x^*$ is the optimal solution. Now, $\delta x(t_0) = 0$ and $\delta x(t_f)$ is arbitrary. Choosing $\delta x(t_f) = 0$, we get Euler-Lagrange equation $\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$, and $x^*$ must satisfy this equation since it is an optimal solution. Now consider the case where $\delta x(t_f) \neq 0$. From the previous case, $x^*$ satisfies Euler-Lagrange equation and $\delta J|_{x^*} = 0$. This implies that $\frac{\partial L}{\partial x}(t_f, x^*(t), \dot{x}^*(t)) = 0$ which is a natural boundary condition.

3. Final time $t_f$ free, endpoint $x_f$ specified: Note that in these type of problems, $t_f$ is not specified. Thus, an extremal curve $x^*$ is extremal in the interval $[t_0, t_f^*]$ and not defined in the interval outside $t_f^*$ where $t_f^*$ is such that $x^*(t_f^*) = x_f$. If $x = x^* + \delta x$ is a curve in a
neighbourhood of $x^*$, then $x(t_f) = x_f$ where $t_f = t_f^* + \delta t$

$$\Delta J = J(x) - J(x^*) = \int_{t_0}^{t_f} L(t, x, \dot{x}) dt - \int_{t_0}^{t_f} L(t, x^*, \dot{x}^*) dt$$

$$= \int_{t_0}^{t_f} [L(t, x, \dot{x}) - L(t, x^*, \dot{x}^*)] dt + \int_{t_0}^{t_f} L(t, x, \dot{x}) dt$$

Integrating the first integral by parts as done in the previous case and using $\dot{x}(t_0) = 0$,

$$\Delta J = \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \delta x(t_f^*) + \int_{t_0}^{t_f} \left[ \frac{\partial L(t, x, \dot{x})}{\partial x} - \frac{d}{dt} \left( \frac{\partial L(t, x, \dot{x})}{\partial \dot{x}} \right) \right] \delta x dt + \int_{t_0}^{t_f} L(t, x, \dot{x}) dt + h.o.t.$$ 

We now simplify the integral $\int_{t_f}^{t_f^*} L(t, x, \dot{x}) dt$. Observe that $t_f = t_f^* + \delta t$. Therefore, $\int_{t_f}^{t_f^*} L(t, x, \dot{x}) dt = L(t_f^*, x(t_f^*), \dot{x}(t_f^*))dt + o(\delta t)$ (this is approximating area under a curve by a rectangle). This is justified as follows. Using $x = x^* + \delta x$, we can expand $L(t_f^*, x(t_f^*), \dot{x}(t_f^*))\delta t$ using the Taylor expansion as

$$L(t_f, x(t_f), \dot{x}(t_f))\delta t = L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*))\delta t + \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial x} \delta x \delta t + \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \delta \dot{x} \delta t + h.o.t.$$ 

Thus using only first order terms,

$$\int_{t_f}^{t_f^*} L(t, x, \dot{x}) dt = L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*))\delta t + h.o.t.$$ 

$$\Rightarrow \Delta J = \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \delta x(t_f^*) + \int_{t_0}^{t_f^*} \left[ \frac{\partial L(t, x^*, \dot{x}^*)}{\partial x} - \frac{d}{dt} \left( \frac{\partial L(t, x^*, \dot{x}^*)}{\partial \dot{x}} \right) \right] \delta x dt + L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*))\delta t + h.o.t.$$ 

$$\Rightarrow \delta J = \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \delta x(t_f^*) + \int_{t_0}^{t_f^*} \left[ \frac{\partial L(t, x^*, \dot{x}^*)}{\partial x} - \frac{d}{dt} \left( \frac{\partial L(t, x^*, \dot{x}^*)}{\partial \dot{x}} \right) \right] \delta x dt + L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*))\delta t$$ (15)

Note that $\delta x(t_f^*) = x(t_f^*) - x^*(t_f^*)$. Moreover, $x^*(t_f^*) = x_f = x(t_f) = x(t_f^* + \delta t)$. Therefore, $\delta x(t_f^*) = x(t_f^*) - x(t_f^* + \delta t)$. Using Taylor expansion,

$$\delta x(t_f^*) = x(t_f^*) - x(t_f^*) - \frac{dx(t_f^*)}{dt} \delta t - h.o.t.$$ (16)

Thus using Equation (16) in Equation (15),

$$\delta J = \int_{t_0}^{t_f^*} \left[ \frac{\partial L(t, x^*, \dot{x}^*)}{\partial x} - \frac{d}{dt} \left( \frac{\partial L(t, x^*, \dot{x}^*)}{\partial \dot{x}} \right) \right] \delta x dt + \left( - \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \right) \dot{x}(t_f^*) + L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*))\delta t = 0.$$ (17)

Now using the fact that $\dot{t}$ is arbitrary, choosing $\delta t = 0$, we obtain Euler-Lagrange equation and from Euler-Lagrange equation and the first order necessary condition $\delta J = 0$, we get the following natural boundary condition

$$\frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \dot{x}(t_f^*) = L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*)).$$ (18)

4. **Final time free $t_f$, endpoint $x_f$ free:** In such problems, $t_0$ and $x(t_0)$ are specified but both $t_f$ and $x(t_f)$ are free. Let $x^*$ be an extremal trajectory. Thus $x^*(t_f^*) = x_f$. Let $x$ be any other
trajectory in a neighbourhood of \( x^* \). Thus, \( x(t_f) = x(t_f^* + \delta t) = x_f \). Using \( \Delta J = J(x) - J(x^*) \) and following the same procedure used in the previous case from Equation (15) we obtain,

\[
\delta J = \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial x} \delta x(t_f^*) + \int_{t_0}^{t_f^*} \left[ \frac{\partial L(t, x^*, \dot{x}^*)}{\partial x} - \frac{d}{dt} \frac{\partial L(t, x^*, \dot{x}^*)}{\partial \dot{x}} \right] \delta x dt + L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*)) \delta t
\]

Recall that \( \delta x(t_f^*) = x(t_f^*) - x^*(t_f^*) \). Let \( \delta x_f := x(t_f) - x^*(t_f^*) = x_f - x_f^* \). Therefore,

\[
\delta x(t_f^*) = x(t_f^*) - x^*(t_f^*) + x(t_f) - x(t_f) = \delta x_f - (x(t_f) - x(t_f^*))
\]

(x) \quad \delta x(t_f^*) = x(t_f^*) + \delta t = x(t_f^*) + \dot{x}(t_f^*) \delta t + \text{h.o.t.}

Using Equation (90) in Equation (88),

\[
\delta x(t_f^*) = \delta x_f - \dot{x}(t_f^*) \delta t - \text{h.o.t.}
\]

Taking first order terms in the previous equation and substituting in Equation (15),

\[
\delta J = \int_{t_0}^{t_f} \left[ \frac{\partial L(t, x^*, \dot{x}^*)}{\partial x} - \frac{d}{dt} \frac{\partial L(t, x^*, \dot{x}^*)}{\partial \dot{x}} \right] \delta x dt + (- \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \dot{x}(t_f^*) + L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*))) \delta t
\]

\[
+ \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \delta x_f = 0.
\]

Since \( \delta t \) and \( \delta x_f \) is arbitrary, choosing them to be equal to zero, we obtain the Euler-Lagrange equation. The boundary conditions are

\[
(- \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \dot{x}(t_f^*) + L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*))) \delta t + \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \delta x_f = 0.
\]

(a) Suppose that the final value for any admissible curve is free to lie anywhere without any constraints. Hence, \( \delta t \) and \( \delta x_f \) are independent. This implies that by the previous equation,

\[
\frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \dot{x}(t_f^*) = 0 \quad \text{which implies that} \quad L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*)) = 0.
\]

(b) Suppose that the final value is a moving object i.e. \( x(t_f^*) = S(t_f^*) \) where \( S \) defines a moving target. Then, \( \delta x_f = x(t_f) - x^*(t_f^*) = S(t_f^*) + \delta t - S(t_f^*) \) \quad \text{h.o.t.} \). Thus, using first order term \( \delta x_f = \frac{dS(t_f^*)}{dt} \delta t \) in Equation (23), we get the following boundary conditions

\[
(- \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \dot{x}(t_f^*) + L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*))) + \frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \frac{dS(t_f^*)}{dt} \delta t = 0
\]

Since \( \delta t \) is arbitrary,

\[
\frac{\partial L(t_f^*, x^*, \dot{x}^*)}{\partial \dot{x}} \left[ \frac{dS(t_f^*)}{dt} - \dot{x}(t_f^*) \right] + L(t_f^*, x^*(t_f^*), \dot{x}^*(t_f^*)) = 0
\]

is the boundary condition which is also called transversality condition.

Examples of these boundary conditions will be considered in the next section on calculus of variations and optimal control.

**Functionals with multiple degrees of freedom:** For cost functionals \( J(x_1, \ldots, x_n) \) with multiple functions and time as independent variable, there are Euler-Lagrange equations for each \( x_i \) and appropriate boundary conditions as seen above. Let \( x \) a vector such that its \( i \)-th component is \( x_i \). We have a vectorial Euler-Lagrange equation

\[
\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0
\]

and corresponding vectorial boundary conditions.
3.2 Functionals with constraints

Here, we use Lagrange multipliers to define a new cost functional and solve the new unconstrained variational problem. For simplicity, we don’t give any boundary conditions but they will be considered in full detail in the next section on optimal control and calculus of variations.

1. Holonomic constraints: Consider the following cost functional

\[ J(x) = \int_{t_0}^{t_f} L(t, x, \dot{x})\,dt \]  

such that \( f_i(t, x) = 0, \; 1 \leq i \leq m. \) (25)

Let \( f(t, x) \) denote a vector whose \( i \)-th component is \( f_i(t, x). \) Consider the following augmented cost functional

\[ J_a(x, p) = \int_{t_0}^{t_f} [L(t, x, \dot{x}) + p^T(t)f(t, x)]\,dt. \]  

(26)

\[ \Delta J_a = J_a(x + \delta x, p + \delta p) - J_a(x, p) = J_a(x + \delta x, p + \delta p) - J_a(x + \delta x, p) + J_a(x + \delta x, p) - J_a(x, p) \]

\[ = \int_{t_0}^{t_f} [f^T(t, x)\delta p(t) + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\dot{\delta x} + p^T(t)\frac{\partial f}{\partial x}\delta x]\,dt. \]

Integrating \( \int_{t_0}^{t_f} \frac{\partial L}{\partial x}\delta x \) by parts, we obtain

\[ \delta J_a = \frac{\partial L}{\partial x}\delta x \bigg|_{t_0}^{t_f} + \int_{t_0}^{t_f} [f^T(t, x)\delta p(t) + \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right)\delta x + p^T(t)\frac{\partial f}{\partial x}\delta x]\,dt \]  

(27)

As seen before, choosing zero variation for \( \delta x, \) from the first order necessary condition, the term inside the integral must be equal to zero. If constraints are satisfied, then \( f(t, x) = 0. \) Therefore, coefficients of \( \delta p \) are zero. Since constraints are satisfied, we can choose \( m \) Lagrange multipliers arbitrarily. Choose them such that \( m \) of the components (out of \( n \)) of \( \delta x \) are zero. Therefore, there are \( n - m \) independent components of \( \delta x \) and their coefficients must be zero to satisfy the first order necessary condition. Therefore,

\[ \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) + p^T(t)\frac{\partial f}{\partial x} = 0. \]  

(28)

2. Non-holonomic constraints: Consider the following cost functional

\[ J(x) = \int_{t_0}^{t_f} L(t, x, \dot{x})\,dt \]  

such that \( f_i(t, x, \dot{x}) = 0, \; 1 \leq i \leq m. \) (29)

Consider the following augmented cost functional

\[ J_a(x, p) = \int_{t_0}^{t_f} [L(t, x, \dot{x}) + p^T(t)f(t, x, \dot{x})]\,dt. \]  

(30)

\[ \Delta J_a = J_a(x + \delta x, p + \delta p) - J_a(x, p) = J_a(x + \delta x, p + \delta p) - J_a(x + \delta x, p) + J_a(x + \delta x, p) - J_a(x, p) \]

\[ = \int_{t_0}^{t_f} [f^T(t, x)\delta p(t) + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\dot{\delta x} + p^T(t)\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}\delta \dot{x}]\,dt. \]

Integrating \( \int_{t_0}^{t_f} \frac{\partial L}{\partial x}\delta x + p^T(t)\frac{\partial f}{\partial x}\delta x \) by parts, we obtain

\[ \delta J_a = \left( \frac{\partial L}{\partial x} + p^T(t)\frac{\partial f}{\partial x} \right)\delta x \bigg|_{t_0}^{t_f} + \int_{t_0}^{t_f} [f^T(t, x)\delta p(t) + \left( \frac{\partial L}{\partial x} + p^T(t)\frac{\partial f}{\partial x} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} + p^T(t)\frac{\partial f}{\partial \dot{x}} \right) \right] \delta x\,dt \]  

(31)

As seen before, choosing zero variation for \( \delta x, \) from the first order necessary condition, the term inside the integral must be equal to zero. If constraints are satisfied, then \( f(t, x, \dot{x}) = 0. \) By the same arguments used in the previous case,

\[ \left( \frac{\partial L}{\partial x} + p^T(t)\frac{\partial f}{\partial x} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} + p^T(t)\frac{\partial f}{\partial \dot{x}} \right) = 0. \]  

(32)
3. Isoperimetric constraints (integral constraints): Suppose that we have the following integral constraints

$$\int_{t_0}^{t_f} f_i(t, x, \dot{x}) dt = c_i \quad 1 \leq i \leq m$$

(33)
on the cost functional. Let $$y_i(t) = \int_{t_0}^{t} f_i(t, x, \dot{x}) dt$$. Thus, in vectorial notion, $$\dot{y} = f(t, x, \dot{x})$$. Consider an augmented Lagrangian:

$$L_a(t, x, \dot{x}, p, \dot{y}) := L(t, x, \dot{x}) + p^T(t) [f(t, x, \dot{x}) - \dot{y}]$$

We have the following set of $$n + m$$ differential equations as a consequence of the first order necessary conditions of optimality

$$\frac{\partial L_a}{\partial \dot{x}} - \frac{d}{dt} \left( \frac{\partial L_a}{\partial x} \right) = 0$$

(34)

$$\frac{\partial L_a}{\partial \dot{y}} - \frac{d}{dt} \left( \frac{\partial L_a}{\partial y} \right) = 0.$$ 

(35)

The optimal solution trajectories satisfy boundary conditions $$\dot{y}^*(t_f) = c$$. Since $$L_a$$ is independent of $$y$$, $$\frac{\partial L_a}{\partial y} = 0$$. Observe that $$\frac{\partial L_a}{\partial y} = -p^*(t)$$ hence, $$\frac{d}{dt} \frac{\partial L_a}{\partial y} = -p^*(t) = 0$$. Therefore, the Lagrange multipliers are constant.

**Example 3.1 (Roughan).** (Holonomic constraints: Geodesics on a sphere) Find the shortest path curves between any two points on the unit sphere in $$\mathbb{R}^3$$. The cost function with constraints is given by

$$\min \quad J = \int_{t_0}^{t_1} \sqrt{x^2 + \dot{y}^2 + \dot{z}^2} dt$$

subject to $$x^2 + y^2 + z^2 = 1.$$ 

(36)

The initial and final points are $$(x(t_0), y(t_0), z(t_0))$$ and $$(x(t_1), y(t_1), z(t_1))$$ respectively. The augmented cost function is given by

$$J_a = \int_{t_0}^{t_1} \left( \sqrt{x^2 + \dot{y}^2 + \dot{z}^2} + p(t)(x^2 + y^2 + z^2) \right) dt.$$ 

Observe that

$$\frac{\partial f}{\partial x} = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x, 2y, 2z)$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = -\frac{d}{dt} \left( \frac{\dot{x}}{\sqrt{x^2 + \dot{y}^2 + \dot{z}^2}}, \frac{\dot{y}}{\sqrt{x^2 + \dot{y}^2 + \dot{z}^2}}, \frac{\dot{z}}{\sqrt{x^2 + \dot{y}^2 + \dot{z}^2}} \right).$$

Using Equation (28), we get the following equation for $$x$$ variable:

$$2xp = \frac{\ddot{x}}{\sqrt{x^2 + \dot{y}^2 + \dot{z}^2}} - \frac{\dot{x}(\dot{x} \ddot{x} + \dot{y} \ddot{y} + \dot{z} \ddot{z})}{(\dot{x}^2 + \dot{y}^2 + \dot{z})^2}$$

(37)

and similar equations for $$y$$ and $$z$$ variables. Due to the symmetry in variables $$(x, y, z)$$, we can represent the differential equation as

$$2up = \frac{\ddot{u}}{\sqrt{x^2 + \dot{y}^2 + \dot{z}^2}} - \frac{u(\dot{x} \ddot{x} + \dot{y} \ddot{y} + \dot{z} \ddot{z})}{(\dot{x}^2 + \dot{y}^2 + \dot{z})^2}$$

(38)
which is true for \( u = x, y, z \). Since, this is a second order ode in \( u \), it has 2 linearly independent solutions. But \( x(t), y(t), z(t) \) all satisfy this ode, hence they must be dependent i.e., there exists \( (a, b, c) \) such that \( ax + by + cz = 0 \). But this represents a plane passing through origin (a subspace with \( (a, b, c) \) as its normal). Therefore, along with the equation of the unit sphere, \( (x, y, z) \) should satisfy \( ax + by + cz = 0 \) for shortest distance. Intersection of these two give a great circle on the unit sphere. Thus the shortest path between two points on a sphere lies on the great circle.

**Example 3.2 (Sastry).** (Minimum energy control of non-holonomic integrator:) Consider a non-holonomic integrator as follows

\[
\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1u_2 - x_2u_1.
\]

We want to find the minimum energy input to drive the state from the origin to a specified point \((0, 0, a)\) from \( t = 0 \) to \( t = 1 \). The cost function is \( J = \int_0^1 (u_1^2 + u_2^2)dt \) subject to system dynamics. Let’s try to bring this problem into the second type mentioned above. Using system equations to eliminate \( u_1 \) and \( u_2 \), we obtain the cost function \( \int_0^1 (x_1^2 + x_2^2)dt \) subject to \( \dot{x}_3 = x_1\dot{x}_2 + x_2\dot{x}_1 = 0 \). Therefore, the augmented cost function is

\[
J_a = \int_0^1 (\dot{x}_1^2 + \dot{x}_2^2 + p(t)(\dot{x}_3 - x_1\dot{x}_2 + x_2\dot{x}_1))dt.
\]

Applying Equations (32), we obtain

\[
-p(t)\dot{x}_2 = \frac{d}{dt}(2\dot{x}_1 + p(t)x_2)
\]

\[
\dot{x}_1 = \frac{d}{dt}(2\dot{x}_2 - p(t)x_1)
\]

\[
0 = \frac{d}{dt}p(t).
\]

Thus, \( p(t) = c \) and we obtain the following second order system of equations

\[
\dot{x}_1 + cx_2 = 0
\]

\[
\ddot{x}_2 - cx_1 = 0.
\]

Now using \( \dot{x}_1 = u_1 \) and \( \dot{x}_2 = u_2 \), we have the following first order ode

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \begin{bmatrix}
  0 & -c \\
  c & 0
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \begin{bmatrix}
  \cos ct & -\sin ct \\
  \sin ct & \cos ct
\end{bmatrix} \begin{bmatrix}
  u_1(0) \\
  u_2(0)
\end{bmatrix}.
\]

We need to find \( u(0) \) and \( c \) using initial and final conditions. Let’s write \( \dot{u} = Hu \) for first order equations in \( u_1, u_2 \). Hence, \( u(t) = e^{Ht}u(0) \). Note that \( e^{Ht} \) is orthogonal, hence the norm of \( ||u(t)|| = ||u(0)|| \) remains constant for all time. Integrating for \( x \) in the original system using the expression for \( u \),

\[
\begin{bmatrix}
  x_1(t) \\
  x_2(t)
\end{bmatrix} = (e^{Ht} - I)H^{-1}u(0).
\]

Substituting \( x_1(1) = 0 \) and \( x_2(1) = 0 \), if \( e^H - I \) is non singular, then \( u(0) = 0 \) which implies \( u(t) = 0 \) throughout since we observed before that \( ||u(t)|| \) remains constant for all \( t \). Hence \( e^H - I \) must be singular which implies that its determinant is equal to zero i.e., \( 2 - 2\cos(c) = 0 \Rightarrow c = 2n\pi \) where \( n = 0, \pm 1, \pm 2, \ldots \). For these values of \( c \), \( e^H = I \).
Observe that \( \dot{x}_3 = x_1u_2 - x_2u_1 \) can be written as
\[
\dot{x}_3 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u^T(0)(H^T)^{-1}(e^{H^T} - I) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} e^{Ht} u(0).
\] (40)

Suppose \( c \neq 0 \). Therefore, \((H^T)^{-1} = \begin{bmatrix} 0 & -\frac{1}{c} \\ \frac{1}{c} & 0 \end{bmatrix}\) and \((H^T)^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{c} & 0 \\ 0 & \frac{1}{c} \end{bmatrix} \). Hence, Equation (40) becomes
\[
\dot{x}_3 = \frac{1}{c} u^T(0)(I - e^{H^T}) e^{Ht} u(0) = \frac{1}{c} u^T(0)(I - e^{H}) u(0)
\]
since \( e^{H^T} e^{Ht} = I \). Integrating \( \dot{x}_3 \),
\[
x_3(1) = \frac{1}{c} u^T(0) \int_0^1 (I - e^{Ht}) dt u(0) = \frac{1}{c} u^T(0)(I - H^{-1}(e^H - I)) u(0) = \frac{1}{c} u^T(0) u(0)
\]
where we have used the fact that \( e^H = I \). Therefore, \( u_1(0)^2 + u_2(0)^2 = ca \). Since \( e^{Ht} \) is an orthogonal matrix and \( u(t) = e^{Ht} u(0) \), the norm of \( u(t) \) remains the same for all \( t \). The cost is \( \int_0^1 (u_1^2 + u_2^2) dt = u_1(0)^2 + u_2(0)^2 \int_0^1 dt = ca \). The cost is always positive. Suppose \( a > 0 \), then the cost is minimum when \( n = 1 \) and \( ||u|| = 2\pi a \) with direction of \( u \) being arbitrary.

(Note that \( \frac{dx_3}{dt} = x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \Rightarrow dx_3 = x_1 dx_2 - x_2 dx_1 \). Thus, by Green’s theorem, \( x_3 \) denotes the area traced out in the \( x_1 - x_2 \) plane. This can be explained as follows. Let \( (x(t), y(t)) \) be the projection of the curve \( (x(t), y(t), z(t)) \) on the \( x-y \) plane. The integral \( \int_0^1 x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \) in the \( x-y \) plane denotes the line integral from the origin to the point \( (x(t), y(t)) \) along the projected curve. Draw a straight line from the origin to the point \( (x(t), y(t)) \) to close the curve. The parametric representation of this line is of the form \( (c_1 t, c_2 t) \). Hence, \( \int_0^1 x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} = \int_0^1 (c_1 c_2 - c_1 c_2) = 0 \) along this path. Now, the line integral over the closed contour is the same as the line integral along the projected curve, since over the straight line, it is zero. Therefore, by Green’s theorem, it gives the area enclosed by the closed curve.)

We have already seen an example of isoperimetric constraints. It is finding a curve of given length enclosing the maximum area. The solution can be obtained from the first order conditions. There is one more type of constraints which are inequality constraints. These will be considered later in the applications of the maximum principle.

3.3 Summary

- We studied four types of unconstrained problems based on fixed/free terminal time and end point and corresponding first order conditions.

- We also studied three types of constrained problems namely: holonomic, non-holonomic and isoperimetric and how to construct augmented Lagrangian and obtain first order conditions.

4 Calculus of variations and optimal control

We obtain first order necessary conditions using variational principle and illustrate how it can be used to solve certain type of problems. We will also underline limitations of this approach before moving on to the stronger maximum principle in the next section. (The discussion is borrowed mostly from Liberzon and parts of it are inspired from Kirk.)
4.1 Cost functional and types

Consider a cost functional in the following form

\[ J(u) := \int_{t_0}^{t_f} L(t, x(t), u(t)) dt + K(t_f, x_f) \]  

where \( t_f, x_f \) are terminal time and terminal state, \( L : \mathbb{R} \times \mathbb{R}^n \times U \) is the running cost (\( U \) is the set of admissible inputs) and \( K : \mathbb{R} \times \mathbb{R}^n \) is the terminal cost. This form of the cost functional is called Bolza form or Bolza problem. If there is no terminal cost i.e. \( K = 0 \), then we have the Lagrange problem and if there is no running cost i.e. \( L = 0 \), we have the Mayer problem. It turns out that all the three types are equivalent. Consider a problem in the Mayer form. Note that

\[
K(t_f, x_f) = K(t_0, x_0) + \int_{t_0}^{t_f} \frac{d}{dt} K(t, x(t)) dt \\
= K(t_0, x_0) + \int_{t_0}^{t_f} (K_t(t, x(t)) + K_x(t, x(t)).f(t, x(t), u(t))) dt 
\]

where we have used \( \dot{x}(t) = f(t, x, u) \). Note that the term \( K(t_0, x_0) \) is independent of \( u \) hence can be ignored. Thus we can transform Mayer form to Lagrange form and consequently, we can also transform Bolza form to Lagrange form.

Suppose that the running cost \( L \) satisfies the same regularity conditions satisfied by \( f \). Introducing an extra state variable \( x^0 \) where

\[
x^0 = L(t, x, u), \ x^0(t_0) = 0,
\]

one obtains \( \int_{t_0}^{t_f} L(t, x, u) dt = x^0(t_f) \). This transforms the Lagrange form into Mayer form. The same trick works to transform Bolza form to Mayer form. It is clear that both Lagrange and Mayer forms are special types of Bolza form.

4.2 Target set

Recall from the previous section that the final time \( t_f \) and the final state \( x_f \) can be fixed, free or it may belong to some set. All these possibilities are captured by introducing a target set \( S \subset [t_0, \infty) \times \mathbb{R}^n \) and letting \( t_f \) to be the smallest time such that \((t_f, x_f) \in S \). It can be seen that \( t_f \) defined in this way depends on the input/control \( u \). We take \( S \) to be closed set so that if \((t, x(t))\) ever enters \( S \), the final time \( t_f \) is well defined. If a trajectory is such that \((t, x(t))\) does not belong to \( S \) for any \( t \), then we consider the cost to be infinite or undefined. We do not allow \( t_f = \infty \).

The target set \( S = [t_0, \infty) \times \{x_1\} \) where \( x_1 \in \mathbb{R}^n \) is fixed gives a free-time, fixed-endpoint problem. A generalization of this case is \( S = [t_0, \infty) \times S_1 \) where \( S_1 \in \mathbb{R}^n \) is a surface (manifold). Another natural target set is \( S = \{t_1\} \times \mathbb{R}^n \). This is a fixed-time free-endpoint problem. Suppose we start with a fixed-time free-endpoint problem and consider an auxiliary state \( x_{n+1} := t \), we recover the previous case with \( S \subset \mathbb{R}^{n+1} \) given by \( S := \{x \in \mathbb{R}^{n+1} : x_{n+1} = t_1\} \). A target set \( S = T \times S_1 \) where \( T \subset [t_0, \infty) \) and \( S_1 \) is some surface in \( \mathbb{R}^n \) includes as special cases all target sets mentioned above. It also includes target sets of the form \( \{t_1\} \times \{x_1\} \) which is fixed-time, fixed-endpoint problem. Moreover, it also includes \( S = [t_0, \infty) \times \mathbb{R}^n \) which is free-time free-endpoint problem (type 4a from the previous section). Note that we have seen that \( S = [t_0, \infty) \times S_1 \) where \( S_1 \in \mathbb{R}^n \) is a surface, is also a special type of free-time free-endpoint problem (type 4b from the previous section).

The objective of finding the optimal control input \( u \) is to minimize the cost (Equation (41)). For Mayer problems, \((t_f, x_f)\) is a point where the terminal cost is minimized and if the minimum is unique, then we do not need to specify a target set a priori. In the presence of a running cost \( L \) taking both positive and negative values, it is clear that remaining at rest at the initial state
may not be optimal. Moreover, this problem can also be brought in the Mayer form. Note that we may consider moving targets e.g. \( g(t, x, u) \) \( t \in [t_0, \infty) \) where \( g : \mathbb{R} \to \mathbb{R}^n \) is a continuous function. Furthermore, point target can be generalized to a set target by making \( g \) set valued.

### 4.3 Necessary first order conditions for optimal control using variational approach

Consider a dynamical system \( \dot{x} = f(t, x, u) \) and the problem is to find \( u \) such that (41) is minimized. **Note that there are no constraints on the input \( u \).** Adding constraints \( \dot{x} = f(t, x, u) \) using Lagrange multipliers \( p(t) \), we get an augmented cost functional

\[
J_a(u, p) = K(t_f, x_f) + \int_{t_0}^{t_f} (L(t, x(t), \dot{x}(t)) + p^T(t)(\dot{x} - f(t, x, u)))dt. \tag{43}
\]

Let \( H(t, x, u, p) := \langle p, f(t, x, u) \rangle - L(t, x, u) \). Thus,

\[
J_a(u, p) = K(t_f, x_f) + \int_{t_0}^{t_f} (\langle p(t), \dot{x}(t) \rangle - H(t, x, u, p))dt. \tag{44}
\]

Let \((x^*, u^*)\) correspond to an optimal trajectory, \( t_f^* \) be the optimal final time and let \( p^* \) be the optimal Lagrange multiplier. Therefore, the first variation is

\[
J_a(u^* + \delta u, p^* + \delta p) - J_a(u^*, p^*) = K(t_f, x_f) - K(t_f^*, x_f^*) + \int_{t_0}^{t_f} \langle \dot{x}^* + \delta \dot{x}, p^* + \delta p \rangle dt - \int_{t_0}^{t_f} \langle \dot{x}^*, p^* \rangle dt - \int_{t_0}^{t_f} \langle \dot{x}, p \rangle dt - H(t, x^*, p, u^*)dt \]

Using Taylor expansion, we obtain

\[
= K(t_f, x_f) - K(t_f^*, x_f^*) + \int_{t_0}^{t_f} \langle (\dot{x}^*)^T \delta p + (p^*)^T \delta \dot{x} - H_p(t, x^*, p^*, u^*)^T \delta p - H_x(t, x^*, p^*, u^*)^T \delta x \rangle dt - \int_{t_0}^{t_f} \langle \dot{x}, p \rangle dt - H(t, x^*, p^*, u^*)dt + \int_{t_0}^{t_f} \langle \dot{x}^*, p^* \rangle dt - H(t, x^*, p^*, u^*)dt \]

Integrating first order terms by parts,

\[
= K(t_f, x_f) - K(t_f^*, x_f^*) + (p^*(t))^T \delta x \bigg|_{t_0}^{t_f} + \int_{t_0}^{t_f} \langle (\dot{x}^*)^T \delta p - H_p(t, x^*, p^*, u^*)^T \delta p - (p^*)^T \delta x - H_x(t, x^*, p^*, u^*)^T \delta x \rangle dt - \int_{t_0}^{t_f} \langle \dot{x}, p \rangle dt - H(t, x^*, p^*, u^*)dt + \int_{t_0}^{t_f} \langle \dot{x}^*, p^* \rangle dt - H(t, x^*, p^*, u^*)dt \]

We know from our previous arguments that for the first variation to be zero, the terms inside the integral must be zero i.e.,

\[
H_u = 0, \dot{p}^* = -H_x, \dot{x}^* = H_p. \tag{45}
\]

Moreover, using \( K(t_f, x_f) - K(t_f^*, x_f^*) = K_i(t_f^*, x_f^*) \delta t + K_x(t_f^*, x_f^*)^T \delta x_f \) (where \( \delta x_f = x_f - x_f^* \)) and representing the last integral by first order terms,

\[
K_i(t_f^*, x_f^*) \delta t + K_x(t_f^*, x_f^*)^T \delta x_f + p^*(t_f^*)^T \delta x_f + ((\dot{x}^*, p^*) - H(t_f^*, x^*, p^*, u^*)) \delta t = 0. \tag{46}
\]

Using Equation (21), \( \delta x(t_f^*) = \delta x_f - \dot{x}(t_f^*) \delta t - h.o.t.. \) Substituting this in the previous equation,

\[
(K_i(t_f^*, x_f^*) + \langle \dot{x}^*, p^* \rangle - H(t_f^*, x^*, p^*, u^*)) \delta t + K_x(t_f^*, x_f^*)^T \delta x_f + (p^*(t_f^*))^T (\delta x_f - \dot{x}(t_f^*) \delta t) = 0 \]

\[
\Rightarrow (K_i(t_f^*, x_f^*) - H(t_f^*, x^*, p^*, u^*)) \delta t + (K_x(t_f^*, x_f^*) + p^*(t_f^*))^T \delta x_f = 0. \tag{47}
\]
The above equation gives boundary conditions in most general form. Note that
\[ \dot{p}^* = -H_x = -f_x^T \dot{p}^* + L_x, \quad \dot{x}^* = H_p = f. \] (48)

To solve above first order odes, we need to find \(2n\) constants of integration. This is done by
using boundary conditions. We refer the reader to Appendix for an alternate approach to obtain
boundary conditions. We now enumerate all possible types of different boundary conditions.

1. For fixed time, fixed endpoint problem, both \(\delta t\) and \(\delta x_f = 0\) and \(x^*(t_0) = x_0, \ x^*(t_f) = x_f\) is
   the boundary condition which gives \(2n\) constants.

2. For fixed time free endpoint problems, \(\delta t = 0\), therefore, the boundary conditions are \(x^*(t_0) = x_0\) and
   \[ K_x(t_f^*, x_f^*) + p^*(t_f^*) = 0 \] (49)
   which gives \(2n\) constants of integration.

3. Fixed time constrained end point problem: Suppose that the final state lies on a surface \(S\) i.e.,
   \(x(t_f^*) \in S\). Let \(S\) be an intersection of hyper-surfaces i.e., \(S = S_1 \cap \ldots \cap S_k\). Thus at any point
   \(s \in S\) normal at point \(s\) are \(\nabla S_1|_s, \ldots, \nabla S_k|_s\). Therefore, \(\delta x_f = x(t_f^*) - x^*(t_f^*)\) must be
   orthogonal to all normals \(\nabla S_1|_{x^*(t_f^*)}, \ldots, \nabla S_k|_{x^*(t_f^*)}\). Thus from Equation (47), \(K_x(t_f^*, x_f^*) + p^*(t_f^*)\)
   must be a linear combination of \(\nabla S_1|_{x^*(t_f^*)}, \ldots, \nabla S_k|_{x^*(t_f^*)}\) i.e., \(K_x(t_f^*, x_f^*) + p^*(t_f^*) = d_1 \nabla S_1|_{x^*(t_f^*)} + \ldots + d_k \nabla S_k|_{x^*(t_f^*)}\). These \(n\) equations along with \(x(t_0) = 0\) forms \(2n\) equations.
   Furthermore, since \(x^*(t_f^*) \in S\) and \(S = S_1 \cap \ldots \cap S_k\), \(x^*(t_f^*)\) satisfies \(k\) equations for \(k\) hyper-
surfaces. Thus there are \(2n + k\) equations and \(2n + k\) unknowns (\(2n\) constants of integration
   and \(d_1, \ldots, d_k\)).

4. For free time, fixed endpoint problems, \(x^*(t_0) = x_0, \ x^*(t_f) = x_f\) and
   \[ K_t(t_f^*, x_f^*) - H(t_f^*, x^*, p^*, u^*) = 0. \] (50)
   Thus there are \(2n + 1\) unknowns and \(2n + 1\) equations.

5. For free time free endpoint problems, \(x^*(t_0) = x_0\) and (47) gives boundary conditions. Again
   there are \(2n + 1\) unknowns and \(2n + 1\) equations.

6. For free time free endpoint problems when there is a moving target i.e., \(x(t_f) = S(t_f)\), then
   using \(\delta x_f = \frac{dS(t_f^*)}{dt}\delta t\), we obtain the following boundary condition
   \[ K_t(t_f^*, x_f^*) - H(t_f^*, x^*, p^*, u^*) + (K_x(t_f^*, x_f^*) + p^*(t_f^*)) \frac{dS(t_f^*)}{dt} = 0. \] (51)
   The above equation along with \(x^*(t_0) = x_0\) and \(x(t_f^*) = S(t_f^*)\) forms \(2n + 1\) equations for
   \(2n + 1\) unknowns.

7. Free time constrained endpoint problem: in here we have an additional Equation (50) apart
   from \(2n + k\) equations in Case 3 above.

8. Free time constrained moving endpoint problem: here the additional equation apart from
   \(2n + k\) equations in Case 3 is given by Equation (51).

Observe that along the optimal trajectory, \(\frac{d}{dt} H(t, x^*, p^*, u^*) = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial u} \dot{u} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}\)
(since \(H_u = 0, \ x = H_p, \ p = -H_x\ along the optimal trajectory). Thus, when \(H\) has no explicit time
dependence, then \(\frac{d}{dt} H = 0\) along the optimal trajectory.
Necessary conditions due to second variation and maximum property of the Hamiltonian: The second order necessary condition to minimize the cost functional $J$ is $\delta^2 J_{uu}(\delta x, \delta u) \geq 0$. It turns out that for this to happen, $H_{uu} \leq 0$ (Liberzon, section 3.4.3). This is called Legendre-Clebsch condition (Liberzon). Suppose we are at the optimal trajectory $x^*(t)$, and Lagrange multipliers $p^T(t)$. Since $\delta J|_{x^*, u^*, p^*}(\delta u) = 0$, $H_u = 0$. This condition tells us that for optimal trajectory, $H$ is a candidate for a maximum at the optimal trajectory. It turns out that $H$ is indeed maximum along the optimal trajectory which is Pontryagin’s Maximum principle.

Sufficient second order conditions via variational approach: We refer the reader to Athans and Falb (section 5.8) for second order sufficient conditions for variational control problems.

4.3.1 Examples: Minimum energy problems without input constraints

All examples here consist of open loop control laws. In the next section, we will see that for some cases, by eliminating the co-state variable, we can obtain a closed loop optimal control law for some finite horizon LQR problem. In general, to obtain closed loop control law for examples considered below, we sometimes need to adjust the terminal cost appropriately to apply HJB theory (Refer Section 8, Example 8.2).

Example 4.1. Consider the problem finding the least energy input for a controllable LTI system $\dot{x} = Ax + Bu$ to drive the state from its initial position $x_0$ to the final position $x_f$. We know from linear systems theory that this is obtained using the Controllability Gramian. We now arrive at the same answer using techniques developed in here.

The cost function is

$$J(u) = \frac{1}{2} \int_{t_0}^{t_f} u^T u dt.$$ 

The problem is to minimize $J(u)$ subject to $\dot{x} = Ax + Bu$, $x(t_0) = x_0$, $x(t_f) = x_f$. Note that there is no terminal cost and this is a fixed time fixed endpoint problem. The Hamiltonian is given by $H = p^T(Ax + Bu) - \frac{1}{2}u^T u$. Now from the canonical equations,

$$\dot{x}^* = Ax + Bu, p^* = -A^T p^*, u^* = B^T p^* \Rightarrow p^*(t_f) = e^{-A^T(t_f-t)} p^*(t) \Rightarrow p^*(t) = e^{A^T(t_f-t)} p^*(t_f)$$

Thus, $u^* = B^T e^{-A^T(t-t_f)} p^*(t_f)$. Therefore,

$$x_f = e^{A(t_f-t_0)} x_0 + \int_{t_0}^{t_f} e^{A(t_f-t)} BB^T e^{A^T(t_f-t)} p^*(t_f) dt$$

Thus, we can find the boundary condition $p^*(t_f)$ from the above equation using the inverse of the Controllability Gramian i.e., $p^*(t_f) = W_{t_f}(A,B)^{-1} (x_f - e^{A(t_f-t_0)} x_0)$. This gives the least energy input $u^*(t) = B^T e^{-A^T(t-t_f)} W_{t_f}(A,B)^{-1} (x_f - e^{A(t_f-t_0)} x_0)$.

Example 4.2. Consider fixed time free endpoint problem where the cost function is $J = \frac{1}{2} x_f^T Q_f x_f + \frac{1}{2} \int_{t_0}^{t_f} u^T u dt$. The problem is to drive the state from $x(0) = x_0$ to an endpoint which is free with given terminal cost at time $t_f$. Therefore, using boundary condition $p^*(t_f) = -K_{x_f}(t_f, x_f^*) = -Q_f x_f$ and first order optimality condition, $u^*(t) = B^T p^*(t) = B^T e^{-A^T(t-t_f)} p^*(t_f) = -B^T e^{-A^T(t-t_f)} Q_f x_f$. Therefore, using similar computations done in the previous example, $x_f = e^{A t_f} x_0 - W_{t_f}(A,B) Q_f x_f$. Solving this linear equation gives $x_f$, the terminal state.
**Example 4.3.** Consider Example 4.1 where end time \( t_f \) is free. Then in addition to \( x(t_0) = x_0, x(t_f) = x_f \), we have \( K(t_f) x_f^* - H(t_f^*, x^*, p^*, u^*) = 0 \). Since, \( K = 0, H(t_f^*, x^*, p^*, u^*) = 0 \). Note that \( t_f \) is not specified and we need to find an optimal \( t_f^* \). Let’s compute the Hamiltonian at \( t = t_f^* \). Observe that \( u^*(t_f^*) = B^T p^*(t_f^*) \).

\[
H(t_f^*, x^*, p^*, u^*) = (p^*(t_f^*))^T (Ax^*(t_f^*) + Bu^*(t_f^*)) - \frac{1}{2} (u^*(t_f^*))^T u^*(t_f^*)
\]

\[
= p^*(t_f^*))^T Ax^*(t_f^*) + p^*(t_f^*)^T BB^T p^*(t_f^*) - \frac{1}{2} p^*(t_f^*)^T BB^T p^*(t_f^*)
\]

Substituting for \( p^*(t_f^*) \), the above expression becomes

\[
(x_f - e^{A(t_f^*-t_0)} x_0)^T W_{t_f^*} (A, B)^{-1} Ax_f + \frac{1}{2} (x_f - e^{A(t_f^*-t_0)} x_0)^T W_{t_f^*} (A, B)^{-1} BB^T W_{t_f^*} (A, B)^{-1} (x_f - e^{A(t_f^*-t_0)} x_0) = 0.
\]

From this equation, \( t_f^* \) is determined.

**Example 4.4.** Consider another version of this problem. Suppose \( t_f \) is fixed and \( x_f \) is constrained to lie on the unit sphere in \( \mathbb{R}^n \). Since there is no terminal cost, \( p^*(t_f) = \alpha \nabla S x_f^* = \alpha x_f^* \). Moreover, \((x_f^*)^2 + \ldots + (x_f^n)^2 = 1 \). Suppose \( t_0 = 0 \) for simplicity. Therefore, from Example 4.1,

\[
p^*(t_f^*) = W_{t_f^*} (A, B)^{-1} (x_f^* - e^{At_f} x_0) = \alpha x_f^* \Rightarrow (I - \alpha W_{t_f^*} (A, B)) x_f^* = e^{At_f} x_0.
\]

The previous equation along with equation \((x_f^*)^2 + \ldots + (x_f^n)^2 = 1 \) forms \( n+1 \) equations in \( n+1 \) unknowns i.e., \( x_f^* \) and \( \alpha \). Solving these equations (boundary conditions), we obtain \( p^*(t_f^*) \) from which we obtain \( u^* \) as done in Example 4.1.

**Example 4.5.** Consider a problem where there is a moving target (shooting problem). This is a free time, free endpoint problem with moving target (type 6). Suppose \( x_f \) lies on a curve \( \gamma(t) \). Since there are no terminal costs, \( x_f = \gamma(t_f) \) and from Equation (51),

\[
-H(t_f^*, x^*, p^*, u^*) + p^*(t_f^*)^T \frac{d\gamma}{dt} |_{t_f^*} = 0.
\]

The above equation along with \( x_f^* = \gamma(t_f^*) \) forms \( n+1 \) equations with \( n+1 \) unknowns i.e., \( p^*(t_f^*) \) and \( t_f^* \). Solving these equations, we are again reduced following final steps of problem 4.1.

**Example 4.6.** Consider an example of a Dubin’s car with dynamics \( \dot{x}_1 = \cos \theta, \dot{x}_2 = \sin \theta, \dot{\theta} = u \).

Suppose we want to drive the state from the origin to \( x_f \) such that the input energy \( \frac{1}{2} \int_0^1 u^2 dt \). The Hamiltonian is given by \( p_1 \cos \theta + p_2 \sin \theta + p_3 u - \frac{1}{2} u^2 \). Now applying first order Hamilton’s equations, \( H_u = 0 \Rightarrow p_3 = u \) and \( \dot{p}_i = H_{x_i} = 0, \) \((i = 1, 2) \) implies that \( p_1 = c_1 \) and \( p_2 = c_2 \). Moreover, \( \dot{p}_3 = c_1 \sin \theta - c_2 \cos \theta = \alpha \sin(\theta + \beta) \) where \( \alpha, \beta \) are defined appropriately. Using \( \dot{\theta} = u = p_3 \), one obtains \( \ddot{\theta} = \alpha \sin(\theta + \beta) \). One needs to solve this equation subject to terminal conditions to find the optimal control input.

**Remark 4.7.** One can design many such minimum energy control examples which can be solved by the variational approach. These are problems with unconstrained inputs. As soon as there are constraints on inputs with boundary (i.e., constrained input set is not open), the variational approach does not always work and one needs to use the maximum principle.

For minimum time problems, \( L = 1 \). It turns out that if \( u \) is constrained to some set, then optimum is attained at the boundary of this set. Note further that since the objective function for time optimal control problems in linear, the optimum is always attained at the boundary of the constraint set and if there are no constraints, then the optimal solution becomes unbounded (Refer short notes on optimization).
Example 4.8 (Roughan). Car parking problem in 1-D: Consider a problem of choosing a force/input
$u(t)$ for a car which starts at $t = 0$ with zero initial speed from location A and wants to go to location $B$ (where $A$ and $B$ are joined by a straight line) in minimum time such that the speed at $B$ is zero. Suppose that there are no constraints on the input. Thus, we wish to minimize $\int_0^T dt$ subject
to $m\ddot{x} = u$ and $\dot{x}(0) = \dot{x}(T) = 0$. Let $y = \dot{x}$, therefore, $\dot{y} = \frac{u}{m}$ and our problem becomes

$$
\int_0^T dt, \text{ subject to } \dot{x} = y, \; \dot{y} = \frac{u}{m} \text{ and } \dot{y}(0) = \dot{y}(T) = 0.
$$

The Lagrangian in the above cost function is $L(t,x,y,\dot{x},\dot{y}) = 1$. There are two constraint equations
$\dot{x} = y$, $\dot{y} = \frac{u}{m}$ Therefore, the Hamiltonian for the above problem is $H = p_1y + p_2\frac{u}{m} - 1$. Using
the first order necessary conditions,

$$
\dot{p}_1 = -\frac{\partial H}{\partial x} = 0 \Rightarrow p_1 = c_1
$$

$$
\dot{p}_2 = -\frac{\partial H}{\partial y} = -p_1 \Rightarrow p_2 = -c_1t + c_2
$$

$$
H_u = 0 \Rightarrow p_2 = 0 \Rightarrow c_1 = c_2 = 0.
$$

Note that the terminal time is free and endpoint is fixed, therefore, from boundary conditions, the
coefficient of $\delta t$ must be zero. Thus, $H(T,x^*,p^*,u^*) = 0$. But since $p_1 = p_2 = 0$, we obtain
$H = -1$. Therefore, there is no finite optimal solution to this problem. Observe that one can use
infinite acceleration (i.e., $u = \pm \infty$) in either direction to minimize the required time. It also follows
from the fact that since the cost function is linear, minima is attained at the boundary.

In reality, $u$ is always bounded and we have a constraint $-l \leq u \leq l$ on $u$ for $l > 0$. Such
problems involve inequality constraints on the input. Such problems will be handled by the maximum
principle. We will see that the solution is given by the so called bang-bang control.

4.4 LQR and tracking problems: Necessary conditions

Consider a linear system $\dot{x} = Ax + Bu$ with the cost functional

$$
J(u) = \frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2} \int_0^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt
$$

(52)

where $x(t_f)$ is free and $t_f$ is fixed. From the previous subsection, the Hamiltonian $H = p^T(Ax +
Bu) - \frac{1}{2}(x^TQx + u^TRu)$. Now applying the first order necessary conditions from the previous
subsection,

$$
\dot{x}^* = H_p = Ax^* + Bu^*
$$

(53)

$$
\dot{p}^* = -H_x = -A^Tp^* + Qx^*
$$

(54)

$$
0 = H_u = B^Tp^* - Ru^*.\tag{55}
$$

Thus, $u^* = R^{-1}B^tp^*$. Therefore, we have a following first order ode in $2n$ variables

$$
\begin{bmatrix}
\dot{x}^* \\
\dot{p}^*
\end{bmatrix} =
\begin{bmatrix}
A & BR^{-1}B^T \\
Q & -A^T
\end{bmatrix}
\begin{bmatrix}
x^*
\\
p^*
\end{bmatrix}
$$

Let $\Phi(t_f,t)$ be the state transition matrix. Thus,

$$
\begin{bmatrix}
x^*(t_f) \\
p^*(t_f)
\end{bmatrix} =
\begin{bmatrix}
\phi_{11}(t_f,t) & \phi_{12}(t_f,t) \\
\phi_{21}(t_f,t) & \phi_{22}(t_f,t)
\end{bmatrix}
\begin{bmatrix}
x^*(t) \\
p^*(t)
\end{bmatrix}.
$$
From the boundary conditions (49), \( p^*(t_f) = -Fx^*(t_f) \). Therefore,

\[
\begin{bmatrix}
  x^*(t_f) \\
  -Fx^*(t_f)
\end{bmatrix}
= \begin{bmatrix}
  \phi_{11}(t_f,t) & \phi_{12}(t_f,t) \\
  \phi_{21}(t_f,t) & \phi_{22}(t_f,t)
\end{bmatrix}
\begin{bmatrix}
  x^*(t) \\
  p^*(t)
\end{bmatrix}.
\]

Substituting the upper equation into the lower equation,

\[
-F\phi_{11}(t_f,t)x^* - F\phi_{12}(t_f,t)p^* = \phi_{21}(t_f,t)x^* + \phi_{22}(t_f,t)p^*
\Rightarrow -F\phi_{12}(t_f,t)p^* - \phi_{22}(t_f,t)p^* = F\phi_{11}(t_f,t)x^* + \phi_{21}(t_f,t)x^*
\Rightarrow p^* = -(F\phi_{12}(t_f,t) + \phi_{22}(t_f,t))^{-1}(F\phi_{11}(t_f,t) + \phi_{21}(t_f,t))x^* = -Px^*.
\]

where \( P = (F\phi_{12}(t_f,t) + \phi_{22}(t_f,t))^{-1}(F\phi_{11}(t_f,t) + \phi_{21}(t_f,t)) \). Note that \( \Phi(t_f,t_f) = I_{2n \times 2n} \). Thus, \( \phi_{22}(t_f,t_f) = I_{n \times n} \) and \( \phi_{12}(t_f,t_f) = 0_{n \times n} \). Therefore, \( F\phi_{12}(t_f,t_f) + \phi_{22}(t_f,t_f) = I_{n \times n} \) which implies that by continuity, \( F\phi_{12}(t_f,t) + \phi_{22}(t_f,t) \) remains invertible for \( t \) close to \( t_f \). (It was proved by Kalman that the stated inverse exists.) Note that \( p^* = -\dot{p}x^* - P\dot{x}^* \). Thus

\[
Qx^* - A^T p^* = -\dot{p}x^* - P(Ax^* + BR^{-1}B^T p^*)
\Rightarrow Qx^* + A^T P x^* = -\dot{p}x^* - P(Ax^* - BR^{-1}B^T P x^*)
\Rightarrow \dot{P} = -Q - A^T P - PA + PBR^{-1}B^T P.
\]

Now \( u^* = R^{-1}B^T p^* = -R^{-1}B^T P x^* = -Kx^* \) where \( K = R^{-1}B^T P \) such that \( P \) satisfies the differential Riccati equation and \( P(t_f) = F \). (Recall that in linear systems theory, we have shown that these are sufficient conditions for optimality.)

One can apply similar arguments used above to obtain the first order necessary conditions for a tracking problem as well.

**Limitations of variational approach**: (Liberzon, Athans and Falb)

- Control set: The control set \( U \) may be bounded or in general it may have a boundary. The maximum of \( H \) could be achieved at a boundary point and in this case \( H_{\text{bf}} \) need not be 0 at the boundary point. One needs to modify the necessary condition accordingly. When the set of control inputs is discrete, variational approach does not work.

- Differentiability: Variational approach involves differentiability of \( H \) and \( L \) w.r.t. \( u \). The second order condition involves more differentiability assumptions. This rules out some obvious cost functions like \( J(u) = \int_{t_0}^{t_1} |u| dt \).

- Control perturbations: We want to consider a larger and richer class of control perturbations to obtain sharper necessary conditions.

**Remark 4.9.** Note that as far as the first limitation is concerned, if there are equality constraints on input \( u \) having algebraic equations, then one can modify the first order condition \( H_{\text{bf}} = 0 \) to the condition that \( H_{\text{bf}} \) is orthogonal to the constrained surface. When there are inequality constraints on inputs, one can consider appropriate version of KKT conditions instead of \( H_{\text{bf}} = 0 \).

The next section gives necessary conditions using Pontryagin’s maximum principle which overcomes the limitations of variational approach. Note that there are optimal control problems where there are inequality constraints involving states and inputs. We refer the reader to Kirk (p.237) on how to incorporate these constraints in the Hamiltonian.

### 4.5 Summary

- We studied various types of cost functions: Lagrange, Bolza and Mayer and showed that they are equivalent.
We studied various types of target sets and obtained first order necessary conditions along with boundary conditions for minimum energy problems using variational approach. The Hamilton’s canonical equations form the first order necessary conditions for an optimum.

- We obtained a necessary condition for the finite horizon continuous time LQR problem.
- We listed some limitations of variational approach e.g., constraints on inputs, differentiability of the Lagrangian, restriction on the type of variations and so on.

5 Pontryagin’s maximum principle (PMP): necessary conditions for optimality

We study continuous version of the Maximum principle. There is an analogous discrete version as well for discrete time optimal control problems which can be related to first order conditions for finite dimensional nonlinear programming problems (Refer Appendix).

5.1 Statement of the Maximum principle

Maximum Principle for the Basic fixed-endpoint control problem: Consider a control system \( \dot{x} = f(x, u) \), \( x(t_0) = x_0 \) with an associated cost functional \( J = \int_{t_0}^{t_f} L(x, u) dt \) where \( f, f_x, L, L_x \) are continuous and the target set is \( S = [t_0, \infty] \times \{x_1\} \). Let \( u^* : [t_0, t_f] \rightarrow U \) be an optimal control and let \( x^* : [t_0, t_f] \rightarrow \mathbb{R}^n \) be the corresponding optimal state trajectory. Then there exists a function \( p^* : [t_0, t_f] \rightarrow \mathbb{R}^n \) and a constant \( p^*_0 \leq 0 \) satisfying \( (p_0, p^*(t)) \neq (0, 0) \) for all \( t \in [t_0, t_f] \) and having the following properties:

1. \( x^* \) and \( p^* \) satisfy canonical equations
   \[
   \begin{align*}
   \dot{x}^* &= H_p(x^*, u^*, p^*, p^*_0), \\
   \dot{p}^* &= -H_x(x^*, u^*, p^*, p^*_0)
   \end{align*}
   \]
   with the boundary conditions \( x^*(t_0) = x_0 \) and \( x^*(t_f) = x_f \) where the Hamiltonian \( H : \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) is defined as
   \[
   H(x, u, p, p_0) := \langle p, f(x, u) \rangle + p_0 L(x, u).
   \]

2. For each fixed \( t \), the function \( u \rightarrow H(x^*, u, p^*, p^*_0) \) has a global maximum at \( u = u^*(t) \) i.e., the inequality
   \[
   H(x^*, u^*, p^*, p^*_0) \geq H(x^*, u, p^*, p^*_0)
   \]
   holds for all \( t \in [t_0, t_f] \) and all \( u \in U \).

3. \( H(x^*, u^*, p^*, p^*_0) = 0 \) for all \( t \in [t_0, t_f] \).

Maximum Principle for the Basic variable-endpoint control problem: Consider a control system \( \dot{x} = f(x, u) \), \( x(t_0) = x_0 \) with an associated cost functional \( J = \int_{t_0}^{t_f} L(x, u) dt \) where \( f, f_x, L, L_x \) are continuous. Suppose the target set is of the form \( S = [t_0, \infty] \times S_1 \) where
   \[
   S_1 = \{ x \in \mathbb{R}^n : h_1(x) = h_2(x) = \cdots = h_{n-k}(x) = 0 \}
   \]
where \( h_i : \mathbb{R}^n \rightarrow \mathbb{R} \in C^1, (1 \leq i \leq n - k) \) such that every \( x \in S_1 \) is a regular point. Let \( u^* : [t_0, t_f] \rightarrow U \) be an optimal control and let \( x^* : [t_0, t_f] \rightarrow \mathbb{R}^n \) be the corresponding optimal state trajectory. Then there exists a function \( p^* : [t_0, t_f] \rightarrow \mathbb{R}^n \) and a constant \( p^*_0 \leq 0 \) satisfying \( (p_0, p^*(t)) \neq (0, 0) \) for all \( t \in [t_0, t_f] \) and having the following properties:
1. \( x^* \) and \( p^* \) satisfy canonical equations

\[
\begin{align*}
\dot{x}^* &= H_p(x^*, u^*, p^*, p_0^*), \\
\dot{p}^* &= -H_x(x^*, u^*, p^*, p_0^*)
\end{align*}
\]  
(62)

with the boundary conditions \( x^*(t_0) = x_0 \) and \( x^*(t_f) \in S_1 \) where the Hamiltonian \( H : \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is defined as

\[
H(x, u, p, p_0) := \langle p, f(x, u) \rangle + p_0 L(x, u).
\]  
(63)

2. For each fixed \( t \), the function \( u \to H(x^*, u, p^*, p_0^*) \) has a global maximum at \( u = u^*(t) \) i.e., the inequality

\[
H(x^*, u^*, p^*, p_0^*) \geq H(x^*, u, p^*, p_0^*)
\]  
(64)

holds for all \( t \in [t_0, t_f] \) and all \( u \in U \).

3. \( H(x^*, u^*, p^*, p_0^*) = 0 \) for all \( t \in [t_0, t_f] \).

4. The vector \( p^*(t_f) \) is orthogonal to the tangent space to \( S_1 \) at \( x^*(t_f) \):

\[
\langle p^*(t_f), d \rangle = 0 \quad \forall d \in T_{x^*(t_f)} S_1.
\]  
(65)

5.2 A rough sketch of the proof of the maximum principle

We refer the reader to Liberzon ([2]) for the complete proof. We give a brief sketch of some ideas below from Liberzon. Note that this is not a complete proof but just an overview where we have omitted many intermediate steps and some technical details. Reader may skip this part and refer Liberzon for a rigorous treatment. This is only to keep track of main points once the proof from Liberzon is broadly understood.

1. From Lagrange to Mayer form: Define an additional state variable \( x^0 \) such that \( \dot{x}^0 = L(x, u), x^0(t_0) = 0 \) and consider the following augmented system

\[
\begin{bmatrix}
0 \\
x_0
\end{bmatrix}.
\]

The cost is rewritten as

\[
J(u) = \int_{t_0}^{t_f} \dot{x}^0(t) dt = x^0(t_f).
\]  
(66)

For fixed endpoint problem (with \( x_1 \) being the terminal state), the target set becomes \([t_0, \infty) \times \mathbb{R} \times \{x_1\} =: [t_0, \infty) \times S'\) where \( S' \) is a line in \( \mathbb{R}^{n+1} \) passing through \([0 \ 0 \ 0] \) and parallel to \( x^0 \)-axis.

Define \( y := \begin{bmatrix} x^0 \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \). Therefore,

\[
\dot{y} = \begin{bmatrix}
L(x, u) \\
f(x, u)
\end{bmatrix} =: g(y, u).
\]  
(67)

The optimal trajectory \( x^*(.) \) of the original system in \( \mathbb{R}^n \) corresponds in an obvious way to an optimal trajectory \( y^*(.) \) of the augmented system in \( \mathbb{R}^{n+1} \). The first component \( x^{0,*} \) of \( y^* \) corresponds to the evolution of the cost and the second component \( x^* \) is the optimal
We observe the effect of the optimal input $y^*$ by projection along $x^0$-axis. Let $t^*$ be the terminal time of the optimal trajectory. The Hamiltonian in Equation (63) can be represented as

$$H(x, u, p_0) = \left\langle \begin{bmatrix} p_0 \\ p \end{bmatrix}, \begin{bmatrix} L(x, u) \\ f(x, u) \end{bmatrix} \right\rangle$$  \hspace{1cm} (68)

2. Temporal control perturbation: We observe the effect of the optimal input $u^*$ applied over $[t_0, t^* + \delta t]$ instead of $u^*$ applied over $[t_0, t^*]$. Let $\tau \in \mathbb{R}$ and $\epsilon > 0$ and consider a control $u_\tau(t) := u^*(\min\{t, t^*\})$ where $t \in [t_0, t^* + \epsilon\tau]$. We want to observe the effect of this perturbed input on $y$.

Using Taylor’s expansion of $y(t^* + \epsilon\tau)$ around $t^*$, one obtains

$$y(t^* + \epsilon\tau) = y^*(t^*) + \epsilon\delta(\tau) + h.o.t.$$  \hspace{1cm} (69)

where $\delta(\tau) = g(y^*(t^*), u^*(t^*))\tau$ depends linearly on $\tau$ (refer Appendix for intermediate steps).

3. Spatial control perturbation: In here, we study the effect of perturbing the optimal input $u^*$ over an interval $I := (b - \epsilon a, b) \subset (t_0, t^*)$ (where $\epsilon, a > 0$, $b \neq t^*$ is a point of continuity of $u^*$). The perturbed control is as follows:

$$u_{w, I}(t) := \begin{cases} u^*(t), & \text{if } t \notin I, \\ w, & \text{if } t \in I \end{cases}$$  \hspace{1cm} (70)

where $w$ is a componentwise rectangular pulse over the interval $I$. We observe the effect of this perturbation on the optimal trajectory $y^*$ from $t = b - \epsilon a$ onwards. Using Taylor’s expansion and first order approximation, we can express $y(b)$ in terms of $y^*(b)$ as follows (refer Appendix for missing steps):

$$y(b) = y^*(b) + v_b(w)\epsilon a + h.o.t.$$  \hspace{1cm} (71)

where $v_b(w) := g(y^*(b), w) - g(y^*(b), u^*(b))$.  \hspace{1cm} (72)

The next step is to characterize the evolution of $y$ from $y(b)$ onwards which is the next item.

4. Variational equation: We now see how the new trajectory $y$ evolves after the perturbation in the control input stops acting. Let

$$y(t) = y^*(t) + \epsilon\psi(t) + h.o.t. := y(t, \epsilon)$$  \hspace{1cm} (73)

for $b \leq t \leq t^*$ where $\psi : [b, t^*] \rightarrow \mathbb{R}^{n+1}$ is a quantity we want to characterize. Note from Equation (71) that $\psi(b) = v_b(w)a$. From (73), $\psi(t) = y_\epsilon(t, 0)$. It turns out that one gets the following linear differential equation for $\psi$ (Refer Appendix):

$$\dot{\psi} = g_y(y^*, u^*) \psi = g_y|_\psi.$$  \hspace{1cm} (74)

5. Terminal cone: We now study the combined effect of temporal and spatial perturbations. Due to linearity of evolution of $\psi$, adding spatial perturbations in $u$ results in a trajectory $y$ obtained by adding resultant trajectories due to single perturbations over an interval. These resultant perturbations form a cone having a vertex at $y^*(t^*)$ (not convex). By taking convex combination of these perturbations, one can obtain a convex cone $co(P)$. Add the line obtained in temporal perturbation to $co(P)$. This forms the terminal cone $C_t^*$ which contains direction of perturbations due to spatial and temporal perturbations from $y^*(t^*)$. Refer Liberzon (fig. 4.8 for a picture of this cone.)
6. Topological lemma: The direction $\mu = [-1 \ 0 \ \ldots \ 0]^T \in \mathbb{R}^{n+1}$ along which the cost decreases does not intersect the interior of the terminal cone. This is due to the optimality of $u^*$ and minimality of the cost associated with the optimal trajectory (Refer Appendix).

7. Separating hyperplane: There exists a separating hyperplane which separates $\mu$ and the interior of $C_t$. The normal to this hyperplane determines the costate vector at $t^*$. The normal vector $\begin{bmatrix} p_0^* \\ p^*(t^*) \end{bmatrix} \in \mathbb{R}^{n+1}$ and the separation property can be written as

$$\langle \begin{bmatrix} p_0^* \\ p^*(t^*) \end{bmatrix}, \delta \rangle \leq 0, \ \forall \delta \text{ such that } y^*(t^*) + \delta \in C_t.$$ 

(75)

$$\langle \begin{bmatrix} p_0^* \\ p^*(t^*) \end{bmatrix}, \mu \rangle \geq 0.$$ 

(76)

The definition of $\mu$ says that $p_0^* \leq 0$.

8. Adjoint equation: Determines the evolution of the costate vector with boundary condition obtained by the normal of the separating hyperplane. This gives the canonical equation $\dot{p}^* = -H_x$ (Refer Appendix).

9. Properties of the Hamiltonian: Maximum property and $H|_s = 0$ (Refer Appendix).

10. Transversality condition: costate vector is orthogonal to the tangent space of the constraint surface at $x^*(t^*)$ (Refer Appendix).

5.3 Change of variables and more general cases

- Fixed terminal time: Suppose the terminal time $t_f$ is fixed to be equal to $t_1$. In this case, the Hamiltonian remains constant along the optimal trajectory and need not be zero. Introducing an extra state variable $x_{n+1} := t$, the system becomes

$$\dot{x} = f(x, u), \quad x(t_0) = x_0$$

$$\dot{x}_{n+1} = 1, \quad x_{n+1}(t_0) = t_0.$$ 

(77)

If the original target set was $\{t_1\} \times S_1$, then for the new system, we can write the target set as $[t_0, \infty) \times S_1 \times \{t_1\}$. Now we can apply the maximum principle for the variable-endpoint problem. The Hamiltonian for the new problem is $\bar{H} = \langle p, f \rangle + p_{n+1} + p_0 L = H + p_{n+1}$ where $H = \langle p, f \rangle + p_0 L$. Thus, $\bar{H}|_s = H|_s + p^*_n + 0 = 0$. Moreover, $\dot{p}^*_n = -\bar{H}|_x|_{x_{n+1}, u_{n+1}} = -H|_s = 0$ (since $H$ is independent of $t$). Hence, $p^*_n$ is constant which implies that $H|_s = -p^*_n$ is constant.

- Time dependent system and cost: The same idea of introducing an extra state variable $x_{n+1} = t$ can be used when the system dynamics $\dot{x} = f(t, \ldots)$ has an explicit time dependence or the running cost $L$ has an explicit time dependence. The time dependent Hamiltonian is given by $H(t, x, u, p, p_0) := \langle p, f(t, x, u) \rangle + p_0 L(t, x, u)$. Note that with the additional variable $x_{n+1}$, $\bar{H} = \langle p, f(x_{n+1}, x, u) \rangle + p_{n+1} + p_0 L = H + p_{n+1}$. Since $\bar{H}|_s = 0, H|_s = -p_{n+1}$. From the previous case, $p^*_n = -\bar{H}|_x|_{x_{n+1}, u_{n+1}} = -H|_s \neq 0$ and $H|_s = -p^*_n$ is not constant any more. Instead, we have the differential equation

$$\frac{d}{dt} H|_s = H|_s.$$ 

(78)

with the boundary condition $H|_s(t_f) = -p^*_n(t_f)$ where we have used the fact that $p^*_n = -H|_s \Rightarrow \frac{d}{dt} p_{n+1} = -\frac{d}{dt} H|_s$ and the canonical equation $\dot{p}^*_n = -H|_t$. If the terminal time $t_f$
is free, then the final value of \( x_{n+1} \) is free and the transversality condition yields \( p_{n+1}^*(t_f) = 0 \). In this case, \( H|_s(t_f) = 0 \) and integrating (78), we obtain \( H|_s(t_f) - H|_s(t) = \int_{t_f}^{t_f} H|_s(s)ds \Rightarrow H|_s(t) = -\int_{t_f}^{t_f} H|_s(s)ds \) since \( H|_s(t_f) = 0 \). The maximum principle holds in this case too except \( H|_s \) is not constant in this case.

- **Terminal cost:** Suppose there is terminal cost \( K(x_f) \) present in the cost function which is differentiable arbitrary number of times. Suppose there is no running cost i.e., \( L = 0 \) and we are dealing with free time, free endpoint problem. Therefore, there is no \( x^0 \) coordinate. The terminal cone \( C_{t^*} \) due to temporal and spatial perturbations now lies in \( \mathbb{R}^n \). By optimality, no perturbed trajectory can have a cost lower than \( K(x^*(t^*)) \). Thus, \( K \) should not decrease along any direction in \( C_{t^*} \) i.e.,

\[
\langle -K_x(x^*(t^*)), \delta \rangle \leq 0 \quad \forall \delta \text{ such that } x^*(t^*) + \delta \in C_{t^*} \tag{79}
\]

Geometrically, this implies that \( C_{t^*} \) lies on one side of the hyperplane passing through \( x^*(t^*) \) with normal \(-K_x(x^*(t^*))\) which we assume to be a non-zero vector. Comparing the above equation with (75), it suggests that \( p^*(t^*) := -K_x(x^*(t^*)) \). Let

\[
K(x_f) = K(x_0) + \int_{t_0}^{t_f} \langle K_x(x(t)), f(x, u) \rangle dt.
\]

Ignoring \( K(x_0) \), define \( L = \langle K_x, f \rangle \). For this modified problem, the Hamiltonian \( \tilde{H} = \langle \tilde{p}, f \rangle + p_0 \langle K_x, f \rangle = \langle \tilde{p} + p_0 K_x, f \rangle \) where \( \tilde{p} \) is the co-state. Applying the maximum principle, we obtain the differential equation \( \dot{p}^* = -H_x|_s = -f_x^T|_s \tilde{p}^* - p_0(f_x^T|_s)K_x|_s - p_0(K_{xx}^*|_s)f|_s \) with the boundary condition \( \tilde{p}^*(t_f) = 0 \). Suppose \( p_0^* \) is normalized to be equal to \(-1 \) (it can not be zero since \( (p_0, \tilde{p}) \neq 0 \)) and \( p_0 = p_0(t_f) \) is a constant. Defining the co-state for the original problem as

\[
p^*(t) = \tilde{p}^*(t) - K_x(x^*(t)), \tag{80}
\]

we obtain \( p^*(t_f) := -K_x(x^*(t_f)) \). Moreover,

\[
H(x, u, p) = \langle p, f \rangle = \langle \tilde{p}(t) - K_x(x(t)), f \rangle = \tilde{H}. \tag{81}
\]

Thus, \( H \) is maximized and equal to zero along the optimal trajectory.

Observe that

\[
\dot{p}^* = \tilde{p}^* - \frac{d}{dt}K_x(x^*(t)) = -\tilde{H}_x - \frac{d}{dt}K_x(x^*(t)) = -f_x^T|_s \tilde{p}^* + (f_x^T|_s)K_x|_s + K_{xx}|_s f|_s - K_{xx}|_s f|_s
\]

\[
= -f_x^T|_s \tilde{p}^* + (f_x^T|_s)K_x|_s = -f_x^T|_s (p^* + K_x|_s) + (f_x^T|_s)K_x|_s = -f_x^T|_s p^* = -H_x|_s
\]

- **Initial sets:** Now we consider the case where the initial condition \( x_0 \) is allowed to vary. We may require that

\[
\begin{bmatrix}
  x_0 \\
  x_f
\end{bmatrix} \in S_2 \subseteq \mathbb{R}^{2n}.
\]

The terminal time \( t_f \) can be either free or fixed. The transversality condition turns out to be that the vector

\[
\begin{bmatrix}
  p^*(t_0) \\
  -p^*(t_f)
\end{bmatrix}
\]

must be orthogonal to the tangent space to \( S_2 \) at

\[
\begin{bmatrix}
  x^*(t_0) \\
  x^*(t_f)
\end{bmatrix}.
\]

The total number of boundary conditions is still \( 2n \) as each additional degree of freedom for \( x^*(t_0) \) leads to one additional constraint on \( p^*(t_0) \).
5.4 Summary

1. We started with PMP for continuous time systems with free terminal time for systems with no explicit time dependence and no terminal cost involved in the cost function ($L \neq 0$, $K(x_f) = 0$). There are two cases namely, fixed endpoint and free/constrained endpoint. Three common necessary conditions are (i). first order conditions using the Hamiltonian, (ii). the maximization property of the Hamiltonian for an optimal input and (iii). the Hamiltonian being zero along the optimal trajectory. For free/constrained endpoint problems, there is an additional transversality condition. Other key points to remember are: (a). the costate vector at the terminal time is normal to the separating hyperplane (which separates the direction of decreasing cost and the terminal cone) for both fixed and free/constrained endpoint problems. (b). the costate vector at the terminal time is normal to the tangent plane at $x^*(t^*)$ for constrained endpoint problems.

2. Fixed time problems can be converted to above types using an auxiliary variable. The Hamiltonian remains constant along the optimal trajectory in this type and it need not be zero.

3. For time dependent systems and Lagrangians, the Hamiltonian need not be constant along the optimal trajectory. It turns out that $H$ satisfies $H|_{*}(t) = -\int_{t}^{t_f} H|_{*}(s)ds$.

4. For non zero terminal cost, we can convert these type of cost into a cost function with only running cost involved (Mayer to Lagrange). Thus, problems with terminal costs can be handled.

5. If initial conditions are allowed to change, then we have a slightly different transversality condition.

6. There are recent developments on discrete PMP.

6 Applications of the maximum principle

Now we will see how to use the maximum principle in applications. The necessary conditions allow us to find extremal controls and we need to find optimal control(s) from them. First of all we need to show that an optimal control exists, then we need to find whether it is unique. Uniqueness of optimal control does not imply uniqueness of extremal controls.

In this section, we consider minimum-time, minimum-fuel and minimum energy problems subject to constrained inputs. Such problems are also referred as optimal control problems with inequality constraints.

6.1 Time optimal control

For time optimal control problems, the sole measure of performance is the minimization of the transition time from an initial state to the target set.

$$\min \int_0^t dt$$

subject to $\dot{x} = f(x,u)$, $x(0) = x_0, x(t) = x_f$.

It turns out that this problem involves two aspects (Athans and Falb)

1. Finding the first instant of time at which the reachable states meets the target set.

2. Finding the control which accomplishes this.

Since the objective function is linear, optimal solution lies on the boundary of the constraint set (Refer short notes on optimization). Before we understand this in detail, let’s consider some typical examples.
6.1.1 Example: Double integrator

(Liberzon) Consider a system

\[ \dot{x} = u, \quad u \in [-1, 1]. \]  \hspace{1cm} (82)

The problem is to bring \( x \) to the origin with zero velocity at the origin in minimum time. Let \( x_1 = x \) and \( x_2 = \dot{x} \). Therefore, the optimal control problem is as follows

\[
\min \int_0^t dt \quad \text{subject to} \quad \dot{x} = Ax + bu, \quad |u| \leq 1, \quad x(0) = x_0, \quad x(t) = 0
\]

where \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Clearly, \( L = 1 \) and we have a free time fixed endpoint problem. Thus, the Hamiltonian is \( H = p^T \dot{x} - L = p_1 \dot{x}_1 + p_2 \dot{x}_2 - 1 = p_1 x_2 + p_2 u - 1 \). Let \( u^* \) be the optimal control input. The co-state \( p^* \) must satisfy the adjoint equation \( \dot{p}^* = -H_x \) i.e.,

\[
\begin{bmatrix} \dot{p}^*_1 \\ \dot{p}^*_2 \end{bmatrix} = \begin{bmatrix} -H_{x_1}^* \\ -H_{x_2}^* \end{bmatrix} = \begin{bmatrix} 0 \\ -p^*_1 \end{bmatrix}.
\]

Thus, \( p^*_1 \) is constant say \( c_1 \) and \( p^*_2(t) = -c_1 t + c_2 \) where \( c_2 \) is another constant. Now by the maximization property of the Hamiltonian at the optimal trajectory, choice of \( u^* \) at any time instant depends upon the sign of \( p^*_2 \) at that time instant i.e.,

\[
u^*(t) = \text{sgn}(p^*_2(t)) := \begin{cases} 1, & \text{if } p^*_2(t) > 0, \\ -1, & \text{if } p^*_2(t) < 0, \\ ?, & \text{if } p^*_2(t) = 0. \end{cases}
\]

The third case indicates that when \( p^*_2(t) = 0 \), \( u^*(t) \) can be arbitrary. Note that if \( p^*_2 \) is identically zero over a time interval, then \( p^*_1 \) is also identically zero which implies that \( p_0 = 0 \) since \( H = 0 \) for an optimal input. Therefore, \( p^*_2 \) can only be zero at isolated points and Equation (83) determines \( u^* \) uniquely except at finitely many points. Since \( p^*_2 \) is a linear function of time, it can have at most one zero crossing. Thus, \( u^* \) has values \( \pm 1 \) and switches between these values at most once. Such controls are called bang-bang.

Suppose \( u = 1 \) for \( \ddot{x} = u \). Integrating once, \( \dot{x} = t + a \) and integrating twice, \( x(t) = \frac{1}{2} t^2 + at + b = \frac{1}{2} \dot{x}^2 + c \) where \( c = b - a^2/2 \). This defines a family of parabolas (with appropriate directions) in \((x, \dot{x})\) plane parameterized by \( c \in \mathbb{R} \). In other words, in \( x_1 - x_2 \) plane, \( x_1 = \frac{1}{2} \dot{x}^2 + c \). Note that for \( t \to -\infty \), \( \dot{x} \to -\infty \) and for \( t \to \infty \), \( \dot{x} \to \infty \). This determines the directions of the family of parabolas as \( x \to \infty \) for \( t \to \pm \infty \).

Similarly for \( u = -1 \), we obtain \( x(t) = -\frac{1}{2} \dot{x}^2 + c \) giving another family of (directed) parabolas. Only two parabolas one each from the two families hit the origin. The union of the two curves passing through the origin is called the switching curve \( \Gamma \) defined by \( x = -\frac{1}{2} |\dot{x}| \dot{x} \).

Let \( x_0 \) be the given initial condition. Suppose \( x_0 \) lies above the switching curve. Then, applying \( u = 1 \) takes the state further away from the origin. If we apply the input \( u = -1 \) it takes the state along the parabola corresponding to \( u = -1 \) which passes through \( x_0 \) (refer the figures). Notice that this parabola intersects the switching curve at a point. If we switch the input to \( u = 1 \) at this point, then we hit the origin along the switching curve. Recall that optimal input involves at most one switching. If we choose \( u = 1 \) to start with, at some point, we must apply \( u = -1 \) to go to the origin and avoid going off to infinity. One can check that this requires at most two switchings of \( u \) hence, it cannot be optimal.

If \( x_0 \) lies below the switching curve, one must apply \( u = 1 \) first until the state hits the switching curve, and then change the input to \( u = -1 \) to travel along the switching curve to the origin. If \( x_0 \)
lies on the switching curve, then one must choose \( u = 1 \) if \( x_0 \) lies on the lower part of the switching curve and \( u = -1 \) if \( x_0 \) lies on the upper part of the switching curve. Any other choice of \( u \) would involve more than one switching of \( u \) and the control law won't be optimal.

The optimal control strategy is to apply \( u = 1 \) or \( u = -1 \) depending upon whether the initial point is below or above \( \Gamma \). If the initial point is already on \( \Gamma \), then no switching is required as one hits the origin along the switching curve.

Observe that the optimal control law obtained above can be described in the form of a state feedback law. Let \( x \) hits the origin along the switching curve. If the initial point is below or above \( \Gamma \), if the initial point is already on \( \Gamma \), then no switching is required as one hits the origin along the switching curve.

Question: What are more general classes of systems where the optimal control law can be both bang-bang and also in the state feedback form?

We refer the reader to Athans and Falb (Chapter 7.2) for optimal control problems on double integrator with target sets different from the origin.

### 6.1.2 Double integrator with constrained terminal state

Consider the same time optimal control problem mentioned in the double integrator example above with the additional condition that the terminal state is constrained to lie on the unit circle \( x_1^2 + x_2^2 = 1 \). Therefore, we need to apply PMP with the second set of conditions. The only additional condition is that \( p^*(t_f) \) is orthogonal to the constraint set. A tangent vector to the unit circle at any point say \((x_1, x_2)\) is of the form \((x_2, -x_1)\). Note that \( p^*(t_f) = (c_1, -c_1 t_f + c_2) \). From the boundary conditions, we want \( p^*(t_f) \) to be normal to the unit circle. Now \( u = \text{sgn}(p_2) = \text{sgn}(-c_1 t + c_2) \) by the Hamiltonian maximization property. Therefore, optimal control law again involves at most one switching to hit the unit circle. (Draw a picture with a unit circle and family of parabolas with \( u = 1 \) and \( u = -1 \). It can be seen how to choose \( u \) based on the initial condition such that the unit circle can be hit with at most one switching.) Once the unit circle is hit, choose \( u = -x_1 \) to maintain the state on the unit circle. (With this feedback law, \( \dot{x}_1 = x_2 \) and \( \dot{x}_2 = -x_1 \), from the phase portrait, it is clear that the state trajectory starting on the unit circle remains on the unit circle.) This gives time optimal control law to hit the unit circle in minimum time and then maintain the trajectory on the unit circle for the double integrator.

**Definition 6.1** (Normal optimal control). Consider an affine control system \( \dot{x} = f(x) + G(x)u \) where \( G = [g_1, \ldots, g_m] \) is an \( n \times m \) matrix. Let \( p \) be the costate vector of the adjoint equation. We say that we have a normal time optimal control problem in the interval \([t_0, t^*]\) if \((p^*)^T(t)g_i(x^*(t)) = 0 \) at at most countable times \( t \in [t_0, t^*] \) for \( 1 \leq i \leq m \).

**Definition 6.2** (Singular optimal control). If in the definition above, \((p^*)^Tg_i(x^*) = 0 \) for some \( i \) and \([t_1, t_2] \subset [t_0, t^*]\), then the time optimal control problem is said to be singular.

### 6.1.3 Bang-bang principle for linear systems

Consider an LTI system \( \dot{x} = Ax + Bu \) where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \). We investigate under what conditions, time-optimal controls for this system has bang-bang property. Consider an \( m \)–dimensional hypercube

\[
U = \{ u \in \mathbb{R}^m \mid u_i \in [-1, 1], i = 1, \ldots, m \}.
\]

We define below the set of states reachable from \( x(t_0) = x_0 \) at time \( t \)

\[
R^t(x_0) := \{ e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau, \ u(\tau) \in U, t_0 \leq \tau \leq t \}.
\]
Theorem 6.3 (Existence of optimal control). [Liberzon] Consider an LTI system $\dot{x} = Ax + Bu$ with a compact and convex control set $U$. Suppose the control objective is to steer $x$ from a given initial state $x(t_0) = x_0$ to a given final state $x_1$ in minimal time. Assume that $x_1 \in \mathbb{R}^n(x_0)$ for some $t \geq t_0$. Then there exists a time-optimal control.

From the above theorem, we know that there exists a time-optimal control for an LTI system when $U$ is given by an $m-$dimensional hypercube. Let $H(x, u, p, p_0) = \langle p, Ax + Bu \rangle + p_0$. The Hamiltonian maximization condition implies that for all $t \in [t_0, t^*]$,

$$\langle p^*(t), Bu^*(t) \rangle = \max_{u \in U} \langle p^*(t), Bu(t) \rangle = \max_{u \in U} \sum_{i=1}^m \langle p^*(t), b_i \rangle u_i(t).$$

Since each component $u_i$ of the optimal control can be chosen independently, it is clear that each term in the summation above must be maximized i.e.,

$$\langle p^*(t), b_i \rangle u_i^*(t) = \max_{|u_i| \leq 1} \langle p^*(t), b_i \rangle u_i, \quad 1 \leq i \leq m.$$

Therefore, we must have

$$u_i^*(t) = \text{sgn}(\langle p^*(t), b_i \rangle) := \begin{cases} 1, & \text{if } \langle p^*(t), b_i \rangle > 0, \\ -1, & \text{if } \langle p^*(t), b_i \rangle < 0, \\ 0, & \text{if } \langle p^*(t), b_i \rangle = 0. \end{cases} \quad (86)$$

We now investigate the case when $\langle p^*(t), b_i \rangle = 0$. Consider the adjoint system $\dot{p}^* = -A^T p^*$ which gives $p^*(t) = e^{-A(t-t^*)} p^*(t^*)$. Thus, $\langle p^*(t), b_i \rangle = \langle p^*(t^*), e^{-A(t-t^*)} b_i \rangle$. Since this is a real analytic function, if it vanishes on some time interval, then it vanishes for all $t$ along with its time derivatives. Suppose $\langle p^*(t), b_i \rangle = 0$ on some time interval. Thus, calculating its derivatives at $t = t^*$ and equating them to zero,

$$\langle p^*(t^*), b_i \rangle = \langle p^*(t^*), A b_i \rangle = \ldots = \langle p^*(t^*), A^{n-1} b_i \rangle = 0. \quad (87)$$

We know from our previous discussion on Hamiltonians that $p^*(t^*) \neq 0$. Then from above equation, $p^*(t^*)$ is orthogonal to $b_i, Ab_i, \ldots, A^{n-1}b_i$. Thus, at least one of the pairs $(A, b_i)$ is uncontrollable.

If $(A, b_i)$ is controllable for all $i$, then, $\langle p^*(t), b_i \rangle \neq 0$ on a time interval. Thus, $(p^*(t), b_i) = 0$ at most at countably many points. The value of $u^*$ at these points won’t change the total cost as these are countably many points and changing a function at countably many points does not change its integral. This is the bang-bang principle for LTI systems.

Theorem 6.4 (Singular and Normal optimal control, necessary and sufficient conditions). The following holds.

- The time-optimal control problem for LTI systems is singular $\iff$ the controllability matrix for at least one of the pairs $(A, b_i)$ $1 \leq i \leq m$ is rank deficient. (Athans and Falb)

- The time-optimal control problem for LTI systems is normal $\iff$ the controllability matrix for all the pairs $(A, b_i)$ $1 \leq i \leq m$ is full row rank. (Athans and Falb)

Proof. $(\Rightarrow)$ (Liberzon) Suppose the problem is singular. Then from the discussion above, it follows that at least one of the pairs $(A, b_i)$ is uncontrollable. $(\Leftarrow)$ Conversely, suppose the controllability matrix for at least one of the pairs $(A, b_i)$ $1 \leq i \leq m$ is rank deficient. Thus, by Cayley-Hamilton theorem, $e^{A(t^*-t)}b_i$ is singular. To be completed..

In the second statement, controllability for all pairs implies normality. This proves $(\Leftarrow)$. The proof of $(\Rightarrow)$ is to be completed.
Theorem 6.5 (Uniqueness of time optimal control for LTI). Normal LTI system ⇒ unique time optimal control.

Proof. Athans and Falb

Remark 6.6. For normal LTI systems, if $U$ is a hypercube, then there is a unique time optimal control steering the state from $x_0$ to $x_f$ (Refer Athans and Falb Theorem 6.7, Sontag Theorem 51). The normality assumption mentioned above for bang-bang control is quite strong. Instead of wishing every time-optimal control to be bang-bang, we could ask if every state reachable from $x_0$ by some control is reachable from $x_0$ by bang-bang control. In other words, whether the reachable sets for bang-bang control coincide with the reachable sets for all controls. This would imply that even though not all time-optimal controls are bang-bang, we can always select one that is bang-bang. This modified bang-bang principle holds for every linear control system and every control set $U$ that is a convex polyhedron (Sussmann).

Theorem 6.7 (LTI and number of switchings (Athans and Falb)). Consider an LTI normal system where all $n$ eigenvalues of $A$ are real. Suppose we want to find a time optimal control when the input set is constrained to a hypercube. Then each control input can switch at most $n-1$ times.


Example 6.8. Minimum time linear regulator:

$$\min \int_0^t dt$$

subject to $\dot{x} = Ax + Bu$, $x(0) = x_0$, $x(t^*) = 0$.

Clearly, $H = \langle p, Ax + Bu \rangle - 1$. From Hamilton’s equations, $\dot{p}^* = -A^T p^*$. From the maximum property of $H$, $u^* = \text{sgn}(\langle p^*, b_i \rangle)$. Moreover, $H|_u = \langle p^*, Ax^* + Bu^* \rangle - 1 = 0$. Note that $p^*(t) = e^{-A^T t} p^*(0)$. Therefore,

$$\langle e^{-A^T t} p^*(0), Ax^* + Bu^* \rangle - 1 = 0.$$

We guess $p(0)$ such that $\langle x(0), p(0) \rangle \geq 0$ (Athans and Falb) and compute $p(t)$ from which we find $u$ using $u_i = \text{sgn}(\langle p, b_i \rangle)$. Then solving $\dot{x} = Ax + Bu$, we check if the origin can be reached in finite time. If not, we guess $p(0)$ again and repeat the process (Athans and Falb).

Example 6.9 (Time optimal control of harmonic oscillator (Athans and Falb)). Consider the time optimal control of the harmonic oscillator. Let

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}.$$

The Hamiltonian $H = p_1 \dot{x}_1 + p_2 \dot{x}_2 - 1 = p_1 \omega x_2 - p_2 (\omega x_1 + u) - 1$. By the maximization property, $u^* = -\text{sgn}(p_2(t))$. From the first order conditions, costate equations are $\dot{p} = -A^T p = Ap$. Therefore, $p_2(t) = -p_1(0) \sin(\omega t) + p_2(0) \cos(\omega t) = a \sin(\omega t + \alpha)$. Thus, we observe that

1. Time optimal control law switches between 1 and $-1$.

2. Time optimal control law remains constant for $\frac{\pi}{\omega}$ units of time.

3. There is no upper bound on the number of switchings of $u^*$.

4. $p_2(t)$ cannot be zero over a finite time interval, hence, this is a normal time optimal control problem.
One can also consider a damped harmonic oscillator where similar observations can be made. The only difference is \( p_2(t) \) is an exponentially growing sinusoid (Athans and Falb, Section 7.8).

Note that Theorem 6.7 does not hold in this case since eigenvalues of \( A \) are purely imaginary.

**Example 6.10** (Athans and Falb). Let

\[
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} = \begin{bmatrix}
    \lambda_1 & 0 \\
    0 & \lambda_2
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} + \begin{bmatrix}
    \lambda_1 \\
    \lambda_2
\end{bmatrix} u
\]

where \( \lambda_1 \neq \lambda_2 \in \mathbb{R} \) and \( \lambda_1, \lambda_2 \neq 0 \). The Hamiltonian \( H = p_1 \lambda_1(x_1 + u) + p_2 \lambda_2(x_2 + u) - 1 = p_1 \lambda_1 x_1 + p_2 \lambda_2 x_2 + u(p_1 \lambda_1 + p_2 \lambda_2) - 1 \). By first order equations, \( \dot{p}_1 = -\lambda_1 p_1 \) and \( \dot{p}_2 = -\lambda_2 p_2 \). Therefore, \( p_1(t) = e^{-\lambda_1 t}p_1(0), p_2(t) = e^{-\lambda_2 t}p_2(0) \). By the maximization property, \( u^* = -\text{sgn}\{p_1 \lambda_1 + p_2 \lambda_2\} = -\text{sgn}\{\lambda_1 e^{-\lambda_1 t}p_1(0) + \lambda_2 e^{-\lambda_2 t}p_2(0)\} \). Note that \( \{\lambda_1 e^{-\lambda_1 t}p_1(0) + \lambda_2 e^{-\lambda_2 t}p_2(0)\} \) can have at most one zero. Therefore, \( u^* \) can switch at most once. Hence, possible values of \( u^* \) are \( \{1, -1\}, \{1, -1\}, \{-1, 1\} \).

We refer the reader to Schättler and Ledzewicz ([5]) for more examples especially for time optimal control of planar linear systems with saddles, stable/unstable nodes and so on.

**Example 6.11.** (Dubin’s car Agrachev-Sachkov [6]) Consider time optimal control of the following nonlinear system which models Dubin’s car:

\[
\dot{x}_1 = \cos(\theta), \quad \dot{x}_2 = \sin(\theta), \quad \dot{\theta} = u,
\]

\( x = (x_1, x_2) \in \mathbb{R}^2, \quad \theta \in S^1 \) (the unit circle), \( |u| \leq 1 \).

Suppose the endpoint \((x_f, \theta_f)\) and the initial condition \((x_0, \theta_0)\) is fixed. The Hamiltonian \( H = p_1 \cos(\theta) + p_2 \sin(\theta) + p_3 u - 1 \). Now using canonical equations of PMP, \( \dot{p}_1 = 0, \dot{p}_2 = 0 \) which implies that \( p_1 = c_1, p_2 = c_2, \dot{p}_3 = p_1 \sin(\theta) - p_2 \cos(\theta) = c_1 \sin(\theta) - c_2 \cos(\theta) \). Using the maximization property of the Hamiltonian, \( p_3 u^* \geq p_3 u \) which implies that \( u^* = \text{sgn}\{p_3\} \). Note that \( \dot{p}_3 = \sqrt{c_1^2 + c_2^2}\left(\frac{c_1}{\sqrt{c_1^2 + c_2^2}} \sin(\theta) - \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \cos(\theta)\right) = \alpha \sin(\theta + \beta) \) where \( \alpha = \sqrt{c_1^2 + c_2^2} \). For a detailed analysis, we refer the reader to [\theta].

**Remark 6.12.** Observe that in practical systems the ideal discontinuous switching may not be possible due to physical limitations and it may take a finite amount of time to switch the input between \( \pm 1 \). If the instantaneous switch of inputs is not applied after hitting the switching curve, then instead of following the switching curve to the origin, one travels along the integral curve to the other half of the switching curve and again due to lack of instantaneous switching, same phenomena occurs. Therefore, to hit the origin, one may have to do infinitely many switchings. This is the so called Fuller’s phenomena or chattering. To avoid infinite switchings, we may define an \( \epsilon \) ball around the origin and once the trajectory enters this ball, we are within \( \epsilon \) neighbourhood of the origin. One has to choose a trade off between the number of switchings and the \( \epsilon \) error from the target.

**Example 6.13** (nonholonomic integrator and time optimal control). Consider time optimal control of the nonholonomic integrator:

\[
\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1 u_2 - x_2 u_1,
\]

where \( |u_i| \leq 1 \) and \( x(0) = (0, 0, 0) \) and \( x(T) = (0, 0, a) \) where \( T \) is unknown and we want to find \( u_1, u_2 \) to minimize \( T \). The Hamiltonian \( H = p_1 u_1 + p_2 u_2 + p_3 (x_1 u_2 - x_2 u_1) - 1 \). Applying first

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\[\text{Thanks to Shauvik Das for raising this concern.}\]
order conditions, \( p_3 = \lambda \) (a constant) and \( \dot{p}_1 = -\lambda \dot{x}_2, \dot{p}_2 = \lambda \dot{x}_1 \). Therefore, \( p_1(t) = -\lambda x_2(t) + c_1 \) and \( p_2(t) = \lambda x_1(t) + c_2 \) where \( c_1, c_2 \) are constants. Therefore,
\[
H = (-\lambda x_2(t) + c_1)u_1 + (\lambda x_1(t) + c_2)u_2 + \lambda (x_1u_2 - x_2u_1) - 1 \\
= c_1u_1 + c_2u_2 + 2\lambda (x_1u_2 - x_2u_1) - 1 \\
= c_1\dot{x}_1 + c_2\dot{x}_2 + 2\lambda \dot{x}_3 - 1.
\]

Note that \( H = 0 \) by PMP. Therefore, at \( t = 0 \),
\[
c_1u_1(0) + c_2u_2(0) = 1.
\]
Similarly, at \( t = T \),
\[
c_1u_1(T) + c_2u_2(T) = 1.
\]

Integrating \( H = 0 \) from 0 to \( T \),
\[
0 = 2\lambda a - T \Rightarrow \lambda = \frac{T}{2a}.
\]

By the third property of PMP, \( H = 0 \) along the optimal trajectory, therefore,
\[
c_1u_1^* + c_2u_2^* + \frac{T}{a}(x_1u_2^* - x_2u_1^*) = 1, \forall t.
\]

Suppose \( u_1 = \cos(\frac{2\pi}{T}t), u_2 = \sin(\frac{2\pi}{T}t) \). These inputs must satisfy PMP necessary conditions. Therefore,
\[
H = c_1\cos(\frac{2\pi}{T}t) + c_2\sin(\frac{2\pi}{T}t) + \frac{T}{a}\frac{T}{2\pi}(\sin^2(\frac{2\pi}{T}t) + \cos^2(\frac{2\pi}{T}t)) - \cos(\frac{2\pi}{T}t)) - 1 = 0 \\
= c_1\cos(\frac{2\pi}{T}t) + c_2\sin(\frac{2\pi}{T}t) + \frac{T^2}{2\pi a}(1 - \cos(\frac{2\pi}{T}t)) - 1 = 0.
\]

Substituting \( t = 0 \), \( c_1 = 1 \). Substituting \( t = \frac{T}{4} \), \( c_2 + \frac{T^2}{2\pi a} - 1 = 0 \Rightarrow c_2 = 1 - \frac{T^2}{2\pi a} \). Substituting \( t = \frac{T}{2} \), \(-c_1 + \frac{T^2}{2\pi a} - 2 - 1 = 0 \Rightarrow -2 + \frac{T^2}{2\pi a} = 0 \Rightarrow T = \sqrt{2\pi a}. \) Therefore, sinusoids serve as time optimal inputs.

6.1.4 Geometry of time optimal control problems

Time-optimal intercept problem: Consider the following dynamical system
\[
\dot{x}(t) = f(t, x) + G(t, x)u, \; y = h(x)
\]
where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p \) and \( |u_i| \leq m_i \) (1 \( \leq i \leq m \). (By an appropriate scaling, we can assume that \( |u_i| \leq 1 \) for all \( i \).) Let \( z \) be the desired output vector. Let \( e = y - z \) be the error vector. Let \( t_0 \) and \( x(t_0) \) be the initial time and state respectively. Find the control(s) such that

1. Constraints (91) are satisfied.
2. At the terminal time \( T \), \( e(T) \in E \) where \( E \) is some specified subset of \( \mathbb{R}^p \).
3. \( T - t_0 \) is minimized.

Example 6.14. Suppose we want to control a rocket \( A \) so as to destroy the missile \( B \). At \( t = t_0 \), the position of \( B \) is \( z(t_0) \) and the position of \( A \) is \( y(t_0) \). The rocket is carried by an airplane and is released at \( t_0 \). The problem is to control \( A \) such that it hits \( B \) in shortest possible time. Let \( \delta \in \mathbb{R} \) be an arbitrarily small number. Let \{ \( e(T) \mid ||e(T)||_2 \leq \delta \} \).

We want to convert this to an optimal control problem on the state space with boundary conditions \( x(T) \in S \) where \( S \) is the target set. We assume that the system is observable. Thus, one can find the state vector \( x \) from the knowledge of the output \( y \). Therefore, \( S = \{ x(T) \mid y(T) = h(x(T)); y(T) \in Y \} \).
Definition 6.15. The constraint set $\Omega$ is a set defined by $\Omega = \{ u \mid |u_i| \leq m_i \ (1 \leq i \leq m)\}$. The set of admissible control functions is defined by

$$U_T = \{ u_{[t_0,T]} \mid u_{[t_0,T]}(t) \in \Omega \ \text{for all} \ t \in [t_0,T].$$

Definition 6.16. The set of reachable states from a given state $x_0$ at time $t_0$ is the subset $R^t(x_0) \subseteq \mathbb{R}^n$ such that for each element $x$ of $R^t(x_0)$, there exists an admissible control input which drives the state from $x_0$ to $x$ in time $t$. The boundary of this set is denoted by $\partial R^t(x_0)$.

The set of reachable states $R^t(x_0)$ satisfies the following properties:

1. $R^t(x_0)$ is closed and bounded for every $t \geq t_0$.
2. $x_0 \subseteq R^{t_1}(x_0) \subseteq R^{t_2}(x_0)$ for $t_0 < t_1 < t_2$.

We denote by $S_t$ the target set to emphasize the time dependence and by $\partial S_t$ its boundary assuming that it is closed.

1. If $R^t(x_0) \cap S_t = \emptyset$, then a solution does not exist for a time-optimal control problem.
2. If $R^t(x_0) \cap S_t \neq \emptyset$, then a solution exists for a time-optimal control problem.

Remark 6.17. If $R^t(x_0) \cap S_t = \emptyset$ and there exists $\epsilon > 0$ such that $R^{t+\epsilon}(x_0) \cap S_{t+\epsilon} \neq \emptyset$, then such solutions are called $\epsilon$-optimal solutions.

Remark 6.18. For LTI systems it can be easily verified that if the set of inputs $U$ is convex, then the set $R^t(x_0)$ is also convex.

Note that this geometric approach helps only in understanding the time-optimal control problem but gives no clue on how to solve the problem.

Remark 6.19. One can construct plenty of time optimal control problems by substituting the energy of input in the cost function of problems in the previous section (on variational control with minimum energy problems, Examples 4.1 – 4.4) with time required as the new cost.

6.1.5 Control constrained on a hypersphere

(Athans and Falb Chapter 10) Time optimal control: Suppose the input is constrained to lie on a hypersphere i.e. $\|u\| \leq m$. We consider the time optimal control problem with this constraint on the input for affine control systems $\dot{x} = f(x,t) + B(t)u(t)$. Therefore, by the Hamiltonian maximization property,

$$\langle u^*(t), B^T(t)p^*(t) \rangle \geq \langle u(t), B^T(t)p^*(t) \rangle.$$ 

Note that $B^T(t)p^*(t) = 0$ implies singular control and no information about $u^*$ can be obtained. If $B^T(t)p^*(t) \neq 0$, then the inner product is maximized when $u^*(t) = \frac{B^T(t)p^*(t)}{\|B^T(t)p^*(t)\|}$. This gives time optimal control law.

6.2 Minimum fuel problems

These type of problems arise in aerospace engineering e.g., in attitude control, rendezvous control and so on. They are more complicated than time-optimal control problems. Consider control systems of the form

$$\dot{x} = f(t, x, u).$$
Let \( \phi(t) \) denote the rate of change of fuel at time \( t \). Then the total fuel \( F \) consumed in the interval \([t_0, t_f]\) is

\[
F = \int_{t_0}^{t_f} \phi(t) dt.
\]

In general, \( \phi = h(u) \). We assume that

\[
\phi = \sum_{i=1}^{m} c_i |u_i|, \quad c_i > 0.
\]

A typical fuel optimization problem is to minimize \( F \) such that the state is transferred from \( x_0 \) to the target set \( S \). In some applications, the mass of the fuel is comparable with the mass of the overall system. In such cases, we must include it as a state variable say \( M(t) \). Then clearly, \( \frac{d}{dt} M(t) = -\sum_{i=1}^{m} c_i |u_i| \) and \( M(t) = M(t_0) - \int_{t_0}^{t_f} \sum_{i=1}^{m} c_i |u_i| dt \).

**Problem of fuel-optimal control to a moving target:** Consider the system given by (91) where \( u_i \)s are constrained to lie in the hypercube. Let

\[
J(u) = \int_{t_0}^{t_f} \sum_{i=1}^{m} c_i |u_i| dt, \quad c_i > 0.
\]  

(93)

Find \( u \) to transfer the state from \( x_0 \) to the target \( S \) such that (93) is minimized.

The Hamiltonian \( H(t, x, p, u) = p_0 \sum_{i=1}^{m} c_i |u_i| + \langle p, f(x) \rangle + \langle p, \sum_{i=1}^{m} g_i(x) u_i \rangle \). For an optimal control trajectory, the Hamiltonian maximization property must hold. Therefore,

\[
p_0 \sum_{i=1}^{m} c_i |u_i^*| + \langle p^* , \sum_{i=1}^{m} g_i(x^*) u_i^* \rangle \geq p_0 \sum_{i=1}^{m} c_i |u_i| + \langle p^* , \sum_{i=1}^{m} g_i(x) u_i \rangle.
\]  

(94)

Now similar to the analysis done for time-optimal problems, we choose \( u_i^* \) according to the value of \( \langle p^*, g_i(x^*) \rangle \). Suppose \( m = 1 \) for simplicity. Thus,

\[
p_0^* |u^*| + \langle p^*, g(x^*) u^* \rangle \geq p_0^* |u| + \langle p^*, g(x) u \rangle.
\]

\[
\Rightarrow |u^*| + \frac{1}{cp_0^*} \langle p^*, g(x^*) \rangle u^* \leq |u| + \frac{1}{cp_0^*} \langle p^*, g(x) \rangle u \quad \text{(since \( p_0 < 0 \)).}
\]

Consider \( \frac{1}{cp_0^*} \langle p^*, g(x^*) \rangle \).

1. If \( \frac{1}{cp_0^*} \langle p^*, g(x^*) \rangle > 1 \), then choose \( u^* = -1 \).
2. If \( \frac{1}{cp_0^*} \langle p^*, g(x^*) \rangle < -1 \), then choose \( u^* = 1 \).
3. If \( -1 < \frac{1}{cp_0^*} \langle p^*, g(x^*) \rangle < 1 \), then choose \( u^* = 0 \).
4. If \( \frac{1}{cp_0^*} \langle p^*, g(x^*) \rangle = 1 \), then on the lhs we have \( |u^*| + u^* \) which is minimum and equal to zero when \(-1 \leq u^* \leq 0 \).
5. If \( \frac{1}{cp_0^*} \langle p^*, g(x^*) \rangle = -1 \), then on the lhs we have \( |u^*| - u^* \) which is minimum and equal to zero when \( 0 \leq u^* \leq 1 \).

Similarly, \( u_i \)s are determined depending upon the values of \( \frac{1}{cp_0^*} \langle p^*, g_i(x^*) \rangle \). To capture properties listed above, we define a dead zone function as follows (Athans and Falb)
Definition 6.20. A function is said to be a dead zone function denoted by \( \text{dez} \{ \} \) if it satisfies the following properties: \( a = \text{dez}\{b\} \) implies that

\[
a = \begin{cases} 
0, & \text{if } |b| < 1, \\
\text{sgn}\{b\}, & \text{if } |b| > 1, \\
0 \leq a \leq 1, & \text{if } b = 1, \\
-1 \leq a \leq 0 & \text{if } b = -1.
\end{cases}
\]  

(95)

Definition 6.21. If the quantity \( \frac{1}{c_i p^*}(p^*, g_i(x^*)) \) has magnitude equal to one only on countable time instances, then we say that the fuel optimal problem is normal. If \( |\frac{1}{c_i p^*}(p^*, g_i(x^*))| = 1 \) on one or more time intervals, then we say that we have a singular optimal control problem and these intervals are called singularity intervals.

Theorem 6.22 (The On-Off principle (Athans and Falb)). Let \( U^* \) be a fuel optimal control and let \( x^* \) and \( p^* \) be the corresponding optimal state and co-state trajectories for a system defined by (??). If the problem is normal, then \( u_i^* = -\text{dez}\{\frac{1}{c_i p^*}(p^*, g_i(x^*))\} \) for \( 1 \leq i \leq m \).

Thus, for normal fuel optimal problems, the components of the fuel optimal control are piecewise constant functions of time and can switch between values 1, 0 and -1.

Remark 6.23. We refer the reader to Athans and Falb (Theorem 6.13) for sufficient conditions for normality and necessary conditions of singularity for fuel optimal control problems for LTI systems. It turns out that for LTI systems, if the fuel optimal problem is normal, then the fuel optimal control is unique (if it exists) (Theorem 6.14).

6.2.1 Minimum fuel: Examples

The On-Off principle holds for LTI systems as well. We refer the reader to Athans and Falb for conditions on normality for LTI systems. It is also shown in Athans and Falb that for normal LTI systems, fuel optimal control is unique (Theorem 6.14).

(The following discussion is from Athans and Falb Chapter 8.) For time optimal control problems, we saw that for second order systems, there is a switching curve dividing the state space into two halves. We now ask if a similar behaviour occurs for fuel optimal problems as well. It turns out that fuel-optimal solution may not exist for some systems. However, it may be possible to find \( \epsilon \)-optimal solutions. Furthermore, fuel-optimal solutions may not be unique for some systems.

Example 6.24. Consider an example of an integrator. Suppose we want to drive the state to the origin such that the fuel cost is minimum. Let \( J = \int_0^t |u(t)| dt \) and \( u(t) \in [-1, 1] \) for all \( t \) and \( t_f \) is not specified. It turns out that this system is not normal in the sense of minimum-fuel normal systems (Athans and Falb, p.666). As a consequence, it turns out that the optimal solution is not unique.

The Hamiltonian \( H = p.u - |u| \). Thus, \( \dot{p} = -H_x = 0 \) and \( p(t) = c, \) a constant. Thus, \( H = c.u - |u| \). Now we need to choose \( u \) such that the maximization property is satisfied. We may consider \( -H = |u| - cu \) and choose \( u \) to minimize \( -H \).

1. Choose \( u = 0 \) if \( |c| < 1 \).
2. Choose \( u = \text{sgn}(c) \) if \( |c| > 1 \).
3. Choose \( 0 \leq u \leq 1 \) if \( c = 1 \).
4. Choose \( -1 \leq u \leq 0 \) if \( c = 1 \).
Thus, we have one optimal control law. Now, integrating the state equation for the integrator with initial condition $x_0$ and final state 0 at the unspecified time $t_f$,

$$
\int_0^{t_f} u(t) dt = -x_0 \Rightarrow |x_0| \leq \int_0^{t_f} |u(t)| dt = J.
$$

This implies that the cost of fuel required to force $x_0$ to 0 cannot be smaller than $|x_0|$. Note that if $u^*$ is an optimal control, then $\int_0^{t_f} u^* dt = -x_0$ for some $t_f$. One can always find a function $u(t) \in [-1, 1]$ for all $t$ and a real number $t_f$ such that $\int_0^{t_f} u^* dt = -x_0$. All these control laws solve the fuel-optimal problem.

Example 6.25. Consider a double integrator system with the same cost function and input bounds as in the previous example where we want to drive the state to the origin minimizing the fuel cost. Suppose the end time is not specified. It turns out that one obtains switching curves to solve this optimization problem. The Hamiltonian for this problem is given by $H = p_1 x_2 + p_2 u - |u|$. Therefore, $\dot{p}_1 = 0 \Rightarrow p_1 = c_1$ and $\dot{p}_2 = -p_1 = -c_1 \Rightarrow p_2 = c_1 t + c_2$. Now consider $-\langle p, b \rangle = -p_2 = c_1 t - c_2$. Now $u^* = -\text{dez}\{-p_2\} = \text{dez}\{p_2\}$. Since $p_2$ is linear, the problem is normal. We refer the reader to Athans and Falb Section 8.5 and 8.6 for a detailed solution to this problem.

We also refer the reader to Kirk for fuel-optimal control examples.

### 6.3 Minimum energy problems

LQR problems and tracking problems are a typical example of minimum energy optimal control problems where the cost function is quadratic in $u$. Some problems involve more constraints on $u$ e.g., $u$ is constrained to lie in a hypercube. In this case, one uses the maximum principle with given constraints to obtain a candidate for optimal control (Athans and Falb).

#### 6.3.1 Unconstrained problems (LQR)

Suppose there are no constraints on the input. Consider a fixed time, fixed endpoint problem (LQR) where $x_f = 0$ for an LTV system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ (Athans and Falb).

$$
\min_u J(u) = \frac{1}{2} \int_0^{t_f} (x^T Q(t)x + u^T R(t)u + 2x^T M(t)u) dt
$$

where $Q, R > 0$ and $N = \begin{bmatrix} Q(t) & M(t) \\ M^T(t) & R(t) \end{bmatrix} > 0$. The Hamiltonian $H = p^T (A(t)x(t) + B(t)u(t)) - L(x, u)$. To solve the optimization problem, from the maximization property of PMP, we need to find $\text{argmax}_u \{H(x^*, p^*, .)\}$. Since $H$ is quadratic in $u$, we can use second order conditions $H_u = 0$ and $H_{uu} < 0$ to find the optimal input $u^*$. Observe that $H(x^*, p^*, u^*) \geq H(x^*, p^*, u) \Rightarrow \phi(u(t)) = -\frac{1}{2} \langle u^*, R(t)u \rangle - \langle x^*, M^T u \rangle - \langle p^*, B(t)u \rangle \geq -\frac{1}{2} \langle u, R(t)u \rangle - \langle x^*, Mu \rangle + \langle p^*, B(t)u \rangle$ (96)

Let $\phi(u(t)) = -\frac{1}{2} \langle u, R(t)u \rangle - \langle x^*, Mu \rangle + \langle p^*, B(t)u \rangle$. To have a maximum of $\phi$ at $u^*$, $\phi_u |_{u^*} = 0$ and $\phi_{uu} |_{u^*} < 0$. Note that $\phi_{uu} = -R < 0$. Thus, $u^*$ such that $\phi_u |_{u^*} = 0$ gives an optimal input. Equating $\phi_u = 0$,

$$
-R(t)u - M(t)x^*(t) + B^T P^*(t) = 0 \Rightarrow u^* = R^{-1}(t)(-M(t)x^*(t) + B^T P^*(t)).
$$

Recall that since this is a time varying system, $H$ is not constant along the optimal trajectory. Now from first order equations using Hamiltonian,

$$
\dot{x}^* = (A(t) - B(t)R^{-1}(t)M(t))x^*(t) + B(t)R^{-1}(t)B^T(t)p^*(t)
$$

$$
\dot{p}^* = (Q(t) - M(t)R^{-1}(t)M(t))x^*(t) - (A^T(t) + M(t)R^{-1}(t)B^T(t))p^*(t).
$$
Define the following matrices

\[ W_{11}(t) := (A(t) - B(t)R^{-1}(t)M(t)), \quad W_{12}(t) := B(t)R^{-1}(t)B^T(t), \]
\[ W_{21}(t) := (Q(t) - M(t)R^{-1}(t)M(t)), \quad W_{22}(t) := (-A^T(t) + M(t)R^{-1}(t)B^T(t)). \] (100) (101)

Therefore,

\[
\begin{bmatrix}
  x^*(t) \\
p^*(t)
\end{bmatrix} =
\begin{bmatrix}
  W_{11}(t) & W_{12}(t) \\
  W_{21}(t) & W_{22}(t)
\end{bmatrix}
\begin{bmatrix}
x^*(t) \\
p^*(t)
\end{bmatrix}.
\]

Let \( \Psi(t, t_0) \) be the state transition matrix for the ode above. Therefore,

\[
\begin{bmatrix}
x^*(t) \\
p^*(t)
\end{bmatrix} =
\begin{bmatrix}
  \Psi_{11}(t, t_0) & \Psi_{12}(t, t_0) \\
  \Psi_{21}(t, t_0) & \Psi_{22}(t, t_0)
\end{bmatrix}
\begin{bmatrix}
x^*(t_0) \\
p^*(t_0)
\end{bmatrix}.
\]

Now use \( x^*(t_0) = x_0 \) and \( x^*(t_f) = 0 \). Therefore, from the top equation in the above matrices, \( \Psi_{12}(t_f, t_0)p^*(t_0) = -\Psi_{11}(t_f, t_0)x_0 \). If \( \Psi_{12}(t_f, t_0) \) is invertible, then \( p^*(t_0) = -\Psi_{12}^{-1}(t_f, t_0)\Psi_{11}(t_f, t_0)x_0 \). Therefore, if \( \Psi_{12}(t_f, t_0) \) is invertible, then there exists a unique optimal control.

**Example 6.26.** Consider a double integrator (LTI system) with minimum input energy state transfer \( J(u) = \frac{1}{2} \int_0^T u^2(t) \, dt \) with \( t_f \) specified. The Hamiltonian \( H = p_1x_2 + p_2u - \frac{1}{2}u^2 \). To satisfy the maximization property, let \( H_u = 0 \Rightarrow u = p_2 \). Moreover, \( H_{uu} = -1 \) hence, \( u^* = p_2^* \) is the optimal input. From first order equations, \( \dot{p}_1 = -H_{x_1} = 0 \Rightarrow \dot{p}_1 = c_1 \text{ and } \dot{p}_2 = -H_{x_2} = p_1 \Rightarrow p_2 = c_1 + c_2. \)

Therefore, \( u^* = c_1t + c_2 \). Since this is a fixed time problem, \( H \) is constant along the optimal trajectory. The system equations become

\[
\begin{align*}
\dot{x}_2 &= u^* = c_1t + c_2 \Rightarrow x_2(t) = \frac{1}{2}c_1t^2 + c_2t + x_20 \\
\dot{x}_1 &= x_2 = \frac{1}{2}c_1t^2 + c_2t + x_20 \Rightarrow x_1(t) = \left(\frac{1}{2}c_1t^2 + c_2t + x_20\right)t + x_{10}.
\end{align*}
\]

Substituting \( x_1(t_f) = x_2(t_f) = 0 \), we can obtain \( c_1, c_2 \) and \( u^* \) as well.

**Example 6.27 (Tracking problem).** Finite horizon tracking problem with unconstrained inputs can be handled in a similar manner

**Question:** How to handle infinite horizon unconstrained regulation/tracking problem using PMP?

### 6.3.2 Constrained problems (LQR)

Consider the same LQR problem consider in 6.3.1 with input constrained to lie inside the unit hypercube (Athans and Falb 6.20). By the maximization property, Equation (96) must be satisfied subject to constraints \( |u_i| \leq 1, \ i = 1, \ldots, m \). Define

\[ w^*(t) = R^{-1}(t)(-M(t)x^*(t) + B^Tp^*(t)). \] (102)

Therefore, Equation (96) can be written as

\[ -\frac{1}{2}\langle u^*, R(t)u^* \rangle + \langle u^*, R(t)w^* \rangle \geq -\frac{1}{2}\langle u, R(t)u \rangle + \langle u, R(t)w^* \rangle \] (103)

Adding \(-\frac{1}{2}\langle w^*, R(t)w^* \rangle \) to both sides,

\[ -\frac{1}{2}\langle u^*, R(t)u^* \rangle + \langle u^*, R(t)w^* \rangle - \frac{1}{2}\langle w^*, R(t)w^* \rangle \geq -\frac{1}{2}\langle u, R(t)u \rangle + \langle u, R(t)w^* \rangle - \frac{1}{2}\langle w^*, R(t)w^* \rangle \]

\[ \Rightarrow -\langle (u^* + w^*), R(t)(u^* + w^*) \rangle \geq -\langle (u + w^*), R(t)(u + w^*) \rangle \]

\[ \Rightarrow \langle (u^* + w^*), R(t)(u^* + w^*) \rangle \leq \langle (u + w^*), R(t)(u + w^*) \rangle \] (104)
where $|u_j| \leq 1$. Since $R(t)$ is symmetric positive definite, there exists $P(t)$ such that $P^T(t)P(t) = I$ and $P^T(t)R(t)P(t) = D(t)$ where $D(t)$ is a diagonal matrix having eigenvalues of $R(t)$. Define $\psi(u(t)) = \langle (u + w^*), R(t)(u + w^*) \rangle$ where $u$ satisfies the given constraints. We know that $\psi$ attains its minimum at $u^*$. Note that $\psi(u) = 
abla_i u(t) = \begin{cases} -w_i^*(t) & \text{if } |w_i^*(t)| \leq 1 \\ 1 & \text{if } w_i^*(t) < -1 \\ -1 & \text{if } w_i^*(t) > 1 \end{cases}$ (106)

This is denoted by $u_i^* = -\text{sat}\{w_i^*\}$ where $x_i = \text{sat}\{y_i\}$ means

$x_i = \begin{cases} y_i & \text{if } |y_i| \leq 1 \\ \text{sgn}(y_i) & \text{if } |y_i^*| > 1 \end{cases}$ (107)

**Example 6.28.** Double integrator with minimum input energy state transfer. Input constrained to be in the interval $[-1, 1]$. Use Sat function to find $\text{argmax}_u H$.

**Example 6.29.** LQR for LTI systems with box constraints on the input.

**Question:** How to handle infinite horizon constrained regulation/tracking problem using PMP?

**Remark 6.30.** Observe that the fuel optimal control problems can be thought of as constrained optimization problems over the function space $L_2[t_0, t_f]$ w.r.t. $L_1$-norm and minimum energy problems can be thought of as constrained optimization problems over the function space $L_2[t_0, t_f]$ w.r.t. $L_2$-norm. These norms are not equivalent and the function spaces are also not the same. Therefore, minimum fuel and minimum energy problems are not the same.

### 6.4 Mixed type of problems

There are optimal control problems where the cost function involves time and energy, time and fuel, energy and fuel, and so on.

**Example 6.31 (minimum time and energy control).** Consider a double integrator $\ddot{x} = u$ with $|u| \leq 1$. Suppose we want to find minimum time and energy control such that $J = \int_0^t (1 + \frac{\alpha}{2} u^2) dt$ is minimized ($\alpha > 0$) where $x(0) = x_0$ and $x(t_f) = 0$ and $t_f$ is free. Clearly, $H = p_1 x_2 + p_2 u - (1 + \frac{\alpha}{2} u^2)$. Hence, $p_1 = c_1$ and $p_2 = c_2 - c_1 t$. By the maximization property, $p_2 u^* - (1 + \frac{\alpha}{2} (u^*)^2) \geq p_2 u - (1 + \frac{\alpha}{2} u^2) \Rightarrow p_2 u^* - \frac{\alpha}{2} (u^*)^2 \geq p_2 u - \frac{\alpha}{2} u^2$. Completing the squares on both sides, $-\frac{\alpha}{2} (\sqrt{\alpha} u^* + p_2)^2 \geq -\frac{\alpha}{2} (\sqrt{\alpha} u + p_2)^2$. Therefore, $u^* = -\text{sat}\{\frac{1}{\sqrt{\alpha}} p_2\}$ gives the optimal control law. Observe that by tuning $\alpha$ we can design our cost function depending upon whether energy consumption or time required is more important.

There are time-optimal control control problems with constraints on the fuel (which are effectively additional constraints on inputs) and fuel-optimal problems with time constraints. Some problems also involve a cost function which is a linear combination of time-optimal and fuel-optimal cost functions. For a slightly more detailed discussion on this, we refer the reader to Athans and Falb (section 6.15).

**Example 6.32.** Consider the double integrator system with input constraints $|u| \leq 1$. We may consider $J = \int_0^t (1 + \alpha|u|) dt$ as a cost function which combines fuel and time optimality by tuning $\alpha$. We may have $J = \int_0^t (u^2 + \alpha|u|) dt$ which combines energy and fuel optimality or $J = \int_0^t (1 + \alpha_1 u^2 + \alpha_2 |u|) dt$ which combines all three aspects. The optimal control law is obtained by the maximization property.

**Example 6.33.** Athans and Falb Section 8.8, 8.9 and 8.10 (for non linear second order systems).
6.5 Summary

Bang-bang control (time optimal), on-off control (fuel optimal), singular control.

1. Three applications of PMP as follows: Time-optimal control, fuel-optimal control, minimum energy control (LQR).

2. Time-optimal control: Existence of optimal control for LTI systems. Time optimal control with input constraints: Bang-bang control a consequence of PMP. Optimal control may be singular or normal. Singular optimal control is not uniquely defined using PMP and requires more work. In general, unique time control exists for some specific cases e.g., normal LTI systems.

3. Fuel optimal control with input constraints: On-off control a consequence of PMP. These are the hardest problems. For LTI systems, normality $\Rightarrow$ unique optimal solution provided it exists.

4. Minimum energy problems (LQR/tracking) with and without input constraints. These are “easier” amongst the three types.

5. There are mixed type of optimal control problems as well where the cost function is a combination of time/fuel/energy optimal functions.

6. It is not clear how to handle infinite horizon LQR problems using PMP.

7. There is a discrete version of PMP for discrete time systems which can be applied to discrete time systems.

7 Dynamic programming, the principle of optimality and HJB equation

In this section we introduce the idea of dynamic programming and principle of optimality to derive HJB equations which provide necessary and sufficient conditions for optimal control. We will compare HJB theory with the maximum principle studied in the previous section. As far as applications are concerned, we will study how to use HJB for LQR/tracking problems and in time-optimal control of double integrator.

Consider a discrete time system

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots, T - 1$$

where $x_k \in X$ a finite set of $N$ elements, $u_k \in U$ a finite set of $M$ elements and $T, N, M > 0$. Suppose each transition from $x_k$ to $x_{k+1}$ has a cost assigned to it and there is also a terminal cost function on $X$. For each trajectory, the total cost accumulated at time $T$ is the sum of all transition costs plus the terminal cost at $x_T$. For a given initial state $x_0$, we want to minimize this cost where the terminal state $x_T$ is free.

The most naive approach is to enumerate all possible trajectories going forward from $x_0$ up to time $T$, calculate the cost for each of them and select the optimal one. Note that there are $M^T$ possible trajectories and we need to compute $T$ additions for each of them which results in computations roughly of the order of $O(M^T T)$. 
7.1 Cost to go

Consider an alternate approach. At \( k = T \), terminal costs are known for each \( x_T \). At \( k = T - 1 \), we find to which \( x_{k+1} \) one should move so as to have the smallest two step cost (the running cost plus the terminal cost). We write this cost-to-go at each \( x_k \) and mark the selected paths. If more than one paths give the same cost, select any one of them at random. Repeat this for \( k = T - 2, \ldots, 0 \) working with the cost-to-go computed in the previous step. We claim that in this way, we have generated an optimal trajectory from each \( x_0 \) to some \( x_T \). The justification for this follows from the principle of optimality.

7.2 Principle of optimality

Lemma 7.1. If there is an optimal path \( p \) from \( x_0 \) to \( x_T \) which passes through a state \( x_l \), then for all paths from \( x_l \) to \( x_T \), the path obtained by restricting \( p \) from \( x_l \) to \( x_T \) is the optimal one.

Proof. Let \( p' \) be an optimal path from \( x_l \) to \( x_T \). Then concatenating \( p' \) with \( p \) restricted from \( x_0 \) to \( x_l \), we obtain an optimal path which contradicts the optimality of \( p \). \( \square \)

From the above lemma, it is clear that the paths discarded while going backwards can not form a part of an optimal trajectory. On the other hand, in the first approach, we could not discard any path until we reach the terminal time and do the computations. In the alternate method, at each time \( k \), for each state \( x_k \) and control \( u_k \), we need to add the cost of the transition to the cost-to-go computed for \( x_{k+1} \). Thus, at \( k = T \), we check the terminal cost for \( N \) possibilities. For \( k = T - 1 \), at each \( x_k \), there are \( M \) possible inputs available. Therefore, we need to do \( \Theta(NM) \) operations at \( k = T - 1 \) to compute the sum of the running cost and the terminal cost for all possibilities which forms cost-to-go values at each state. Note that now we can treat the cost-to-go values at the step \( k - 1 \) as terminal costs and repeated the process for \( k = T - 2 \) to obtain cost-to-go at \( T - 2 \) and so on. Thus, the number of operations required to find an optimal path are of the order \( \Theta(NMT) \) which is better than \( \Theta(M^2T) \). But as the size of \( N \) and \( M \) increases, the number of computations become extremely large. This is also known as the curse of dimensionality. Note that dynamic programming also works in the same manner for multi-source multi-sink type of problems.

Note the iterative nature of the backward propagation of cost-to-go at each step. This approach provides much more information than the forward brute force approach. It finds the optimal policy for every initial condition \( x_0 \) and also tells the optimal decision at every \( x_k \) for all \( k \). Thus, it gives an optimal control policy in the form of a state feedback law. This recursive scheme serves as an example of the general method of dynamic programming. Discrete time optimal control problems can be handled by dynamic programming.

7.2.1 Continuous time optimal control

Suppose we are dealing with a fixed-time, free-endpoint problem, i.e., the target set is \( S = \{t_1\} \times \mathbb{R}^n \). Consider a cost functional

\[
J(u) = \int_{t_0}^{t_1} L(t, x, u)dt + K(x(t_1)).
\]

Note that the cost functional also depends upon the initial condition \( x_0 \) and the initial time \( t_0 \). Consider a family of minimization problems associated with cost functionals

\[
J(t, x, u) = \int_{t}^{t_1} L(t, x(\tau), u(\tau))d\tau + K(x(t_1))
\]  \hfill (108)

where \( t \in \{t_0, t_1\} \) and \( x \in \mathbb{R}^n \). Note that this is the cost-to-go from \( t \) to \( t_1 \) for any time instant \( t \) with the state value \( x(t) \). To be more precise, define the value function as follows

\[
V(t, x) := \inf_{u_{t_0, t_1}} J(t, x, u)
\]  \hfill (109)
where \( u_{[t,t_1]} \) indicates that the control \( u \) is restricted to the interval \([t, t_1]\). Thus, \( V(t, x) \) is the optimal cost or cost-to-go from \((t, x)\). Note that an existence of an optimal control and hence the optimal cost is not assumed, that’s why we use the infimum rather than the minimum. If an optimal control exists, then the infimum turns into a minimum and \( V \) coincides with the optimal cost-to-go. In general, infimum need not exist and may be equal to \(-\infty\) for some \((t, x)\).

It is clear that the value function \( V \) must satisfy boundary condition

\[
V(t_1, x) = K(x) \quad \forall x \in \mathbb{R}^n. \tag{110}
\]

If the problem involves a more general target set \( S \subset [t_0, \infty) \times \mathbb{R}^n \), then the boundary condition would be \( V(t, x) = K(x) \) for \((t, x) \in S\).

Now we can apply the principle of optimality for continuous time problems. For every \( \Delta t \in (0, t_1 - t] \), the value function \( V \) satisfies the relation

\[
V(t, x) = \inf_{u_{[t,t+\Delta t]}} \left\{ \int_t^{t+\Delta t} L(\tau, x(\tau), u(\tau))d\tau + V(t + \Delta t, x(t + \Delta t)) \right\}. \tag{111}
\]

Thus, we have split the problem on the interval \([t, t_1]\) into two intervals \([t, t + \Delta t]\) and \([t + \Delta t, t_1]\). We now prove that \( V \) satisfies Equation (111). Let \( \tilde{V}(t, x) \) denote the rhs of Equation (111). Note that by definition (Equation (109)), \( V(t, x) \leq \tilde{V}(t, x) \). It can be shown that \( V(t, x) \geq \tilde{V}(t, x) \) (Liberzon p.160).

### 7.3 Hamilton-Jacobi-Bellman equation: necessary and sufficient conditions for optimality

**Necessity**: We now work with Equation (111) using Taylor expansions to obtain HJB equation which is in the form of a partial differential equation (PDE). Observe that

\[
x(t + \Delta t) = x(t) + \dot{x}\Delta t + o(\Delta t) = x(t) + f(t, x, u)\Delta t + o(\Delta t) \tag{112}
\]

and

\[
V(t + \Delta t, x(t + \Delta t)) = V(t, x) + V_t(t, x)\Delta t + \langle V_x(t, x), f(t, x, u)\rangle\Delta t + o(\Delta t). \tag{113}
\]

Moreover,

\[
\int_t^{t+\Delta t} L(\tau, x(\tau), u(\tau))d\tau = L(t, x(t), u(t))\Delta t + o(\Delta t). \tag{114}
\]

Substituting (113) and (114) in (111),

\[
V(t, x) = \inf_{u_{[t,t+\Delta t]}} \{ L(t, x(t), u(t))\Delta t + V(t, x) + V_t(t, x)\Delta t + \langle V_x(t, x), f(t, x, u)\rangle\Delta t + o(\Delta t) \}
\]

\[
\Rightarrow 0 = \inf_{u_{[t,t+\Delta t]}} \{ L(t, x(t), u(t))\Delta t + V_t(t, x)\Delta t + \langle V_x(t, x), f(t, x, u)\rangle\Delta t + o(\Delta t) \}.
\]

Now dividing by \( \Delta t \) and taking limit as \( \Delta t \to 0 \), we obtain,

\[
-V_t(t, x) = \inf_{u \in U} \{ L(t, x(t), u(t)) + \langle V_x(t, x), f(t, x, u)\rangle \}. \tag{115}
\]

We can rewrite this equation in a more insightful way as follows:

\[
V_t(t, x) = \sup_{u \in U} \{ -L(t, x(t), u(t)) - \langle V_x(t, x), f(t, x, u)\rangle \}.
\]

Recall that \( H(t, x, u, p) = \langle p, f(t, x, u) \rangle - L(t, x, u) \). Thus in the above equation, \(-V_x(t, x)\) plays the role of a co-state. Therefore,

\[
V_t(t, x) = \sup_{u \in U} \{ H(t, x, u, -V_x(t, x)) \}. \tag{116}
\]
So far we did not assume an existence of an optimal control. When it does exists, the infimum becomes minimum and the supremum becomes maximum.

**Sufficiency:** Note that so far we have proved necessity of HJB equations for optimality. Assuming that an optimal control exists, we now prove the sufficiency of HJB equation. Suppose \( \hat{V}(t, \mathbf{x}) \) satisfies HJB equation with boundary conditions. Suppose that a control \( \hat{u} : [t_0, t_1] \to U \) and the corresponding trajectory \( \hat{\mathbf{x}} \) satisfy

\[
L(t, \hat{\mathbf{x}}, \hat{u}) + \langle \hat{\mathbf{V}}_x(t, \hat{\mathbf{x}}), \mathbf{f}(t, \hat{\mathbf{x}}, \hat{u}) \rangle = \min_{u \in U} \{ L(t, \mathbf{x}, u) + \langle \mathbf{V}_x(t, \mathbf{x}), \mathbf{f}(t, \mathbf{x}, u) \rangle \}. \tag{117}
\]

We need to show that \( \hat{V}(t_0, \mathbf{x}_0) \) is the optimal cost and \( \hat{u} \) is the optimal control.

Applying HJB equation for \( \mathbf{x} = \hat{\mathbf{x}} \),

\[
- \hat{V}_t(t, \hat{\mathbf{x}}) = L(t, \hat{\mathbf{x}}, \hat{u}) + \langle \hat{\mathbf{V}}_x(t, \hat{\mathbf{x}}), \mathbf{f}(t, \hat{\mathbf{x}}, \hat{u}) \rangle.
\]

Moving the lhs term to rhs and using the total time derivative of \( \hat{V} \) along \( \hat{x} \),

\[
0 = L(t, \hat{x}, \hat{u}) + \frac{d}{dt} \hat{V}(t, \hat{x}).
\]

Integrating this w.r.t. \( t \) from \( t_0 \) to \( t_1 \),

\[
0 = \int_{t_0}^{t_1} L(t, \hat{x}, \hat{u}) dt + \hat{V}(t_1, \hat{x}(t_1)) - \hat{V}(t_0, \hat{x}(t_0)) \Rightarrow \hat{V}(t_0, \hat{x}(t_0)) = \int_{t_0}^{t_1} L(t, \hat{x}, \hat{u}) dt + K(\hat{x}(t_1)) = J(t_0, x_0, \hat{u}) \tag{118}
\]

where we have used the boundary condition \( \hat{V}(t_1, \hat{x}(t_1)) = K(\hat{x}(t_1)) \).

Suppose \( \mathbf{x} \) is another trajectory for an arbitrary input \( u \). Then,

\[
- \hat{V}_t(t, \mathbf{x}) \leq L(t, \mathbf{x}, u) + \langle \hat{\mathbf{V}}_x(t, \mathbf{x}), \mathbf{f}(t, \mathbf{x}, u) \rangle \Rightarrow 0 \leq L(t, \mathbf{x}, u) + \frac{d}{dt} \hat{V}_t(t, \mathbf{x}(t)).
\]

Integrating w.r.t. \( t \) from \( t_0 \) to \( t_1 \) as we did above, we obtain

\[
0 \leq \int_{t_0}^{t_1} L(t, \mathbf{x}, u) dt + \hat{V}(t_1, \mathbf{x}(t_1)) - \hat{V}(t_0, \mathbf{x}(t_0)) \Rightarrow \hat{V}(t_0, \mathbf{x}(t_0)) \leq \int_{t_0}^{t_1} L(t, \mathbf{x}, u) dt + K(\mathbf{x}(t_1)) = J(t_0, x_0, u). \tag{119}
\]

This inequality shows that \( \hat{u} \) gives the cost \( \hat{V}(t_0, \mathbf{x}_0) \) while no other control \( u \) can produce a smaller cost. Thus, \( \hat{V}(t_0, \mathbf{x}_0) \) is the optimal cost and \( \hat{u} \) is an optimal control.

**Remark 7.2.** Note that HJB equation involves solving a PDE in the value function \( V \) as well as the functional equation of taking infimum over \( u \). For some cost functions e.g., those which are quadratic in \( u \), we can use first order optimality conditions to write the optimal input \( u^* \) in in terms of the value function \( V \). In terms of the Hamiltonian (116) formulation, we can apply first order optimality conditions on the Hamiltonian i.e., \( \frac{\partial H(t, x, u^*, V_x(t, x))}{\partial u} = 0 \) to find \( u^* \) as a function of \( (t, x, V_x(t, x)) \). Once this explicit closed loop state feedback optimal input is found, substitute for \( u^* \) in the HJB equation. Now we are left with solving a PDE whose solution is the value function \( V \). Therefore, after solving the PDE, one has an explicit representation of \( V \) which in turn gives explicit state feedback law. It is clear from the construction that HJB always gives optimal closed loop state feedback law. It works for any initial and terminal condition.

**Remark 7.3.** Observe that for HJB conditions, we need the value function \( V \) to be differentiable. However, it may not always be differentiable ([2]). To handle such cases, one needs viscosity solutions of PDEs ([2]).
7.4 HJB equation vs the Maximum principle

Recall that for the maximum principle, the optimal input is given by

\[ u^*(t) = \arg\max_{u \in U} H(t, x^*, u, p^*) \]  

which is an open loop specification because \( u^* \) depends on the state \( x^* \) and the co-state \( p^* \). For HJB equation, the optimal input is given by

\[ u^*(t) = \arg\max_{u \in U} H(t, x^*, u, -V_x(t, x^*)) \]  

which is a closed loop specification. The ability to generate an optimal control policy in the form of a state feedback law is an important feature of the dynamic programming approach. However, solving a PDE given by the HJB equation is a difficult task. Thus, from computational point of view, the maximum principle has some advantages as it allows one to solve problems for which the HJB equation is analytically intractable especially for higher dimensions. But the dynamic programming approach provides more information in principle.

One may try to derive the maximum principle from HJB theory. We may define co-state via \( p^* = -V_x(t, x^*) \). Then, the maximization condition for Hamilton follows. Moreover, one can check that the boundary conditions and the canonical equations are also satisfied. However, note that we need that the value function \( V \) has a well-defined gradient which can be differentiated w.r.t. time (to satisfy the canonical equation \( \dot{p}^* = -H_x \)). This may not be true in general (Liberzon p.170).

**Remark 7.4.** What happens if instead of using cost to go and going backwards, we use cost accumulated and go forward in dynamic programming? Of course in this case we don’t have a fixed initial condition. When we go backwards, initial condition remains fixed and when we go forward, terminal condition remains fixed (for fixed endpoint problems). This is fine for fixed endpoint problems. But for free/constrained endpoint problems, if we decide to go forward, then both initial and final conditions are free in the intermediate steps of dynamic programming. However, for discrete time problems, one can convert free endpoint problems into a fixed endpoint problem by adding an auxiliary terminal node which converts the problem into free initial and fixed terminal condition. Thus, going backwards, the initial condition is held fixed and only the final state becomes free in the intermediate stages whereas while going forward, initial condition is free and terminal condition can be made fixed. For continuous time problems, going forward and adding an auxiliary terminal state for free endpoint problems makes it more complicated compared to having fixed initial state and going backwards.

7.5 Summary

- The idea of dynamic programming is to divide a problem into smaller sub-problems of similar type and solve the smaller problems iteratively.
- The principle of optimality says that every sub-path of an optimal path is itself optimal.
- These two ideas lead to HJB necessary and sufficient conditions for optimal control.
- HJB gives a closed loop optimal control law i.e., optimal control in terms of state feedback where as the maximum principle gives open loop control.
- HJB involves partial differential equation and one uses viscosity solutions for non differentiable functions.

**Short summary:** Now we give a quick summary and a bird’s eye view of variational approach, Maximum principle and HJB theory.
1. Using variational approach, one obtains necessary and sufficient conditions for weak and strong minima. A solution is a weak minimum if it minimizes the cost function over differentiable perturbations. It is said to be a strong minimum if it minimizes the cost function over continuous perturbations with at most finitely many corners. Euler-Lagrange equations give first order necessary conditions for weak minima. For strong minima, one needs Weierstrass-Erdmann corner conditions for necessity. There are second order sufficient conditions as well. We need the cost function to be differentiable.

2. To handle perturbations with jump discontinuities, to allow non differentiable cost functions and to allow the possibility that the minima may be achieved on the boundary of the constraint sets, one uses Pontryagin’s Maximum principle which provides necessary conditions for optimal control. These conditions help us to limit our search space for optimal control. However, there are cases e.g., infinite horizon LQR problems which can not be handled (as far as we know?) by PMP.

3. For sufficient conditions, when one considers the completely general problem described in the previous item, dynamic programming approach leads to HJB equations which give necessary and sufficient conditions for optimal control. One needs viscosity solutions of PDEs to handle non differentiable cases. These PDEs are hard to solve in general therefore, the Maximum principle is used in most of the practical cases.

4. We mostly considered continuous time optimal control. One can relate discrete time optimal control problems to finite dimensional optimization problems (Refer Appendix). There is a discrete version of the Maximum principle and dynamic programming gives necessary and sufficient conditions for optimality for discrete time optimal control problems. Dynamic programming approach is used to handle discrete time optimal control problems. The PMP and dynamic programming (HJB) complement one another.

8 Applications of HJB and dynamic programming

HJB equation and dynamic programming gives necessary and sufficient conditions for optimal control problem for (minimum energy problems such as) continuous and discrete time LQR, tracking and time optimal control problems. We can handle infinite horizon LQR and time optimal control of discrete systems as well. For fuel optimal control problems, differentiability of the Lagrangian running cost function is an issue hence, PMP seems more suitable. For continuous time problems, we need a solution of a PDE and for discrete time problems, we need a solution of a recursive equation. It turns out that for unconstrained LQR problems one can guess the solution of a PDE/recursive equation. But for more general problems e.g., energy minimization with constrained inputs, unconstrained energy minimization for non linear systems, time optimal control, we may not be able to guess the form of a solution of a PDE and need to solve the PDE analytically to find the optimal control law. That is why PMP is preferred in applications.

Dynamic programming has plenty of applications. For more detailed applications of dynamic programming, we refer the reader to Bertsekas ([15]).

8.1 Dynamic programming and the shortest path problem

Consider a single source and single sink directed graph with edge weights. The length of a path between any two nodes is given by the sum of the edge weights of the edges along that path. We want to find the shortest path from the source $s$ to the sink $t$. The dynamic programming approach works as follows:

- For each node $v$ in the graph, find the length of the shortest path from $v$ to $t$. This gives the minimum cost to go at each node.
8.2 Finite horizon LQR: continuous time

We have already seen the necessary conditions for the finite horizon LQR problem in section 4.4. This is as far the maximum principle can take us. We need to employ other tools to investigate whether a solution to Riccati differential equation exists and whether the control law is indeed optimal.

We now show that this control law is optimal using HJB theory. For a fixed time free endpoint finite horizon LQR problem for an LTI system $\dot{x} = Ax + Bu$, the HJB equation becomes

$$-V_t(t,x) = \inf_u \{ \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \langle V_x, Ax + Bu \rangle \}$$  \hspace{1cm} (122)

and the boundary condition is

$$V(t_f, x) = \frac{1}{2} x^T(t_f) F x(t_f).$$  \hspace{1cm} (123)

Since $R > 0$, by second order conditions, infimum in Equation (122) is in fact minimum and achieved at

$$u = -R^{-1} B^T V_x(t,x).$$  \hspace{1cm} (124)

Substituting this in Equation (122),

$$-V_t(t,x) = \frac{1}{2} x^T Q x + \frac{1}{2} V_x(t,x)^T BR^{-1} R B^T V_x(t,x) + V_x(t,x)^T Ax - V_x(t,x)^T BR^{-1} B^T V_x(t,x)$$

$$= \frac{1}{2} x^T Q x + V_x(t,x)^T Ax - \frac{1}{2} V_x(t,x)^T BR^{-1} B^T V_x(t,x).$$  \hspace{1cm} (125)

We need to $V$ satisfying the above equation and it must satisfy (109). Let $V(t,x) = \frac{1}{2} x^T P(t)x$ such that $V(t_f, x) = x^T F x$. Thus,

$$V_x(t,x) = P(t)x$$  \hspace{1cm} (126)

$$V_t(t,x) = \frac{1}{2} x^T \dot{P}(t)x.$$  \hspace{1cm} (127)

Note that since we are taking a partial derivative of $V$ w.r.t. to $t$, (127) holds. (For total derivative of $V$ w.r.t. $t$, we would have $\frac{d}{dt} V = \frac{1}{2} (\dot{x}^T P(t)x + x^T \dot{P}(t)x + x^T P(t) \dot{x}).$) Substituting (126) in (125),

$$-V_t(t,x) = \frac{1}{2} x^T Q x + x^T P(t) Ax - \frac{1}{2} x^T P(t) BR^{-1} B^T P(t)x$$

$$= \frac{1}{2} x^T Q x + \frac{1}{2} x^T (P(t)A + A^T P(t))x - \frac{1}{2} x^T P(t) BR^{-1} B^T P(t)x$$

$$= \frac{1}{2} x^T (Q + P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t))x.$$  \hspace{1cm} (128)

From (127) and (128),

$$-\frac{1}{2} x^T(\dot{P})x = \frac{1}{2} x^T (Q + P(t)A + A^T P(t) - P(t)BR^{-1}B^T P(t))x$$

$$\Rightarrow \frac{1}{2} x^T(A^T P + PA + \dot{P} - P(t)BR^{-1}B^T P(t) + Q)x = 0.$$  

Thus, $P$ must satisfy the differential Riccati equation with boundary conditions. Now by sufficiency of HJB equations, this indeed gives an optimal control law. (We showed sufficiency of this using completion of squares in linear systems theory without referring to HJB conditions).
Remark 8.1. For a free time LQR problem, we need to find the terminal time first. This can be found from the necessary conditions of variational approach or PMP. One can also find a candidate for an optimal control law by variational approach/PMP. Once terminal time and an optimal candidate is found, sufficiency can be checked by using HJB equations. This policy can be followed in general. One obtains possible candidates for optimal solution using variational principle/PMP and then sufficiency can be checked using HJB. We also refer the reader to Remark 8.4 for an alternate approach. An optimal solution may not always exist for free time problems ([4] Section 6.19).

Example 8.2. Recall Example 4.1. Suppose we want to solve this problem using a state feedback law i.e., closed loop control. Note that the optimal input is $u = B^T P(t)x$ where $P(t)$ satisfies DRE $\dot{P} = PA + A^T P - PBB^T P$. Since there is no terminal cost, $P(t_f) = 0$ but this is an equilibrium point of the DRE so we cannot solve the DRE backwards in time to obtain a state feedback law.

Now shift the origin to the target state $x_f$ and $\bar{x} := x - x_f$. Thus, we want to drive the state from $\bar{x}(0) = -x_f$ to $\bar{x} = 0$ in time $t_f$. Let $\bar{x}^T Q_f \bar{x}$ be the terminal cost to penalize the deviation of $\bar{x}$ from $x_f$. Now this is a regulator problem which can be solved using the same DRE with $P(t_f) = Q_f$.

8.3 Infinite horizon LQR: continuous time

The arguments for this problem are slightly technical and we refer the reader to Liberzon. Note that we proved sufficiency in linear systems theory using completion of squares arguments and feedback invariants.

It turns out that the closed loop system is stable when $(Q,A)$ is observable and $(A,B)$ is controllable. The optimal feedback control is given in terms of a solution of algebraic Riccati equation $P \succeq 0$ and the optimal cost is $x_0^T P x_0$.

Remark 8.3. We list the following observations for LQR problems.

1. The method of completion of squares and feedback invariants can be used to solve both finite and infinite horizon LQR problems (continuous as well as discrete). This gives sufficient conditions (existence of a solution of Riccati equation) for optimal input (Refer short notes on linear systems).

2. By variational methods, we obtain necessary conditions for finite horizon LQR for continuous time systems. For finite horizon discrete LQR, finite dimensional optimality conditions can be used. Variational approach or PMP doesn't work for infinite horizon LQR (as far as my knowledge goes but this could be wrong or debatable). PMP gives necessary conditions for finite horizon LQR.

3. Dynamic programming and HJB gives necessary and sufficient conditions for both finite and infinite horizon, discrete and continuous time systems.

8.4 LQR: discrete time

Finite/infinite horizon discrete LQR follows similarly (We refer the reader to EE363, Stanford lecture notes). Note that finite horizon discrete LQR problem can be considered as a finite dimensional optimization problem and can be solved independently without using any of these methods.

Consider finite horizon discrete LQR using dynamic programming. We want to minimize

$$J(U) = \sum_{k=0}^{N-1} (x^T(k)Qx(k) + u^T(k)Ru(k)) + x^T(N)Q_f x(N)$$

(129)
where \( U = (u(0), \ldots, u(N-1)) \), \( Q, Q_f \geq 0 \), \( R > 0 \). For \( t = 0, \ldots, N \), define the value function as follows:

\[
V_t(z) = \min_{u(t), \ldots, u(N-1)} \sum_{k=t}^{N-1} (x^T(k)Qx(k) + u^T(k)Ru(k)) + x^T(N)Q_fx(N)
\]

subject to \( x(t) = z \), \( x(k + 1) = Ax(k) + Bu \) \( \quad (130) \)

where \( V_t(z) \) is the minimum cost to go. Let \( V_{t+1}(z) = z^TP_{t+1}z \) where \( P_t \geq 0 \) and \( P_t \) is symmetric. Note that \( V_N(z) = z^TQ_fz \) which implies that \( P_N = Q_f \). Suppose we know \( V_{t+1} \), then we want to know the optimal choice for \( u(t) \). Let \( w = u(t) \). Then,

\[
V_t(z) = \min_{w} (z^TQz + w^TRw + V_{t+1}(Az + Bw))
\]

where \( z^TQz + w^TRw \) is the cost incurred at time \( t \) and \( V_{t+1}(Az + Bw) \) is the minimum cost to go at \( t + 1 \). Therefore, \( u^*(t) = \text{argmin}_w (w^TRw + V_{t+1}(Az + Bw)) \). Observe that

\[
V_t(z) = z^TQz + \min_{w} (w^TRw + (Az + Bw)^TP_{t+1}(Az + Bw)).
\]

Setting the derivative of the above equation w.r.t. \( w \) equal to zero,

\[
2w^TR + 2(Az + Bw)^TP_{t+1}B = 0 \Rightarrow w^* = -(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}Az.
\] \( \quad (131) \)

Substituting (131) in the expression \( V_t(z) \),

\[
(w^*)^TRw = z^T(A^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}R(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}Az
\]

\[
V_{t+1}(Az + Bw^*) = z^T(A^TP_{t+1}A - 2A^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}A + A^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}A)z.
\]

Using the previous two equations one obtains

\[
V_t(z) = z^T(Q + A^TP_{t+1}A - 2A^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}A + A^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}R(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}A)z
\]

\[
= z^T(Q + A^TP_{t+1}A - 2A^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}A + A^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}A)z
\]

\[
= z^T(Q + A^TP_{t+1}A - 2A^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}A)z
\]

\[
= z^TP_tz
\]

where \( P_t = Q + A^TP_{t+1}A - 2A^TP_{t+1}B(R + B^TP_{t+1}B)^{-1}B^TP_{t+1}A \) and \( P_N = Q_f \). Note that \( P_t \) is said to satisfy difference Riccati equation.

**Remark 8.4.** How to solve free time finite horizon discrete LQR problems? We can find an optimal control law for a some arbitrarily fixed terminal time and compute the total cost. Now this cost is a function of the terminal time. For free time problems, optimal time can be found by varying the terminal time and observing how the total cost changes. We need to find the time for which the cost is the smallest.

**Infinite horizon discrete LQR:** For a discrete LTI system, choose \( u(0), u(1), \ldots \) to minimize

\[
J = \sum_{k=0}^{\infty} (x^T(k)Qx(k) + u^T(k)Ru(k))
\] \( \quad (132) \)
assuming that \((A, B)\) is controllable. Define the value function as follows:

\[
V(z) = \min_{u(0)\ldots}\sum_{k=0}^{\infty}(x^T(k)Qx(k) + u^T(k)Ru(k))
\]

subject to \(x(0) = z, x(k + 1) = Ax(k) + Bu.\) \((133)\)

Note that the value function is independent of time to go which is always infinite. Let \(V(z) = z^TPz.\) Observe that

\[
V(z) = \min_w(z^TQz + w^TRw + V(Az + Bw))
\]

\[\Rightarrow z^TPz = \min_w(z^TQz + w^TRw + (Az + Bw)^TP(Az + Bw)).\]

The minimizing \(w\) is \(w^* = -(R + B^TPB)^{-1}B^TPAz.\) Therefore,

\[
z^TPz = z^TQz + (w^*)^TRw^* + (Az + Bw^*)^TP(Az + Bw^*)
 = z^T(Q + A^TPA - A^TPB(R + B^TPB)^{-1}B^TPA)z.
\]

Therefore, \(P\) must satisfy discrete ARE \(P = Q + A^TPA - A^TPB(R + B^TPB)^{-1}B^TPA.\)

### 8.5 Optimal control of affine nonlinear systems and HJB

Consider an affine nonlinear control system

\[
\dot{x} = f(x) + G(x)u
\]

which is controllable. Suppose we want to minimize the energy \(\frac{1}{2} \int_0^T u^T u dt\) to go from \(x_0\) to \(x_f\) in time \(T.\) Using HJB equation,

\[
-V_i(t, x) = \inf_u \{\frac{1}{2} u^T u + \langle V_x, f(x) + G(x)u \rangle\}.
\]

Using first order optimality conditions, \(u^* = -G(x)^T V_x(t, x)\). Substituting in the above equation one obtains the following PDE,

\[
-V_i(t, x) = \frac{1}{2} V_x(t, x)^T G(x) G(x)^T V_x(t, x) + \langle V_x, f(x) - G(x) G(x)^T V_x(t, x) \rangle.
\]

\[\Rightarrow \langle V_x, f(x) \rangle - \frac{1}{2} V_x(t, x)^T G(x) G(x)^T V_x(t, x).\] \((135)\)

For general Lagrangians, substitute \(L(t, x, u)\) instead of \(\frac{1}{2} u^T u\) in the above equations. One can observe that these are hard pdes in general. For LQR type, one can guess the form of the solution of pdes but not in general.

For drift free systems, e.g., nonholonomic integrator, \(f = 0.\) The HJB equations are

\[
V_i(t, x) = \frac{1}{2} \begin{bmatrix} V_{x_1} & V_{x_2} & V_{x_3} \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_2 \\
0 & 1 & x_1 \
-x_2 & x_1 & x_2^2 + x_3^2 \end{bmatrix} \begin{bmatrix} V_{x_1} \\
V_{x_2} \\
V_{x_3} \end{bmatrix}.
\]
8.6 HJB equation and time-optimal control: Double integrator

Recall time-optimal control problem and the example of the double integrator. For each point \((x_1, x_2)\) in the state space, there exists a minimum time \(t^* = t^*(x_1, x_2)\) required to force \((x_1, x_2)\) to \((0, 0)\). It can be shown that this minimum time \(t^*\) is a solution of HJB equation (Athans and Falb).

**Definition 8.5** (Athans and Falb). The set of states which can be forced to zero in time \(t^*\) are called \(t^*\) minimum isochrone and this set is denoted by \(S(t^*)\).

To see isochrones for double integrator, refer Athans and Falb (p.515).

The minimum cost for time optimal control law for the double integrator is \(J(t^*, x^*, u^*) = V(t^*, x^*) = t^*(x_1, x_2)\). The HJB equation becomes

\[
\frac{\partial t^*}{\partial t} = H(t^*, x^*, u^*, -t_{xx}^*(x_1, x_2))
\]

where \(H(t^*, x^*, u^*, -t_{xx}^*(x_1, x_2)) = \sup_{u \in U} \{H(t, x, u, -t_{xx}(x_1, x_2))\}\). Note that \(\frac{\partial t^*}{\partial t} = 0\) and \(H = -1 + p_1 x_2 + p_2 u\) where \(p = -t_{xx}(x_1, x_2)\). Thus, \(p_1^* = -\frac{\partial t^*}{\partial x_1}\) and \(p_2^* = -\frac{\partial t^*}{\partial x_2}\). Therefore, \(H = -1 - \frac{\partial t^*}{\partial x_1} x_2 - \frac{\partial t^*}{\partial x_2} u = 0\).

Recall the time optimal control law obtained using the maximum principle for this problem (section 6.1.1). We observed that if \(x_1 \geq -\frac{1}{2}|x_2|x_2\), then \(u^* = -1\) and if \(x_1 < -\frac{1}{2}|x_2|x_2\), then \(u^* = 1\). The time required to force the state to the origin is the sum of the time required to force the state to the switching curve and the time required to go to the origin on the switching curve. It turns out that by elimination of variables,

\[
t^*(x_1, x_2) = \begin{cases} 
  x_2 + \sqrt{4x_1 + 2x_2^2}, & \text{if } x_1 > -\frac{1}{2}|x_2|x_2, \\
  -x_2 + \sqrt{-4x_1 + 2x_2^2}, & \text{if } x_1 > -\frac{1}{2}|x_2|x_2, \\
  |x_2|, & \text{if } x_1 = -\frac{1}{2}|x_2|x_2.
\end{cases}
\]

Substituting the above expression for \(t^*\), we can check that the HJB equations are satisfied and the control law proposed earlier using the maximum principle is indeed optimal.

**Time optimal control of discrete systems:** Question: How to solve time optimal control problem for discrete systems using dynamic programming?

First, we give an answer without using any dynamic programming. Consider a time optimal problem for discrete time LTI systems driving the state from the origin to \(x_f\) in minimum time with input constraints \(|u_i| \leq 1\). Let \(C_t(A, B) = [B \ AB \ \cdots \ A^tB]\) be the controllability matrix at time \(t\). Note that \(x(t) = C_{t-1}(A, B)u\) where \(u = [u^T(0) \ \cdots \ u^T(t-1)]^T\). For the state transfer to be possible, \(x_f\) must lie in the column span of \(C_t(A, B)\) and input constraints must be satisfied. Let \(t'\) be the first instance when \(x_f\) lies in the column span of \(C_{t'}(A, B)\). Find the least norm solution w.r.t. \(\infty\)-norm i.e.,

\[
\min \|u\|_{\infty} \quad \text{subject to} \quad x_f = C_{t'}(A, B)u.
\]

If the input constraints \(|u_i| \leq 1\) are violated in the solution of the above problem, then consider \(t' + 1\) and so on until we get a least \(\infty\)-norm solution satisfying the required input constraints. **Why does this algorithm terminate after finite iterations?** The answer lies in existence of time
optimal control for discrete systems. It can be shown that time optimal control exists for discrete
time LTI systems by using similar arguments used for continuous LTI systems. Note that in the
statement of the existence theorem, there is an assumption that the state is reachable in finite time
using inputs from the control set \( U \). This guarantees the termination of the above iterative process.

Now we give a solution using dynamic programming. Starting from the terminal state, travel
backwards iteratively. If the initial condition lies \( i \) hops away from the terminal state (with \( i \) being
the least of such instances), then the corresponding path with the corresponding set of inputs forms
a time optimal trajectory for discrete systems. Consider an LTI system \( x(t+1) = Ax(t) + Bu(t) \) where \( A \)
is invertible and \( |u_i(t)| \leq 1 \). Suppose \( x(0) = 0 \) and \( x(T) = x_f \) where \( T \) is the minimum
time (i.e., assume time optimal control exists). Therefore, \( x(T) = A x(T-1) + Bu(T-1) \). Since
\( A \) is invertible,

\[
x(T-1) = A^{-1} x(T) - A^{-1} Bu(T-1)
\]

\[
\Rightarrow x(T-2) = A^{-2} x(T) - A^{-2} Bu(T-1) - A^{-1} Bu(T-2)
\]

\[
\Rightarrow x(T-i) = A^{-i} x(T) - A^{-i} Bu(T-1) - A^{-i+1} Bu(T-2) - \ldots - A^{-1} Bu(T-i).
\]

This turns out to be the same problem with as the one before. We need to find the minimum \( i \) and
the input sequence such that \( 0 = x(T-i) = A^{-i} x(T) - A^{-i} Bu(T-1) - A^{-i+1} Bu(T-2) - \ldots - A^{-1} Bu(T-i) \).

**Example 8.6.** Consider \( x(t+1) = x(t) + u(t) \) where \( |u(t)| \leq 1 \). Let \( x(0) = 0 \). Find time optimal
control to go from 0 to \( x_f = 4.5 \). Clearly, an input sequence \( u(0) = u(1) = u(2) = u(3) = 1 \) and
\( u(4) = 0.5 \) does the required transfer. No transfer is possible for \( t < 4 \) and the optimal control is
not unique.

**Remark 8.7.** One can assign unit cost to an edge between any two nodes and convert the shortest
time problem into a shortest path problem.  

### 8.7 Estimation, Stochastic control and LQG

Stochastic LQR involves regulation of dynamical systems in presence of disturbances. Optimal
input can be obtained using dynamic programming in terms of state feedback laws. LQG involves
presence of disturbances in state dynamics and presence of noise in observed outputs. One wants
to build state estimates and implement feedback laws using these estimates to solve LQR problem.
This section is based on Lecture notes by Boyd ([11]).

**Discrete LQR in the presence of disturbances:** Consider discrete stochastic control problem.
Let \( x(t+1) = Ax(t) + Bu(t) + w(t) \) where \( w \) is the disturbance. Let \( E(w(t)) = 0 \) and
\( E(w(t)w^T(t)) = W \). Suppose \( x(0) \) is independent of \( w(t) \) and \( E(x(0)x^T(0)) = X \). The objective is
to minimize

\[
J = E(x^T(N)Q_f x(N) + \sum_{t=0}^{N-1} x^T(t)Q x(t) + u^T(t)R u(t))
\]

where \( Q, Q_f, R > 0 \). Thus, we want to choose control inputs \( u(t) \) to minimize the cost function.

Let \( V_t(z) \) be the value function where \( z = x(t) \),

\[
V_t(z) = \min_u E(x^T(N)Q_f x(N) + \sum_{t=0}^{N-1} x^T(t)Q x(t) + u^T(t)R u(t))
\]

subject to \( x(t+1) = Ax(t) + Bu(t) + w(t) \). Note that \( V_N(z) = z^T Q_f z \). Let \( V_t(x(t)) = x^T(t)P(t)x(t) + q(t) \), \( P(t) \geq 0 \), \( P(N) = Q_f \), \( q(N) = 0 \). Let \( v = u(t) \) Therefore,

\[
V_t(z) = z^T Q z + \min_v \{ v^T R v + E(V_{t+1}(x(t+1))) \}
\]

\[
u^*(t) = \underset{v}{\arg\min} \{ v^T R v + E(V_{t+1}(x(t+1))) \}
\]

---

\(^6\)Thanks to Kewal Bajaj for this important observation.
Therefore, using $V_t(x(t)) = x^T(t)P(t)x(t) + q(t),
\[ V_t(z) = z^TQz + \min_{\nu}\{v^TRv + E(x^T(t+1)P(t+1)x(t+1) + q(t+1))\} \]
$ 
$V_t(z) = z^TQz + \min_{\nu}\{v^TRv + E((Az + Bv + w(t))^TP(t+1)(Az + Bv + w(t)) + q(t+1))\} \)

Using $E(w^T(t)P(t+1)w(t)) = \text{Tr}(WP(t+1))$, 
\[ V_t(z) = z^TQz + \text{Tr}(WP(t+1)) + q(t+1) + \min_{\nu}\{v^TRv + (Az + Bv)^TP(t+1)(Az + Bv)\} \]
Thus, we get the same recursion as deterministic LQR with an added constant. It turns out that just like the deterministic LQR, the optimal input is given by $u^*(t) = K(t)x(t)$ where $K(t) = -(B^TP(t+1)B + R)^{-1}B^TP(t+1)A + Q$
\[ P(t) = A^TP(t+1)A - A^TP(t+1)B(B^TP(t+1)B + R)^{-1}B^TP(t+1)A + Q \]
$q(t) = q(t+1) + \text{Tr}(WP(t+1))$. 
Thus, the optimal control is same as that of deterministic LQR, independent of $X$ and $W$. The optimal cost is $E(V_0(x(0))) = \text{Tr}(WP(0)) + q(0) = \text{Tr}(WP(0)) + \sum_{t=1}^{N} \text{Tr}(WP(t))$.

One can also consider infinite horizon cost $J = \lim_{N \to \infty} \frac{1}{N} E \sum_{t=0}^{N-1} (x^T(t)Qx(t) + u^T(t)Ru(t))$. Moreover, continuous time problems can also be handled in a similar manner. **Discrete LQG:** Consider a linear stochastic system $x(t+1) = Ax(t) + Bu(t) + w(t), \ t = 0, 1, \ldots, N - 1$ where $w(t)$ is the disturbance or process noise. Let $y(t) = Cx(t) + v(t), \ t = 0, 1, \ldots, N$ be the output where $v(t)$ is the measurement noise. Suppose $x(0), w(t), v(t)$ are all zero mean, Gaussian and independent with covariance matrices $X, W, V$ respectively. Let $Y(t) = (y(0), \ldots, y(t))$ be the entire output history.

Consider the following optimization problem
\[ \min_u J = E(\sum_{t=0}^{N-1} x^T(t)Qx(t) + u^T(t)Ru(t) + x^T(N)Qx(N)) \] (137)
with $Q \geq 0, R > 0$. This is a discrete LQG problem. We now give steps towards building the solution.

1. (Towards a solution): We want to obtain output feedback laws $u(t) = \phi_t(Y(t))$ for $t = 0, \ldots, N - 1$ to minimize the cost function.

One wants to have state estimates $\hat{x}$ using output measurements $Y(t)$ which are then used to implement state feedback laws. From linear systems theory (refer short notes on linear systems), one expects optimal control $\phi_t(Y(t)) = K(t)E(x(t)|Y(t))$ where $K(t)$ is the optimal feedback matrix gain matrix for the associated LQR problem and $E(x(t)|Y(t))$ is the MMSE estimate of $x(t)$ from $Y(t)$ (Refer section 11.3 from Appendix on estimation and Kalman filter). The separation principle holds for LQG as well.

2. (LQR control): Define $P(N) = Q$. It turns out that $P(t-1) = A^TP(t)A + Q - A^TP(t)B(R + B^TP(t)B)^{-1}B^TP(t)A$ and $K(t) = -(R + B^TP(t)B)^{-1}B^TP(t)A$ for $t = 0, \ldots, N - 1$. If states were directly accessible, we know from LQR problem that $u(t) = K(t)x(t)$ gives optimal feedback laws. Since states are not directly accessible, we need Kalman filter to obtain state estimates.

3. (Kalman filter and state estimates): Let $V_t(Y_t)$ be the optimal cost to go from $t$ onwards i.e.,
\[ V_t(Y_t) = \min_{\phi_1, \ldots, \phi_N} E(\sum_{t=0}^{N-1} x^T(t)Qx(t) + u^T(t)Ru(t) + x^T(N)Qx(N) \mid Y(t)). \] (138)
We show that $V_i(Y_t)$ is a quadratic function plus a constant, in fact, $V_i(Y_t) = \hat{x}^T(t)P(t)\hat{x}(t) + q(t)$, $t = 0, \ldots, N$, where $P(t)$ is the LQR cost to go matrix. Suppose $x(N)|Y(N)$ is Gaussian with mean $\hat{x}(N)|Y(N)$ and the error covariance $\Sigma_N$. Let $e(t) = x(N)|Y(N) - \hat{x}(N)|Y(N)$. Note that

$$V_N(Y_N) = E(x^T(N)Qx(N)|Y_N) = \hat{x}^T(N)Q\hat{x}(N) + e_N^TQe_N$$

$$= \hat{x}^T(N)Q\hat{x}(N) + \text{Tr}(Q\Sigma_N) \quad (139)$$

Let $P_N = Q$ and $q_N = \text{Tr}(Q\Sigma_N)$. The dynamic programming equation is

$$V_i(Y_t) = \min_{u(t)} E(x^T(t)Qx(t) + u^T(t)R_u(t) + V_{i+1}(Y_{t+1})|Y(t)).$$

Assuming $V_{i+1}(Y_{t+1}) = \hat{x}^T(t + 1)P(t + 1)\hat{x}(t + 1) + q(t + 1)$,

$$V_i(Y_t) = \min_{u(t)} E(x^T(t)Qx(t) + u^T(t)R_u(t) + \hat{x}^T(t + 1)P(t + 1)\hat{x}(t + 1) + q(t + 1)|Y(t))$$

$$= E(x^T(t)Qx(t)|Y(t)) + q(t + 1) + \min_{u(t)} (u^T(t)R_u(t) + E(\hat{x}^T(t + 1)P(t + 1)\hat{x}(t + 1)|Y(t))) \quad (140)$$

Suppose $x(t)|Y(t)$ has mean $\hat{x}(t)$ and the error covariance is $\Sigma_{t|t}$. Therefore, in the previous equation,

$$E(x^T(t)Qx(t)|Y(t)) = \hat{x}^T(t)Q\hat{x}(t) + \text{Tr}(Q\Sigma_{t|t}). \quad (141)$$

For Kalman filter, analogous to (197), state estimates satisfy

$$\hat{x}(t + 1) = \hat{x}(t) + A\hat{x}(t) + B(t) + L_t(y(t) - \hat{y}(t)) \quad (142)$$

where $L_t = A\Sigma_{t|t-1}C^T(C\Sigma_{t|t-1}C^T + V)^{-1}$. To find the covariance of the error between observed and estimated output,

$$E(y(t) - \hat{y}(t))(y(t) - \hat{y}(t))^T = C\Sigma_{t|t-1}C^T + V. \quad (143)$$

We now expand the term $E(\hat{x}^T(t + 1)P(t + 1)\hat{x}(t + 1)|Y(t))$ from (140) using (142) and (143)

$$E(\hat{x}^T(t + 1)P(t + 1)\hat{x}(t + 1)|Y(t)) = \hat{x}^T(t)A^TP(t + 1)A\hat{x}(t) + u^T(t)B^TP(t + 1)Bu(t) + 2\hat{x}^T(t)A^TP(t + 1)Bu(t) + \text{Tr}(L_t^TP(t + 1)L_t)(C\Sigma_{t|t-1}C^T + V)$$

Substituting for $L_t$ in the last term, $P(t + 1)L_t(C\Sigma_{t|t-1}C^T + V) = P(t + 1)A\Sigma_{t|t-1}C^T$. Therefore, $\text{Tr}(L_t^TP(t + 1)L_t)(C\Sigma_{t|t-1}C^T + V) = \text{Tr}(P(t + 1)A\Sigma_{t|t-1}C^T)(C\Sigma_{t|t-1}C^T + V)^{-1}C\Sigma_{t|t-1}A^T$. Recall that (Appendix 11.3) from (194),

$$A\Sigma_{t|t}A^T = A\Sigma_{t|t-1}A^T - A\Sigma_{t|t-1}C^T(C\Sigma_{t|t-1}C^T + V)^{-1}C\Sigma_{t|t-1}A^T \Rightarrow$$

$$\text{Tr}(P(t + 1)A\Sigma_{t|t-1}C^T)(C\Sigma_{t|t-1}C^T + V)^{-1}C\Sigma_{t|t-1}A^T = \text{Tr}(P(t + 1)A(\Sigma_{t|t-1} - \Sigma_{t|t})A^T) \quad (144)$$

Therefore,

$$E(\hat{x}^T(t + 1)P(t + 1)\hat{x}(t + 1)|Y(t)) = \hat{x}^T(t)A^TP(t + 1)A\hat{x}(t) + u^T(t)B^TP(t + 1)Bu(t) + 2\hat{x}^T(t)A^TP(t + 1)Bu(t) + \text{Tr}(P(t + 1)A(\Sigma_{t|t-1} - \Sigma_{t|t})A^T) \quad (145)$$

Substituting (145) in (140),

$$V_i(Y_t) = \hat{x}^T(t)(Q + A^TP(t + 1)A)\hat{x}(t) + q(t + 1) + \text{Tr}(Q\Sigma_{t|t}) +$$

$$\text{Tr}(P(t + 1)A(\Sigma_{t|t-1} - \Sigma_{t|t})A^T + \text{min}_{u(t)}(u^T(t)(R + B^TP(t + 1)B)u(t) +$$

$$2\hat{x}^T(t)A^TP(t + 1)Bu(t)). \quad (146)$$
Now we are in the same situation as in discrete deterministic LQR. The optimal control is given by \( \phi^*_i(Y_i) = u^*(t) = K(t)\dot{x}(t) \) where \( K(t) = -(R + B^TP(t + 1)B)^{-1}B^TP(t + 1)A \). Suppose following equations are satisfied

\[
P(t) = A^TP(t + 1)A + Q - A^TP(t + 1)B(R + B^TP(t + 1)B)B^TP(t + 1)
\]

\[
q(t) = q(t + 1) + \text{Tr}(Q\Sigma_{q|t}) + \text{Tr}P(t + 1)A(\Sigma_{q|t-1} - \Sigma_{q|t})A^T.
\]

Then, \( V_i(Y_i) = \ddot{x}^T(t)P(t)\dot{x}(t) + q(t) \). Thus, we need the above equations to be satisfied so that the guessed form of \( V_i(Y_i) \) is indeed true.

For infinite horizon LQG problem, one obtains steady state values and equations (AREs), \( K \) is the steady state feedback gain and \( L \) is the steady state Kalman filter gain. **Continuous time LQG**: Consider a linear system \( \dot{x} = Ax + Bu + w, \ y = Cx + v \) where \( w, v \) are uncorrelated zero mean, Gaussian white noise stochastic processes with covariance matrices \( W \) and \( V \) respectively. We showed in short notes on linear systems ([14], Kalman-Bucy filter) that the optimal estimator has the form \( \dot{x} = Ax + Bu + L(t)(y - Cx) \) where \( L(t) = P(t)C^TV^{-1}, \ P(t) = E(x - \dot{x})(x - \dot{x})^T \) and \( \dot{P} = AP + PA^T - PC^TV^{-1}CP + W^{-1}, \ P(0) = E(x(0)x^T(0)). \)

Suppose we want to minimize

\[
\min_u J = E\left[ \frac{1}{2} \int_{t_0}^{t_f} x^T(t)Qx(t) + u^T(t)Ru(t) \, dt + \frac{1}{2} E(x^T(t_f)Fx(t_f)) \right]
\]  

(147)

Now just like the discrete LQG case, we use HJB equation to obtain optimal control law using Kalman filter and continuous time LQR.

### 8.8 Other applications

Other applications of dynamic programming include the traveling salesman problem, knapsack problem and so on. Dynamic programming can be used to solve many combinatorial optimization problems. Furthermore, it is also used in optimal control involving differential games e.g., two player differential games where one player tries to maximize the cost functional and the other tries to minimize it.

### 8.9 Summary

- Dynamic programming has applications in optimal control and combinatorial (discrete) optimization problems.

- Both finite and infinite horizon continuous and discrete time tracking and regulator problems can be handled. Note that for LQR/LQG problems for linear systems, we guessed the form of the cost to go function \( V(t, x) \) and verified that it is indeed a solution of HJB equation which is a PDE. So we did not really solve the PDE to find the optimal control law. For regulator problems with non linear systems or for time optimal control problems, it may not be obvious to guess a solution of a PDE and verify that it is indeed a solution. For discrete systems, the cost to go function is a recursive function which is guessed in discrete LQR/LQG problems. But it is not obvious to guess this function for an arbitrary discrete non linear system.

- Time optimal control problems for both continuous and discrete time systems can be handled.

- All discrete time optimal control problems can be handled by dynamic programming approach. Fixed time problems can be handled using finite dimensional optimization methods. Free time problems are handled using dynamic programming. Stochastic control, state estimation and control e.g., LQG problems can be handled by dynamic programming approach.
9 Maximum principle on manifolds

Liberzon, Manifolds: locally look like $\mathbb{R}^n$, can attach local coordinates. Curves form one dimensional manifolds. Curves can be parameterized using a map from the real line. The derivative of this map gives a tangent vector at any point. Tangent and cotangent spaces: Consider a point $x$ on a manifold $M$ of dimension $n$ and consider all possible curves passing through $x$. The collection of tangent vectors to all these curves forms a tangent space $T_xM$ of $M$ at $x$. Standard basis vectors $e_1, \ldots, e_n$ form a basis for $T_xM$. Moreover, $TM = \cup_{x \in M} T_xM$ is called the tangent bundle of $M$. This is also a manifold in itself of dimension $2n$. Let $f : M \to N$ be a map between two manifolds. Then $df : T_xM \to T_{f(x)}N$ is a linear map between the corresponding tangent spaces. If we attach to each point $x$ in $M$ an element $v$ from $T_xM$, then this forms a vector field on $M$. The element $(x, v)$ lives in the tangent bundle $TM$. A vector field defines a flow on a manifold i.e., starting from any initial point $x_0$, if one travels along the path specified by a vector field, one obtains an integral curve. Let $f : M \to TM$ where $f(x) = (x, f(x)) = (x, v)$ for some smooth function $f$. The integral curves form solutions of ODE $\dot{x} = f(x)$ on $M$.

The dual space of $T_xM$ is $T^*_xM$ which forms a set of all linear maps from $T_xM$ to $\mathbb{R}$. This is called the cotangent space and $T^*M = \cup_{x \in M} T^*_xM$ is called the cotangent bundle of $M$. Let $g : M \to \mathbb{R}$ be a smooth function. Then at any point $x \in M$, the directional derivative of $g$ in the direction of a vector $v \in T_xM$ is $\nabla g |_x \cdot v = \frac{\partial g}{\partial x^1} v_1 + \ldots + \frac{\partial g}{\partial x^n} v_n$. Thus, $\nabla g |_x \in T^*_xM$. If $x_1, \ldots, x_n$ are local coordinates of $M$, then $dx_1, \ldots, dx_n$ form a basis for $T^*_xM$ and for any $g : M \to \mathbb{R}$, $dg = \frac{\partial g}{\partial x^1} dx_1 + \ldots + \frac{\partial g}{\partial x^n} dx_n \in T^*_xM$. Note that $dx_i(e_j) = \delta_{ij}$. Moreover, $(x_1, \ldots, x_n, dx_1, \ldots, dx_n)$ form local coordinates on cotangent bundle which is a manifold of dimension $2n$.

Now for a dynamical system $\dot{x} = f(x, u)$ on $M$, it is clear that $f(x, u)$ lives in the tangent space $T_xM$ at each point $x \in M$. Thus, the state $x$ live in $M$ and the velocity vector $\dot{x}$ lives in the tangent space. We want to figure out where the co-state vector $p$ that $p$ lives in the cotangent space since it maps elements $x$ in the tangent space to $\mathbb{R}$.

Note that any vector field on a manifold defines a flow $\Phi_{\tau} : M \to M$ for each $\tau > 0$. If $x \in M$, then the vector field defines an ode and corresponding integral curves on $M$. Once an initial condition is determined, one can uniquely determine the evolution of that initial point along the integral curve. The point $\Phi_{\tau}(x)$ denotes the evolution of $x$ starting at time $t = 0$ to time $t = \tau$. The differential $d\Phi_{\tau} : T_xM \to T_{\Phi_{\tau}(x)}M$. Thus, $d\Phi_{\tau}$ pushes the tangent vectors at $x$ forward at $\Phi_{\tau}(x)$.

Suppose $p|_{\Phi_{\tau}(x)} \in T^*_{\Phi_{\tau}(x)}M$ be a cotangent vector at $\Phi_{\tau}(x)$. One can define its pull back $p|_x \in T^*_xM$ as follows. For $v \in T_xM$,

$$p|_x(v) = p|_{\Phi_{\tau}(x)}(d\Phi_{\tau}|_x(v)).$$

Recall that for co-state vectors we always have terminal conditions, and solutions are obtained by solving the ode backwards.

For the Mayer type cost functions (all three of them are equivalent), the Hamiltonian $H = H(x, p, u) = p(f(x, u))$. Note that cotangent vectors are also known as 1–forms (refer some differential Geometry book) which take a single vector and spit out a scalar in other words they form all linear maps from $T_xM \to \mathbb{R}$. There are 2–forms as well which take two vectors and spit out a scalar. These 2–forms are linear in both variables and antisymmetric, in specific, let $\omega$ be a two form. Then $\omega(v, w) = -\omega(w, v) \Rightarrow \omega(v, v) = 0$ with $\omega$ linear in both variables. Two forms form maps of the form $T_xM \times T_xM \to \mathbb{R}$. There are more general $k$–forms which take $k$ vectors and give a scalar. The determinant is an example of $n$–form. These forms are useful in computing area elements. Consider a 2–form $dx_1 \wedge dx_2$ defined on $\mathbb{R}^2$ (which is tangent space to itself) as follows

$$(dx_1 \wedge dx_2)(v, w) := dx_1(v)dx_2(w) - dx_2(v)dx_1(w) = v_1w_2 - w_1v_2.$$
which is the area of the parallelogram spanned by vectors \( v \) and \( w \). The term \( dx_1 \wedge dx_2 \) is known as the wedge product of 1–forms \( dx_1 \) and \( dx_2 \). One can similarly define \( k \)--forms. A wedge product of a \( k \)--form and \( l \)--form produces \((k+l)\)--form as long as \( k+l \leq n \). These \( k \)--forms form a vector space. (They are more than a vector space but we don’t bother about that right now.)

Let \((x_1,\ldots,x_n,p_1,\ldots,p_n)\) be local coordinates of a point \((x,p)\in T^*M\). Then \(dx_1,\ldots, dx_n, dp_1, \ldots, dp_n\) form basis vectors of cotangent space \(T^*_{(x,p)} M\) of \(M\). Consider the following 2–form on \(T^*M\)

\[
\omega^2 := dx_1 \wedge dp_1 + \ldots + dx_n \wedge dp_n.
\]

Let \((v_1, w_1), (v_2, w_2) \in T_{(x,p)}T^*M\). Note that \(\omega^2 : T_{(x,p)}T^*M \times T_{(x,p)}T^*M \to \mathbb{R}\).

\[
\omega^2((v_1, w_1), (v_2, w_2)) = \sum_{i=1}^{n} v_1(i)w_2(i) - w_1(i)v_2(i).
\]

Observe that if we fix one argument of \(\omega^2\) fixed, then we get one form i.e., an element of the cotangent space. Suppose we fix an element \((v, w) \in T_{(x,p)}T^*M\). Then \(\omega^2((v, w), \cdot)\) belongs to \(T^*_{(x,p)}T^*M\). Thus, by fixing one argument \(\omega^2\) induces a linear map

\[
\omega^2((v, w), \cdot) : T_{(x,p)}T^*M \to T^*_{(x,p)}T^*M
\]

\[
(v, w) \mapsto \omega^2_{(v, w)}
\]

where \(\omega^2_{(v, w)}((y, z)) = \omega^2((v, w), (y, z))\). Now we find the matrix representation of the linear map

\[
\omega^2((v, w), \cdot) : T_{(x,p)}T^*M \to T^*_{(x,p)}T^*M
\]

\[
(v, w) \mapsto \omega^2_{(v, w)}
\]

Note that the basis for \(T_{(x,p)}T^*M\) is \(\{e_1, \ldots, e_{2n}\}\) and the basis for \(T^*_{(x,p)}T^*M\) is \(\{dx_1, \ldots, dx_n, dp_1, \ldots, dp_n\}\). We find the image of \(e_i\) (\(1 \leq i \leq n\)) using Equation (149). Suppose \(1 \leq j \leq n\), then

\[
\omega^2((e_j, 0), (v_2, w_2)) = w_2(j) = dp_j(v_2, w_2).
\]

Now for \(n+1 \leq j \leq 2n\), then

\[
\omega^2((0, e_{j-n+1}), (v_2, w_2)) = -v_2(j-n+1) = -dx_{j-n+1}(v_2, w_2)
\]

Thus, the matrix is given by

\[
\begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix}
\]

We can view the Hamiltonian \(H\) as a function on the cotangent bundle \(T^*M\) (i.e., a function of variables \(x, p\)) parameterized additionally by controls \(u\). The differential \(dH\) gives a co-vector (element of the cotangent space) on \(T^*M\) at each point. Applying the inverse of the linear map constructed above to \(dH\), we obtain a tangent vector at each point in \(T^*M\) which forms a vector field on \(T^*M\). This is called the Hamiltonian vector field denoted by \(\tilde{H}\). Note that \(dH = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial p} dp\). Therefore,

\[
\tilde{H}(x, p) = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}^{-1} dH = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial x} \\ -\frac{\partial H}{\partial x} \end{bmatrix}.
\]

The ODE for this vector field is

\[
\begin{bmatrix}
\dot{x} \\
\dot{p}
\end{bmatrix} = \tilde{H}(x, p) = \begin{bmatrix}
\frac{\partial H}{\partial x} \\
-\frac{\partial H}{\partial x}
\end{bmatrix}
\]

and we obtain the same canonical equations. Now we can apply the maximum principle to obtain optimal control candidates for systems over manifolds.
10 \( \mathcal{L}_2 \) gain and \( H_\infty \) control

(Liberzon) Consider an LTI system \( \dot{x} = Ax + Bu, y = Cx \) and the cost functional

\[
J(u) = \int_0^\infty (\gamma u(t)^T u(t) - \frac{1}{\gamma} y(t)^T y(t)) dt
\]

(154)

\( \gamma > 0 \). The problem is to find an input which minimizes this cost functional. Observe that for \( u = 0 \), \( J(u) < 0 \), thus the optimal cost is negative. Suppose there exists \( P \in \mathbb{R}^{n \times n} \) satisfying the following properties:

1. \( P = P^T \geq 0 \).
2. \( P \) is a solution of the ARE

\[
PA + A^T P + \frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P = 0.
\]

(155)

3. The matrix \( A + \frac{1}{\gamma} BB^T P \) is Hurwitz.

We claim that the optimal cost is

\[
V(x_0) = -x_0^T P x_0
\]

(156)

and the optimal control is the linear state feedback

\[
u^*(t) = \frac{1}{\gamma} B^T P x^*.
\]

(157)

To prove this claim, define \( \hat{V}(x) = x^T P x \).

\[
\frac{d}{dt} \hat{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T (PA + A^T P)x + 2x^T PB u
\]

\[
= x^T (PA + A^T P + \frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P)x + 2x^T PB u - x^T (\frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P)x
\]

where we have added and subtracted some new terms to get the expression for ARE mentioned above. We now complete the square in \( u \) to obtain the claimed expression for an optimal input. Note that

\[
2x^T PB u - \frac{1}{\gamma} x^T PBB^T P x + \gamma u^T u - \gamma u^T u = -\gamma (u - \frac{1}{\gamma} B^T P x)^T (u - \frac{1}{\gamma} B^T P x) + \gamma u^T u.
\]

Using the above expression and \( y = Cx \) in the expression for \( \frac{d}{dt} \hat{V}(x) \),

\[
\frac{d}{dt} \hat{V}(x) = x^T (PA + A^T P + \frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P)x - \gamma (u - \frac{1}{\gamma} B^T P x)^T (u - \frac{1}{\gamma} B^T P x) + \gamma u^T u - \frac{1}{\gamma} y^T y.
\]

(158)

Since \( P \) satisfies the algebraic Riccati equation,

\[
\frac{d}{dt} \hat{V}(x) = -\gamma (u - \frac{1}{\gamma} B^T P x)^T (u - \frac{1}{\gamma} B^T P x) + \gamma u^T u - \frac{1}{\gamma} y^T y.
\]

(159)

Consider the following auxiliary finite horizon cost

\[
\bar{J}^{t_1}(u) := \int_{t_0}^{t_1} (\gamma u^T u - \frac{1}{\gamma} y^T y) dt - x(t_1)^T P x(t_1).
\]

(160)
Using (159),

\[ \bar{J}^1(u) = \int_{t_0}^{t_1} (\gamma|u - \frac{1}{\gamma}B^TPx|^2 + \frac{d}{dt}V(x))dt - x(t_1)^TPx(t_1) \]
\[ = \int_{t_0}^{t_1} \gamma|u - \frac{1}{\gamma}B^TPx|^2dt - x_0^TPx_0. \] (161)

Therefore, it is clear that (156) and (157) are optimal cost and optimal input respectively for the cost functional (160). The next step is to show that they are also optimal for the cost functional (154). Note that since \( P \geq 0 \),

\[ J(u) = \lim_{t_1 \to \infty} \int_{t_0}^{t_1} (\gamma u^TP - \frac{1}{\gamma}y^Ty)dt \geq \lim_{t_1 \to \infty} (\int_{t_0}^{t_1} (\gamma u^TP - \frac{1}{\gamma}y^Ty)dt - x(t_1)^TPx(t_1)) = \lim_{t_1 \to \infty} \bar{J}^1(u). \]

We have already shown that \( \lim_{t_1 \to \infty} \bar{J}^1(u) \geq \lim_{t_1 \to \infty} \bar{J}^1(u^*) = -x_0^TPx_0. \) Note that

\[ J(u^*) = \lim_{t_1 \to \infty} \int_{t_0}^{t_1} (\gamma(u^*)^TPu^* - \frac{1}{\gamma}(y^T)^Ty^*)dt - x(t_1)^TPx(t_1) + x(t_1)^TPx(t_1) \]
\[ = \lim_{t_1 \to \infty} (\bar{J}^1(u^*) + x(t_1)^TPx(t_1)) = -x_0^TPx_0 + \lim_{t_1 \to \infty} x(t_1)^TPx(t_1) \]

Now with the control input \( u^*(t) = \frac{1}{\gamma}B^TPx^* \), the feedback matrix becomes \( A + \frac{1}{\gamma}B^TP \) which is Hurwitz by the hypothesis. Thus, the system is exponentially stable for this control law and \( \lim_{t_1 \to \infty} x(t_1) = 0 \). Therefore, \( J(u^*) = -x_0^TPx_0 \) thus, the lower bound is achieved for the input \( u^* \) which proves the claim.

Note that the cost function is not designed to make both \( u \) and \( y \) small. It is designed to make \( y \) large relative to \( u \). One can regard \( u \) as a disturbance rather than control which tries to make the output large, with the optimal input being the worst case disturbance. Let’s try to understand this better. We showed that \( J(u) \geq J(u^*) = -x_0^TPx_0 \). Suppose \( x_0 = 0 \). Therefore,

\[ 0 \leq \int_{t_0}^{\infty} (\gamma|u|^2 - \frac{1}{\gamma}|y|^2)dt \Rightarrow \int_{t_0}^{\infty} \gamma|u|^2 dt \geq \int_{t_0}^{\infty} \frac{1}{\gamma}|y|^2 dt \Rightarrow \gamma \geq \frac{\sqrt{\int_{t_0}^{\infty} |y|^2 dt}}{\int_{t_0}^{\infty} |u|^2 dt} \] (162)

The quantity \( \gamma \) is called the \( \mathcal{L}_2 \) gain of the system. For a given \( \gamma \), if there exists a symmetric matrix \( P \) satisfying the three properties listed above, then the system’s \( \mathcal{L}_2 \) gain is less than or equal to \( \gamma \). Converse is also true, i.e., if the system’s \( \mathcal{L}_2 \) gain is less than or equal to \( \gamma \), then a matrix \( P \) with indicated properties exists (we don’t prove this here). Suppose instead of the optimal control problem mentioned above, we seek for sufficient conditions for \( \mathcal{L}_2 \) gain to be less than or equal to \( \gamma \). It turns out that the conditions on matrix \( P \) can be relaxed. Suppose that \( P \geq 0 \) satisfies the following algebraic Riccati inequality

\[ PA + A^TP + \frac{1}{\gamma}C^TC + \frac{1}{\gamma}PBB^TP \leq 0. \] (163)

Then, from (159),

\[ \frac{d}{dt}V(x) \leq \gamma u^TPu - \frac{1}{\gamma}y^Ty. \] (164)

Integrating from \( t_0 \) to \( T \),

\[ V(x(T)) - V(x_0) \leq \gamma \int_{t_0}^{T} |u|^2 dt - \frac{1}{\gamma} \int_{t_0}^{T} |y|^2 dt. \]
Letting $T \to \infty$, we obtain $-\dot{V}(x_0) \leq \gamma \int_{t_0}^{\infty} \|u\|^2 dt - \frac{1}{\gamma} \int_{t_0}^{\infty} \|y\|^2 dt$ which is the same inequality obtained before. For $x_0 = 0$, we get (162).

In frequency domain, the transfer matrix $G(s) = C(sI - A)^{-1}B$. Using Parseval’s theorem, one can show that the $L_2$ gain is equal to $\sup_{\omega \in \mathbb{R}} \sigma_1(G(j\omega))$ where $\sigma_1(G(j\omega))$ is the largest singular value of $G(j\omega)$. The $L_2$ gain is also called the $H_\infty$ norm of the system.

10.1 $H_\infty$ control

Suppose that an input $u$ and a disturbance $w$ both are present in the system. We want to stabilize the system attenuating the unknown disturbances. This disturbance attenuation will be measured by the $L_2$ gain of the closed loop system. Consider a system

$$\dot{x} = Ax + Bu + Dw, y = Cx, z = Ex$$

where $y$ is the measured output and $z$ is the controlled output. Consider a simple case where $C = I$ and suppose that our controllers only take static state feedback i.e., of the form $u = Kx$. The control objective is to design the feedback gain matrix $K$ so that

1. The closed loop system matrix $A_{cl} := A + BK$ is Hurwitz.
2. The $L_2$ gain of the closed loop system from $w$ to $z$ (which is the $H_\infty$ norm of the closed loop transfer function) is less than or equal to the specified value $\gamma$.

Note that this is not an optimal control problem since we are not looking to minimize $\gamma$ (although we want $\gamma$ to be as small as possible). Controls solving such kind of problems are called suboptimal.

It is clear from the discussion on $L_2$ gain that we need a matrix $P \geq 0$ such that $PA_{cl} + A_{cl}^TP + \frac{1}{\gamma}E^TE + \frac{1}{\gamma}PDD^TP \leq 0$ so that the second property gets satisfied. To satisfy the first property as well, we need $P > 0$ and above ARI must be strict i.e., $PA_{cl} + A_{cl}^TP + \frac{1}{\gamma}E^TE + \frac{1}{\gamma}PDD^TP < 0$. The strict inequality implies that

$$PA_{cl} + A_{cl}^TP + \frac{1}{\gamma}E^TE + \frac{1}{\gamma}PDD^TP + \epsilon Q = 0$$  \hspace{1cm} (165)$$

where $Q > 0, \epsilon > 0$. Thus $PA_{cl} + A_{cl}^TP < 0$ which implies that $A_{cl}$ is Hurwitz by Lyapunov condition. Let $R > 0$ and suppose there is a solution $P > 0$ to the following ARE

$$PA_{cl} + A_{cl}^TP + \frac{1}{\gamma}E^TE + \frac{1}{\gamma}PDD^TP - \frac{1}{\epsilon}PBR^{-1}B^TP + \epsilon Q = 0.$$  \hspace{1cm} (166)$$

Then defining $K := -\frac{1}{\epsilon}R^{-1}B^TP$, $u = Kx$ enforces (165) and achieves both control objectives. Conversely, it can be shown that if a system is stabilizable with an $L_2$ gain less than or equal to $\gamma$, then (166) is solvable for $P > 0$.

The general case where full state is not measured is more complicated (Liberzon). One needs a state estimator (LQG estimator in the presence of noise) to implement state feedback laws. For an ideal case without any noise, if the system is observable or detectable, one can build an asymptotic state estimator so that $\hat{x} \to x$ as $t \to \infty$ and $e = x - \hat{x} \to 0$ (refer short notes on linear systems theory). In the presence of noise, we build an LQG state estimator (Kalman-Bucy filter) (refer short notes on linear systems theory). In here, we need $\bar{P}(t)$ satisfying a differential Riccati equation to construct $\hat{x}$. 
11 Appendix

11.1 Fixed time discrete optimal control control problems and finite dimensional optimization

Consider the following fixed time discrete time optimal control problem

\[
\min \Phi(x(N)) + \sum_{k=0}^{N-1} f_0(k, x(k), u(k))
\]
subject to \(x(k+1) = f(k, x(k), u(k)), \ h(x(N)) = 0.\) (167)

where \(h(x) = [h_1(x) \ h_2(x) \ \ldots \ h_l(x)]^T\) denotes the constraint on the final state and \(\Phi(x(N))\) denotes the terminal cost. This is a finite dimensional optimization problem. The constraint \(h(x(N)) = 0\) specifies the boundary condition on the terminal state.

Let \(z = [x^T(1) \ \ldots \ x^T(N) \ u^T(0) \ \ldots \ u^T(N-1)]^T.\) Now, we can write the cost function as \(F(z) = \Phi(x(N)) + \sum_{k=0}^{N-1} f_0(k, x(k), u(k)).\) Next, we write down the constraints using the variable \(z.\) Let

\[
H(z) = \begin{bmatrix}
    x(1) - f(0, x(0), u(0)) \\
    \vdots \\
    x(N) - f(N-1, x(N-1), u(N-1)) \\
    h(x(N))
\end{bmatrix}.
\]

The constraints in the original problem can be written as \(H(z) = 0.\) Therefore, we have the following finite dimensional constrained optimization problem

\[
\min \ F(z)
\]
subject to \(H(z) = 0.\) (168)

Now using first order optimality conditions via Lagrange multipliers, \(\nabla_z F + \lambda^T \nabla_z H(z) = 0.\) Therefore, for \(k = 1, \ldots, N - 1,\)

\[
\frac{\partial f_0}{\partial x(k)} + \lambda^T(k - 1) - \lambda^T(k) \frac{\partial f}{\partial x(k)} = 0.\) (169)

For \(k = N,\)

\[
\frac{\partial \Phi}{\partial x(N)} + \lambda^T(N - 1) - \lambda^T(N) \frac{\partial h}{\partial x(k)} = 0.\) (170)

And for \(k = 0, \ldots, N - 1,\)

\[
\frac{\partial f_0}{\partial u(k)} - \lambda^T(k) \frac{\partial f}{\partial x(k)} = 0.\) (171)

These first order conditions (Equations (169), (170) and (171)) give a discrete version of variational methods or Pontryagin’s principle for optimal control problems with fixed terminal time. If there are inequality constraints of states or input, then using KKT conditions, one obtains the first order necessary conditions for fixed terminal time problems.

For discrete control of free terminal time problems, the dimension of the underlying space changes as the terminal time is not fixed.
11.2 Missing steps in the proof of PMP

1. Temporal control perturbation: Recall the expression for $u_\tau(t)$. The new terminal time is $t^* + \epsilon \tau$. For $\tau > 0$,

$$y(t^* + \epsilon \tau) = y(t^*) + \dot{y}(t^*)\epsilon \tau + h.o.t.$$ 

Observe that $y(t^*) = y^*(t^*)$ since up to $t^*$, the new input $u_\tau = u^*$. Therefore,

$$y(t^* + \epsilon \tau) = y^*(t^*) + \dot{y}(t^*)\epsilon \tau + h.o.t.$$ 

$$= y^*(t^*) + g(y^*(t^*), u^*(t^*))\epsilon \tau + h.o.t.$$ 

$$= y^*(t^*) + \epsilon \delta(\tau) + h.o.t.$$ 

where $\delta(\tau) = g(y^*(t^*), u^*(t^*))\tau$. For $\tau < 0$, $y(t^* + \epsilon \tau) = y(t^* + \epsilon \tau)$ and the first order approximation gives the same result. The vector $\epsilon \delta(\tau)$ gives infinitesimal perturbation of the terminal point. Moreover, it depends linearly on $\tau$. As one varies $\tau \in \mathbb{R}$, keeping $\epsilon$ fixed, one obtains a line $\bar{\rho}$ passing through $y^*(t^*)$. The approximation $y(t^* + \epsilon \tau) = y^*(t^*) + \epsilon \delta(\tau)$ is valid only in the limit as $\epsilon \to 0$. Hence, $\delta(\tau)$ gives the direction of change in the terminal point $y^*(t^*)$ due to infinitesimal change in the terminal time.

2. Spatial control perturbation: Recall that the rectangular perturbation in $u$ is applied in the interval $(b - ea, b]$. Taking Taylor expansion of $y^*$ at $t = b$,

$$y^*(b - ea) = y^*(b) - \dot{y}^*(b)ea + h.o.t.$$ 

$$\Rightarrow y^*(b) = y^*(b - ea) + g(y^*(b), u^*(b))ea$$ (172) 

where we have used that $\dot{y}^*(b) = g(y^*(b), u^*(b))$. Similarly, taking Taylor expansion of $y$ at $t = b - ea$,

$$y^*(b - ea) = y(b - ea) + \dot{y}(b - ea)ea + h.o.t.$$ 

where $\dot{y}(b - ea)$, we mean the right hand derivative of $y$ at $b - ea$. Note that $y(b - ea) = y^*(b - ea)$ and $\dot{y}(b - ea) = g(y^*(b - ea), w)$. Therefore,

$$y(b) = y^*(b - ea) + g(y^*(b - ea), w)ea + h.o.t.$$ (173) 

Apply Taylor’s expansion to $g(y^*(b - ea), w)ea$,

$$g(y^*(b - ea), w)ea = g(y^*(b), w)ea + g_\epsilon(y^*(b), w)(y^*(b - ea) - y^*(b))ea + h.o.t.$$ 

Note that the second term above is of the order $\epsilon^2$. Therefore,

$$y^*(b - ea) + g(y^*(b), w)ea + h.o.t.$$ 

$$= y^*(b) - g(y^*(b), u^*(b))ea + g(y^*(b), w)ea + h.o.t.$$

( using (172)

$$= y^*(b) + v_b(w)ea$$ (174) 

where $v_b(w) := g(y^*(b), w) - g(y^*(b), u^*(b))$.

Thus we obtain the value of the perturbed trajectory $y$ at $t = b$ (the point at which input perturbation vanishes). Now starting with this value as initial condition, we investigate how $y(b)$ propagates to $t = t^*$.

3. Variational equation: Recall that

$$y(t, \epsilon) = y^*(t) + \epsilon \psi(t) + h.o.t.$$ (Equation (73)) $\Rightarrow \psi(t) = y_\epsilon(t, 0)$ (175) 

and $\psi(b) = v_b(w)a$. (176)
Rewriting (67) as an integral equation:

$$y(t, \epsilon) = y(b, \epsilon) + \int_b^t g(y(s, \epsilon), u(s)) ds.$$  

Differentiating both sides w.r.t. \(\epsilon\) at \(\epsilon = 0\),

$$\dot{\psi}(t) = y_\epsilon(b, \epsilon)|_{\epsilon=0} + \int_b^t g_y(y(s, 0), u^*(s)) y_\epsilon(s, 0) ds.$$  

Note that \(y_\epsilon(b, \epsilon)|_{\epsilon=0} = \psi(b) = v_b(w)a\) and \(y_\epsilon(s, 0) = \psi(s)\) (from (175)). Therefore,

$$\dot{\psi}(t) = v_b(w)a + \int_b^t g_y(y^*(s), u^*(s)) \psi(s) ds.$$  

Differentiating w.r.t. \(t\),

$$\dot{\psi}(t) = g_y(y^*, u^*) \psi = g_y|_* \psi$$

which is Equation (74). This can be rewritten as

$$\dot{\psi}(t) = A_\ast(t) \psi$$

where \(A_\ast(t) = g_y|_* (t)\). This describes the linearization of the original system around the optimal trajectory \(y^*\). Let \(\psi = \begin{bmatrix} \eta^0 \\ \eta \end{bmatrix}\) and recall that \(g = \begin{bmatrix} L \\ f \end{bmatrix}\). Therefore,

$$A_\ast(t) = \begin{bmatrix} 0 & L_x|_\ast(t) \\ 0 & f_x|_\ast(t) \end{bmatrix}.$$  

(178)

Let \(\Phi_\ast(.,.)\) be the state transition matrix for this linear system. Therefore,

$$\dot{\psi}(t^*) = \Phi_\ast(t^*, b) \psi(b) = \Phi_\ast(t^*, b) v_b(w)a$$

Substituting the previous equation in Equation (73),

$$y(t^*) = y^*(t^*) + c\Phi_\ast(t^*, b) v_b(w)a + h.o.t.$$  

where \(\delta(w, I) = \Phi_\ast(t^*, b) v_b(w)a\).

(179)

4. **Topological lemma**: We refer the reader to Liberzon.

5. **Adjoint equation**: Two linear systems \(\dot{y} = Ay\) and \(\dot{z} = -ATz\) are called adjoint to each other and \(z\) is called adjoint vector. The inner product \(\langle y, z \rangle\) remains constant and it can be verified as follows:

$$\frac{d}{dt} \langle y, z \rangle = \langle \dot{y}, z \rangle + \langle y, \dot{z} \rangle = \langle Ay, z \rangle - \langle y, A^Tz \rangle = 0.$$  

Recall that \(\dot{\psi} = A_\ast \psi\). The adjoint system is given by

$$\dot{z} = -A^T_\ast z = \begin{bmatrix} 0 & 0 \\ -L_x|_\ast(t) & -f_x^T|_\ast(t) \end{bmatrix} z.$$  

Let \(z = \begin{bmatrix} p_0 \\ p \end{bmatrix}\), thus, \(\dot{p}_0 = 0\) and \(\dot{p} = -L_x|_* p_0 - f_x^T|_* p\). Therefore, by the definition of the Hamiltonian, \(\dot{p} = -H_x(x^*, u^*, p, p_0)\). Specifying the terminal condition for the system
\[ \dot{z} = -A^T z, \text{ as the normal } \begin{bmatrix} p_0^* \\ p^*(t^*) \end{bmatrix} \text{ to the separating hyperplane, we obtain } p_0 = p_0^* \text{ and } p^* = -H_x(x^*, u^*, p^*, p_0^*). \] Moreover, the first canonical equation also holds by the definition of the Hamiltonian.

By the property of the inner product of states of adjoint systems,

\[ \langle \begin{bmatrix} p_0^* \\ p^*(t^*) \end{bmatrix}, \psi(t) \rangle = \langle \begin{bmatrix} p_0^* \\ p^*(t^*) \end{bmatrix}, \psi(t^*) \rangle \quad \forall t \in [t_0, t^*]. \tag{180} \]

Note that the vector \( \begin{bmatrix} \dot{p}_0 \\ \dot{p}^*(t) \end{bmatrix} \) which is normal to the separating hyperplane is non-zero.

6. Properties of \( H \): Using the hyperplane separation property (Equation 75) and Equation (179),

\[ \langle \begin{bmatrix} p_0^* \\ p^*(t^*) \end{bmatrix}, \Phi_\ast(t^*, b)v_\ast(w) \rangle \leq 0 \]

(since \( a > 0 \) in (179)). Using the adjoint property (180),

\[ \langle \begin{bmatrix} p_0^* \\ p^*(b) \end{bmatrix}, v_\ast(w) \rangle \leq 0. \tag{181} \]

Recall that \( v_\ast(w) = g(y^*(b), w) - g(y^*(b), u^*(b)) \). Now substituting for \( g \), we get

\[ \langle \begin{bmatrix} p_0^* \\ p^*(b) \end{bmatrix}, \begin{bmatrix} L(x^*(b), w) \\ f(x^*(b), w) \end{bmatrix} \rangle \leq \langle \begin{bmatrix} p_0^* \\ p^*(b) \end{bmatrix}, \begin{bmatrix} L(x^*(b), u^*(b)) \\ f(x^*(b), u^*(b)) \end{bmatrix} \rangle. \tag{182} \]

\[ \Rightarrow H(x^*(b), w, p^*(b), p_0^*) \leq H(x^*(b), u^*(b), p^*(b), p_0^*). \tag{183} \]

Thus, the maximum property of the Hamiltonian holds. One has to be careful about the points where \( u^* \) is discontinuous. Such points can be handled by taking right/left limits (Liberzon).

Another property of the Hamiltonian is that \( H|_* = 0 \). The hyperplane separation property (75) applies to \( \delta(\tau) \in C_{t^*} \) where \( \delta(\tau) = \begin{bmatrix} L(x^*(t^*), u^*(t^*)) \\ f(x^*(t^*), u^*(t^*)) \end{bmatrix} \) is the terminal state perturbation vector corresponding to a temporal perturbation of control.

\[ \delta(\tau) = \begin{bmatrix} L(x^*(t^*), u^*(t^*)) \\ f(x^*(t^*), u^*(t^*)) \end{bmatrix} \tau \]

where \( \tau \) can be positive or negative. Therefore, for (75) to be satisfied,

\[ \langle \begin{bmatrix} p_0^* \\ p^*(t^*) \end{bmatrix}, \begin{bmatrix} L(x^*(t^*), u^*(t^*)) \\ f(x^*(t^*), u^*(t^*)) \end{bmatrix} \rangle = 0 \tag{184} \]

i.e., \( H(x^*(t^*), u^*(t^*), p^*(t^*), p_0^*) = 0 \) which implies that at terminal time, \( H|_* = 0 \). We refer the reader to Liberzon to show that \( H|_* \) is identically zero.

7. Transversality condition: For the extension of topological lemma to this case and for transversality arguments, refer Liberzon.
11.3 State estimation and discrete Kalman filter

[Boyd] A random vector $x \in \mathbb{R}^n$ is said to be Gaussian if it has probability density $p_x(v) = (2\pi)^{-\frac{n}{2}} \det(\Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}(v-x)^T \Sigma^{-1}(v-x)}$ for some $\Sigma = \Sigma^T > 0$, $\bar{x} \in \mathbb{R}^n$. It turns out that $\bar{x} = E(x) = \int v p_x(v) dv$ is the mean of $x$ and $\Sigma = E((x - \bar{x})(x - \bar{x})^T)$ is the covariance matrix of $x$ where $\Sigma(i,j)$ denotes the covariance between $x_i$ and $x_j$. Note that $E\|x - \bar{x}\|^2 = ETr((x - \bar{x})(x - \bar{x})^T) = Tr(\Sigma)$. For a random vector $x$, we denote its covariance matrix by $\Sigma_x$.

Consider an affine transformation $z = Ax + b$ where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Then $\bar{z} = E(z) = A\bar{x} + b$. Observe that $\Sigma_z = E((z - \bar{z})(z - \bar{z})^T) = E((Ax - \bar{x})(x - \bar{x})^T A^T) = A\Sigma_x A^T$.

Consider linear measurements with noise: $y = Ax + v$ where $v$ is the sensor noise and we want to estimate $x$ from measurements $y$. We assume that $x$ and $v$ are independent. Observe that

$$E \left[ \begin{bmatrix} x \\ y \end{bmatrix} \right] = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{x} + \bar{v} \end{bmatrix}.$$ 

Hence, covariance of measurements $y$ is $\Sigma_y = A\Sigma_x A^T + \Sigma_v$ where $A\Sigma_x A^T$ is the signal covariance and $\Sigma_v$ is the noise covariance.

**Minimum mean-square estimation:** We want to find a function $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that $\hat{x} = \phi(y)$ is near $x$. In other words, we want to minimize $E(\|\phi(y) - x\|^2)$. Minimum mean-square estimator (MMSE) $\phi_{\text{mmse}}$ minimizes this quantity. Consider $E(x|y)$ i.e., the conditional expectation of $x$ given $y$. Suppose $x$ and $y$ are jointly Gaussian. Let $\begin{bmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{yx}^T & \Sigma_y \end{bmatrix}$ be the covariance matrix of $\begin{bmatrix} x \\ y \end{bmatrix}$. It turns out (after a lot of algebraic manipulations) that the conditional density is $p_{x|y}(v|y) = (2\pi)^{-\frac{n}{2}} \det(\Lambda)^{-\frac{1}{2}} e^{-\frac{1}{2}(v-w)^T \Lambda^{-1}(v-w)}$ where $\Lambda = \Sigma_x - \Sigma_{xy} \Sigma_y^{-1} \Sigma_{yx}$, $w = \bar{x} + \Sigma_{xy} \Sigma_y^{-1}(y - \bar{y})$. The conditional expectation is

$$E(x|y) = w = \bar{x} + \Sigma_{xy} \Sigma_y^{-1}(y - \bar{y}). \quad (185)$$

It turns out that the MMSE estimator $\hat{x} = \phi_{\text{mmse}}(y) = E(x|y)$. The covariance matrix of the error vector $x - \hat{x}$ is

$$E((x - \hat{x})(x - \hat{x})^T) = \Sigma_x - \Sigma_{xy} \Sigma_y^{-1} \Sigma_{yx}^T. \quad (186)$$

Note that $\Sigma_{xy} \Sigma_y^{-1} \Sigma_{yx}^T \leq \Sigma_x$ i.e., the covariance of the estimation error is always less than the prior covariance of $x$.

**Best linear unbiased estimator:** The estimator $\hat{x} = \phi_{\text{blue}}(y) = \bar{x} + \Sigma_{xy} \Sigma_y^{-1}(y - \bar{y})$ makes sense even when $x$ and $y$ aren’t jointly Gaussian. This estimator is unbiased i.e., $E(\hat{x}) = E(x)$ and has minimum mean-square error among all affine estimators. It works well, in practice and is widely used. It is called best linear unbiased estimator (BLUE).

Recall that $\Sigma_{xy} = \Sigma_x A^T$ and $\Sigma_y = A\Sigma_x A^T + \Sigma_v$. Therefore,

$$\hat{x} = \bar{x} + \Sigma_{x} A^T (A\Sigma_x A^T + \Sigma_v)^{-1}(y - \bar{y}) = \bar{x} + B(y - \bar{y})$$

where $B := \Sigma_x A^T (A\Sigma_x A^T + \Sigma_v)^{-1}$.

- $\bar{x}$ is our best prior guess of $x$.
- $y - \bar{y}$ measures discrepancy between the measured output $y$ and the expected value $\bar{y}$ of the measured output.
- Estimate modifies the guess by $B$ times $y - \bar{y}$.
\[ \Sigma_e = \Sigma_x - \Sigma_x A^T (A \Sigma_x A^T + \Sigma_v)^{-1} A \Sigma_x. \] (187)

Observe that \( \Sigma_e \) can be calculated before output measurements. It is useful for experiment design/sensor selection. It turns out that after a few manipulations, \( \Sigma_e = (A^T \Sigma_v A + \Sigma_x^{-1})^{-1} \). Note that \( \Sigma_x^{-1} \) and \( \Sigma_v^{-1} \) are called information matrices. The last equation says that posterior information matrix \( \Sigma_e^{-1} \) is prior information matrix \( \Sigma_v^{-1} \) for measurement \( A^T \Sigma_v A \).

**Linear stochastic processes:** Consider a discrete linear system \( x(t+1) = Ax(t) + Bu(t) \) where \( x(0), u(0), u(1), \ldots \) are random variables.

\[ \bar{x}(t + 1) = E(A(x(t) - \bar{x}(t)) + B(u(t) - \bar{u}(t))) \] (188)

and similarly define \( \bar{u}(t) \) and \( \Sigma_u(t) \). Taking expectation of \( x(t+1) = Ax(t) + Bu(t) \), \( x(t+1) = A\bar{x}(t) + B\bar{u}(t) \). Therefore, \( x(t+1) - \bar{x}(t+1) = A(x(t) - \bar{x}(t)) + B(u(t) - \bar{u}(t)) \). Hence,

\[ \Sigma_x(t + 1) = E(A(x(t) - \bar{x}(t)) + B(u(t) - \bar{u}(t)))(A(x(t) - \bar{x}(t)) + B(u(t) - \bar{u}(t))^T) \]
\[ = A \Sigma_x A^T + B \Sigma_u B^T + A \Sigma_{xu} B^T + B \Sigma_{ux} A^T \]

where \( \Sigma_{xu} = \Sigma_{ux} = E(\bar{x} - \bar{x})(u - \bar{u})^T \). Thus, the covariance \( \Sigma_x(t) \) satisfies Lyapunov like equation. Suppose \( \Sigma_{xu} = 0 \) i.e., \( x \) and \( u \) are uncorrelated. Therefore, \( \Sigma_x(t+1) = A \Sigma_x(t) A^T + B \Sigma_u B^T \) which is stable if \( A \) is stable. If \( A \) is stable and \( \Sigma_u \) is constant, then \( \Sigma_x(t) \) converges to \( \Sigma_x \) which is called the steady state covariance which satisfies \( \Sigma_x = A \Sigma_x A^T + B \Sigma_u B^T \).

**State estimation and discrete Kalman filter:** We enumerate below the steps to obtain recursive equation for discrete Kalman filter.

1. (Initialization): Consider a linear dynamical system \( x(t + 1) = Ax(t) + w(t), \ y(t) = Cx(t) + v(t) \) where \( x \) is the state, \( w \) is the disturbance/process noise, and \( v \) is the measurement noise. We assume that \( x(0), w(0), w(1), \ldots, v(0), v(1), \ldots \) are jointly Gaussian and independent. Both \( w(t) \) and \( v(t) \) are IID with \( Ew(t) = Ev(t) = 0 \) and \( Ew(t)w^T(t) = W, Ev(t)v^T(t) = V \). Furthermore, \( E(x(0) = x_0, E(x(0) - x_0)(x(0) - x_0)^T = \Sigma_0 \). Let \( X(t) = (x(0), \ldots, x(t)) \). We assume that \( x \) is a Markov process i.e., \( x(t)|x(0), \ldots, x(t-1) = x(t)|x(t-1) \).

Note that \( x(t + 1) = A\bar{x}(t) \). Suppose \( \bar{x}(0) = x_0 \), hence, \( \bar{x}(t) = A^t\bar{x}_0 \) and

\[ \Sigma_x(t + 1) = E((x(t + 1) - \bar{x}(t + 1))(x(t + 1) - \bar{x}(t + 1))^T) \]
\[ = E(A(x(t) - \bar{x}(t) + w(t))(A(x(t) - \bar{x}(t)) + w(t))^T) \]
\[ = A \Sigma_x A^T + W \]

where the last equality follows by assuming that \( x \) and \( w \) are uncorrelated. If \( A \) is stable, then \( \Sigma_x(t + 1) \) converges to steady-state covariance \( \bar{\Sigma} \) which satisfies Lyapunov equation \( \Sigma_x = A \Sigma_x A^T + W \). Let

\[ \bar{x}(t|s) = E(x(t)|y(0), \ldots, y(s)) \]
\[ \Sigma_{t|s} = E(x(t) - \bar{x}(t|s))(x(t) - \bar{x}(t|s))^T \]

where \( x(t|y(0), \ldots, y(s)) \) is random variable with mean \( \bar{x}(t|s) \). Note that \( \bar{x}(t|s) \) is the minimum mean-square estimate of \( x(t) \) based on \( y(0), \ldots, y(s) \) and \( \Sigma_{t|s} \) is the covariance of the error of the estimate \( \bar{x}(t|s) \). We focus on two state estimation problems:

- Finding \( \bar{x}(t|t) \) i.e., estimating the current state, based on the current and past observed outputs.

- Finding \( \bar{x}(t|0) \) i.e., estimating the current state, based on the current observed output.
Finding $\hat{x}(t+1|t)$ i.e., predicting the next state, based on the current and past observed outputs.

2. (MMSE): Since, $x(t)$ and $Y(t)$ are jointly Gaussian, using (185) to find $\hat{x}(t|t)$,

$$\hat{x}(t|t) = \hat{x}(t) + \Sigma_{x(t)Y(t)}^{-1}(Y(t) - \hat{Y}(t))$$

(191)

where $\Sigma_{xY}^{-1}$ is $mt \times mt$ matrix which grows with $t$ ($m$ being the number of outputs).

Kalman filter is a clever method of computing $\hat{x}(t|t)$ and $\hat{x}(t+1|t)$ recursively. We now write $\hat{x}(t|t)$ and $\Sigma_{x|t}$ in terms of $\hat{x}(t|t-1)$ and $\Sigma_{x|t-1}$.

3. (Measurement update): Start with $y(t) = Cx(t) + v(t)$ and condition on $Y(t-1)$: $y(t)|Y(t-1) = Cx(t)|Y(t-1) + v(t)|Y(t-1) = Cx(t)|Y(t-1) + v(t)$ since $v(t)$ and $Y(t-1)$ are independent. Note that $x(t)|Y(t-1)$ and $y(t)|Y(t-1)$ are jointly Gaussian with mean and covariance

$$\begin{bmatrix} \hat{x}(t|t-1) \\ C\hat{x}(t|t-1) \end{bmatrix}, \begin{bmatrix} \Sigma_{x|t-1} & \Sigma_{x|t-1} C^T \\ C\Sigma_{x|t-1} C^T + V \end{bmatrix}.$$  

(192)

We now use the above mean and covariances in (192) in the standard formula $\hat{x} = \bar{x} + \Sigma_{xy}\Sigma_{y}^{-1}(y - \bar{y})$ (Equation (185)) to obtain the mean of $(x(t)|Y(t-1))(y(t)|Y(t-1)) = x(t)|Y(t)$ as

$$\hat{x}(t|t) = \hat{x}(t|t-1) + \Sigma_{x|t-1} C^T(C\Sigma_{x|t-1} C^T + V)^{-1}(y(t) - C\hat{x}(t|t-1))$$

(193)

where from (192), $\hat{x}(t|t-1)$ is the mean of $(x(t)|Y(t-1))$ and other terms are obtained similarly using (192).

Now we find the covariance of the error vector $x - \hat{x}$ using the formula obtained in the MMSE section (Equation (187)) and using corresponding matrices from (192) as

$$\Sigma_{x|t} = \Sigma_{x|t-1} - \Sigma_{x|t-1} C^T(C\Sigma_{x|t-1} C^T + V)^{-1}C\Sigma_{x|t-1}.$$  

(194)

Equations (193) and (194) give $\hat{x}(t|t)$ and $\Sigma_{x|t}$ in terms of $\hat{x}(t|t-1)$ and $\Sigma_{x|t-1}$ which is called measurement update.

4. (Time update): Using $x(t+1) = Ax(t) + w(t)$ and conditioning on $Y(t)$, $x(t+1)|Y(t) = Ax(t)|Y(t) + w(t)|Y(t) = x(t+1)|Y(t) = Ax(t)|Y(t) + w(t)$ since, $w(t)$ is independent of $Y(t)$. Now $x(t+1)|Y(t) = Ax(t)|Y(t) + w(t)$ implies that $\hat{x}(t+1|t) = A\hat{x}(t|t)$ and

$$\begin{align*}
\Sigma_{t+1|t} &= E((\hat{x}(t+1|t) - x(t+1))(\hat{x}(t+1|t) - x(t+1))^T) \\
&= E(A\hat{x}(t|t) - Ax(t) - w(t))(A\hat{x}(t|t) - Ax(t) - w(t))^T \\
&= A\Sigma_{x|t} A^T + W.
\end{align*}$$

(195)

These are called time updates.

5. (Recursion using measurement and time updates): Measurement and time updates together give a recursive solution. Let $\hat{x}(0|1) = \hat{x}_0$ and $\Sigma_{0|0} = \Sigma_0$. Apply measurement updates from (193) and (194) to get $\hat{x}(0|0)$ and $\Sigma_{0|0}$; and then apply time updates $\hat{x}(t+1|t) = A\hat{x}(t|t)$ and (195) to obtain $\hat{x}(1|0)$ and $\Sigma_{1|0}$ and keep repeating measurement and time updates. Therefore, using covariance matrices, substituting (194) in (195), we get

$$\Sigma_{t+1|t} = A\Sigma_{x|t} A^T + W - A\Sigma_{x|t} C^T(C\Sigma_{x|t} C^T + V)^{-1}C\Sigma_{x|t} A^T.$$  

(196)
Example 11.1 (Bryson and Ho). (Persuit evasion: minimum energy) Consider two dynamical systems
\[ \dot{x}_p = A_p x_p + B_p u \]
\[ \dot{x}_e = A_e x_e + B_e v. \]
where \( x_p(t_0) \) and \( x_e(t_0) \) are given. The subscripts \( p \) and \( e \) stand for persuer and evader respectively. Consider \( \|x_p(t_f) - x_e(t_f)\|^2 \) which is the distance between the two where \( t_f \) is the fixed terminal time. The persuer wants to minimize this distance by choosing an input strategy whereas the evader wants to maximize it. Consider the following cost function
\[ J = \frac{1}{2} \|x_p(t_f) - x_e(t_f)\|^2 + \frac{1}{2} \int_{t_0}^{t_f} (u^T R_p u - v^T R_e v) dt. \]

The persuer tries to minimize it and the evader tries to maximize it. Note that the negative sign in the running cost implies that the evader tries to maximize the distance of separation by maximizing \(-\int_{t_0}^{t_f} v^T R_e v dt\) which is same as maximizing the separation by minimizing the input energy. Let
\[ \dot{x}_p(t) := e^{A_p(t_f-t)}x_p(t), \quad \dot{x}_e(t) := e^{A_e(t_f-t)}x_e(t), \quad z(t) := \dot{x}_p(t) - \dot{x}_e(t). \]
Therefore, \( \dot{z} = e^{A_p(t_f-t)}B_p u - e^{A_e(t_f-t)}B_e v \) and \( z(t_f) = x_p(t_f) - x_e(t_f) \). Moreover,
\[ J = \frac{1}{2} \|z(t_f)\|^2 + \frac{1}{2} \int_{t_0}^{t_f} (u^T R_p u - v^T R_e v) dt. \]
We want to find \( \min_u \max_v (J) \). The Hamiltonian

\[
H = p^T (e^{Ap(t_f-t)} B_p u - e^{Ac(t_f-t)} B_e v) - \frac{1}{2} u^T R_p u + \frac{1}{2} v^T R_e v.
\]

From the canonical equations,

\[
p = -H_z = 0
\]

hence, \( p(t) \) is constant. Recall that by Section 5.3 and the item on Terminal cost, \( p^*(t_f) = -K_z(z^*(t_f)) \) where \( K_z(z) = \frac{1}{2} \|z(t_f)\|^2 \). Therefore, \( p^*(t_f) = p^*(t_f) = -z^*(t_f) \). Note further that \( H_u = 0 \Rightarrow u = R_p^{-1} B_p^T e^{Ap}(t_f-t) p^*(t) = R_p^{-1} B_p^T e^{Ap}(t_f-t) z^*(t_f) \) and \( H_v = 0 \Rightarrow v = R_e^{-1} B_e^T e^{Ac}(t_f-t) p^*(t) = R_e^{-1} B_e^T e^{Ac}(t_f-t) z^*(t_f) \).

Let \( p(t) = S(t) z(t) \). Since \( p = 0 \), \( \dot{S} z(t) + S \dot{z}(t) = \dot{S} z(t) + S(e^{Ap(t_f-t)} B_p u - e^{Ac(t_f-t)} B_e v) = 0 \)

\[
\Rightarrow \dot{S} z(t) + S \dot{z}(t) = \dot{S} z(t) + S(e^{Ap(t_f-t)} B_p R_p^{-1} B_p^T e^{Ap}(t_f-t) S z(t) - e^{Ac(t_f-t)} B_e R_e^{-1} B_e^T e^{Ac}(t_f-t) S z(t)) = 0
\]

\[
\Rightarrow \dot{S} + S e^{Ap(t_f-t)} B_p R_p^{-1} B_p^T e^{Ap}(t_f-t) S - S e^{Ac(t_f-t)} B_e R_e^{-1} B_e^T e^{Ac}(t_f-t) S = 0
\]

\[
\Rightarrow S^{-1} \dot{S} S^{-1} = e^{Ap(t_f-t)} B_p R_p^{-1} B_p^T e^{Ap}(t_f-t) - e^{Ac(t_f-t)} B_e R_e^{-1} B_e^T e^{Ac}(t_f-t)
\]

\[
\Rightarrow \frac{d}{dt} S^{-1} = -e^{Ap(t_f-t)} B_p R_p^{-1} B_p^T e^{Ap}(t_f-t) + e^{Ac(t_f-t)} B_e R_e^{-1} B_e^T e^{Ac}(t_f-t). \tag{203}
\]

Since \( p^*(t_f) = S(t_f) z^*(t_f) \) and \( p^*(t_f) = -z^*(t_f) \), \( S(t_f) = -I \). Integrating the previous equation from \( t \) to \( t_f \),

\[
-I - S^{-1}(t) = -\int_t^{t_f} e^{Ap(t_f-\tau)} B_p R_p^{-1} B_p^T e^{Ap}(t_f-\tau) d\tau + \int_t^{t_f} e^{Ac(t_f-\tau)} B_e R_e^{-1} B_e^T e^{Ac}(t_f-\tau) d\tau.
\]

Let \( W_{R_p,A_p,B_p}(t_f, t) := \int_t^{t_f} e^{Ap(t_f-\tau)} B_p R_p^{-1} B_p^T e^{Ap}(t_f-\tau) d\tau \) and \( W_{R_e,A_e,B_e}(t_f, t) := e^{Ac(t_f-\tau)} B_e R_e^{-1} B_e^T e^{Ac}(t_f-\tau) d\tau \). Therefore,

\[
S^{-1}(t) = -I + W_{R_p,A_p,B_p}(t_f, t) - W_{R_e,A_e,B_e}(t_f, t). \tag{204}
\]

Thus, from \( S(t) = (-I + W_{R_p,A_p,B_p}(t_f, t) - W_{R_e,A_e,B_e}(t_f, t))^{-1} \), one can find \( u \) and \( v \). Note that

\[
z^*(t_f) = p(t) = p(0) = (-I + W_{R_p,A_p,B_p}(t_f, 0) - W_{R_e,A_e,B_e}(t_f, 0))^{-1} z(0).
\]

Thus, \( z^*(t_f) \) can be found from the previous equation and from \( z^*(t_f) \), one can find optimal \( u \) and \( v \).

**Example 11.2** (Persuit evasion: time optimal control) Consider the same model for persuer and evader as in the previous example with input constraints \( |u_i| \leq 1, |v_i| \leq 1 \). The condition for an intercept is \( a^T x_p(t_f) = a^T x_e(t_f) \). Define \( z(t) := a^T (x_p(t) - x_e(t)) \) where \( x_p(t), x_e(t) \) are defined in the previous example. The Hamiltonian \( H = p(t) a^T (e^{Ap(t_f-t)} B_p u - e^{Ac(t_f-t)} B_e v) - 1 \).

The evader tries to maximize the capture time whereas the persuer tries to minimize it. By the previous example, \( p^*(t) = -a^T (x_p(t_f) - x_e(t_f)) \). Therefore, \( H = \dot{z}(t_f) (a^T e^{Ap(t_f-t)} B_p u - a^T e^{Ac(t_f-t)} B_e v) - 1 \). Consider a single input persuer and evader. Hence, \( u^* = \text{sgn} \{-z(t_f) a^T e^{Ap(t_f-t)} B_p \} \) and \( v^* = -\text{sgn} \{z(t_f) a^T e^{Ac(t_f-t)} B_e \} \).

For persuit-evasion min-max energy problem with input constraints \( |u_i| \leq 1, |v_i| \leq 1 \), one can use the sat function from constrained minimum energy problems to find the optimal control laws.
11.5 HJB, method of characteristics and canonical equations

(Liberzon) We will see now how to solve a PDE by using an appropriate system of ODEs. This is the method of characteristics. Suppose we want to find a function \( v : \mathbb{R}^n \to \mathbb{R} \), that solves the PDE

\[
F(x, v(x), \nabla v(x)) = 0
\]

where \( F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \) is a continuous function. At first, we are interested in the case when \( F \) depends linearly on \( \nabla v \). Such PDEs are called quasi-linear. For simplicity, we consider the case \( n = 2 \),

\[
a(x_1, x_2, v)v_{x_1} + b(x_1, x_2, v)v_{x_2} = c(x_1, x_2, v)
\]

for some functions \( a, b, c \). The graph of a solution \( v(x_1, x_2) \) to (206) is a surface in the \( (x_1, x_2, v) \)—space defined by the equation \( h(x_1, x_2, v) := v(x_1, x_2) - v = 0 \) (where \( v(x_1, x_2) \) denotes explicit expression involving \( x_1, x_2 \) and \( v \) without arguments denotes just a variable e.g., we may have a function \( f(x, y) \) of two variables and \( (x, y, f(x, y)) \) denote its graph. We might as well write \( z = f(x, y) \) or \( z - f(x, y) = 0 \) to express this graph as a level surface in \( \mathbb{R}^3 \). Instead of denoting the function by \( f \), we could have use \( z \) itself, then the level surface becomes \( z - z(x, y) = 0 \). This is what we were doing above.)

The gradient \( \nabla h = (v_{x_1}, v_{x_2}, -1)^T \) is normal to this level surface \( h \) at all points. The PDE (206) can be written as \( \langle (a, b, c), (v_{x_1}, v_{x_2}, -1) \rangle = 0 \) which implies that the vector \( (a, b, c) \) is tangent to the level surface. In other words, \( (a, b, c) \) form a vector field on the solution surface giving rise to the following system of ODEs

\[
\frac{dx_1}{ds} = a, \quad \frac{dx_2}{ds} = b, \quad \frac{dv}{ds} = c.
\]

These are called the characteristic ODEs of the PDE (206) and their solution curves are called characteristics. Sometimes, by the abuse of terminology, we refer the equations above by characteristics.

The solution surface \( v = v(x_1, x_2) \) is filled with these characteristic curves. Once initial conditions are given, (for odes, it is just a point, for \( n \)—variable pdes, the initial conditions forms \( (n-1) \)—dimensional hyper-surface, for \( n = 2 \), initial conditions form a curve) through which the solution surface passes, the characteristics curves are uniquely determined. This is called Cauchy problem. For each ode, the corresponding point on this curve serves as the initial condition for that ode. An initial curve can be defined in parametric form as \( x_1 = x_1(r), x_2 = x_2(r), v = v(r), r \in \mathbb{R} \). Then, the characteristic curves whose initial conditions lie on this initial curve are given by

\[
x_1 = x_1(r, s), x_2 = x_2(r, s), v = v(r, s)
\]

which gives a description of the solution surface of (206). However, we want a different representation for this surface, namely, \( v = v(x_1, x_2) \). For this we need the map \( (r, s) \mapsto (x_1(r, s), x_2(r, s)) \) to be invertible. From the Inverse function theorem, we need the following Jacobian matrix to be invertible

\[
\begin{bmatrix}
(x_1)_r & (x_1)_s \\
(x_2)_r & (x_2)_s
\end{bmatrix}
\]

along the initial curve. The columns of the above matrix are tangent vectors to the initial curve and the characteristic curve respectively and the non singularity condition says that they must be linearly independent. Thus, if this condition is satisfied, then we can solve (206) in principle by the method of characteristics.
Now consider more general PDEs of the form (205) which may not be quasi-linear. For \( n = 2 \), we have,

\[
F(x_1, x_2, v, v_{x_1}, v_{x_2}) = 0. \tag{209}
\]

For quasi-linear pdes, the tangent vector \((a, b, c)\) doesn’t depend on \( v_{x_1}, v_{x_2} \), as a result, characteristic equations involve \((x_1, x_2, v)\). For general pdes, this is no longer guaranteed and five differential equations are required describing the joint evolution of \(x_1, x_2, v, v_{x_1}, v_{x_2} \) instead of three. The solution of these equations are called characteristic strips. For the ease of notation let \( \xi_1 := v_{x_1} \) and \( \xi_2 := v_{x_2} \). The characteristic strips are defined by following equations (given without derivation here, refer Liberzon p.211 for a partial justification)

\[
\frac{dx_1}{ds} = F_{\xi_1}, \quad \frac{dx_2}{ds} = F_{\xi_2}, \quad \frac{dv}{ds} = \xi_1 F_{\xi_1} + \xi_2 F_{\xi_2}, \quad \frac{d\xi_1}{ds} = -F_{x_1} - \xi_1 F_v, \quad \frac{d\xi_2}{ds} = -F_{x_2} - \xi_2 F_v \tag{210}
\]

(Check that for quasi-linear pdes, the first three equations reduce to the earlier three equations. The last two equations follow from the application of chain rule (Liberzon).)

### 11.5.1 Canonical equations as characteristics of HJB equation

With an abuse of notation, write HJB equation as \( V(t, x) - H(t, x, -V_x(t, x)) = 0 \). We did not include the control \( u \) as an argument of \( H \) assuming that an optimal control is already plugged in achieving the maximum of \( H \). This PDE can be brought in the form (205) by using an additional variable \( x_{n+1} := t \). Now we write down equations (210) for this particular PDE. Since the time \( t \) is present in the Hamiltonian, we can use it as an independent variable instead of \( s \) and write \( \dot{x}_i \) instead of \( \frac{dx_i}{ds} \). Define co-states \( p_i = -\xi_i = -v_{x_i} \). Now from equations (210),

\[
\dot{x}_i = F_{\xi_i} = -\frac{\partial}{\partial p_i} (V_{x_{n+1}}(x) - H(x, p)) = H_{p_i}, \quad 1 \leq i \leq n
\]

where \( H(x, p) = \sum_{i=1}^{n} p_i f_i - L(x) \) and \( \dot{x}_{n+1} = F_{\xi_{n+1}} \). Note that \( \dot{x}_{n+1} = f_{n+1} = 1 \) since \( x_{n+1} = t \). We derive this from \( \dot{x}_{n+1} = F_{\xi_{n+1}} \). Observe that \( V_{x_{n+1}}(x) = -p_{n+1} \). Thus,

\[
\dot{x}_{n+1} = F_{\xi_{n+1}} = -F_{p_{n+1}} = -\frac{\partial}{\partial p_{n+1}} (-p_{n+1} - \sum_{i=1}^{n} p_i f_i + L(x)) = 1
\]

which is a redundant equation due to auxiliary variable \( x_{n+1} = t \).

Note that \( \frac{d\xi_i}{dt} = -\frac{dp_i}{dt} = F_{x_i} + \xi_i F_v \). But \( F_v = 0 \) since \( H \) is independent of \( v \). Therefore,

\[
-p_i = F_{x_i} = H_{x_i}, \quad 1 \leq i \leq n + 1.
\]

Thus we obtain canonical equations. Moreover, we have one more equation

\[
\frac{dV}{dt} = \sum_{i=1}^{n+1} \xi_i F_{\xi_i} = \sum_{i=1}^{n} \xi_i \dot{x}_i + \xi_{n+1} = -\sum_{i=1}^{n} p_i \dot{x}_i - p_{n+1}. \tag{211}
\]

Note that \(-p_{n+1} = V_{x_{n+1}}(x) = V_t \) and \( \dot{x}_i = H_{p_i} \). Therefore, \( \frac{dV}{dt} = V_t - \sum_{i=1}^{n} p_i H_{p_i} \). Now, from HJB equation, \( V_t = H \), therefore,

\[
\frac{dV}{dt} = H - \sum_{i=1}^{n} p_i H_{p_i}. \tag{212}
\]

This allows us to use the method of characteristics to solve HJB equation as follows. The Cauchy problem for HJB is given by the terminal cost \( V(t_f, x) = K(x) \) where \( t_f \) is the terminal time.
To calculate $V(t, x)$ for a specific time $t$ and state $x$, we must find $\tilde{x}$ such that $x = x(t)$ where $(x(.), p(.))$ is the solution of the system of canonical equations with the boundary conditions

$$x(t_f) = \tilde{x}, p(t_1) = -Kx(\tilde{x}).$$

Then, $V(t, x)$ is computed by integrating along the corresponding characteristic

$$V(t, x) = K(\tilde{x}) - \int_t^{t_f} (H(s, x(s), p(s)) - \sum_{i=1}^{n} p_i(s)H_{p_i} s, x(s), p(s))ds. \quad (213)$$

12 Extra topics

Geometric optimal control e.g. optimal control on manifolds and Lie groups (Agrachev-Sachkov, Shâttler-Ladziewicz), Hybrid optimal control (Liberzon), stochastic optimal control (MIT lecture notes, Boyd lecture notes, Kwakernaak and Sivan, Bertsekas), $H_2, H_\infty$ control and robust control (Zhou, Doyle, Glover).

13 Numerical methods

Kirk, MIT lecture notes

14 Quick overview

We summarize the topics from optimization we studied in the following figure.
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References


[14] Sanand D. *Short notes on linear systems*.


