INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR
MA21201/MA31005 - Real Analysis
Problem Sheet 1(To be updated)
Autumn 2021

Problem 1 Let $f : A \to B$ and $g : B \to C$ be functions. Prove the following:

(i) If f and g are bijective, then the composite $g \circ f : A \to C$ is bijective.

(ii) If $g \circ f$ is injective, then f is injective.

(iii) If $g \circ f$ is surjective, then g is surjective.

Problem 2 Prove that, if $A \subseteq \mathbb{N}$, then A is countable.

Problem 3 Prove that a set A is denumerable if and only if there is a bijection from A onto a denumerable set B.

Problem 4 Prove that the set of all finite subsets of $\mathbb{N}$ is countable.

Problem 5 Prove that if A is a countable set and B is an uncountable set, then $B \setminus A$ is uncountable.

Problem 6 Let $\mathbb{R}$ be the set of all real numbers, and let S denote the set of all functions defined on $\mathbb{R}$. Show that there does not exist a bijection between $\mathbb{R}$ and S.

Problem 7 Find an explicit bijection between $\{x \in \mathbb{R} : 0 < x < 1\}$ and $\{x : 0 \leq x < 1\}$.

Problem 8 A real number is called algebraic if it is a root of an algebraic equation $a_0 + a_1 x + \cdots + a_n x^n = 0$ where the coefficients $a_0, a_1, \ldots, a_n$ are integers. Prove that the set of algebraic numbers is countable.

Problem 9 Show that, if $x > -1$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

Problem 10 Show that, if $a \geq 0, b \geq 0$ and $a < b$, then $\sqrt{a} < \sqrt{b}$.

Problem 11 If $a \in \mathbb{R}$ and $0 \leq a \leq \epsilon$ for all $\epsilon > 0$, then show that $a = 0$.

Problem 12 Show that if $a, b \in \mathbb{R}$, then

1. $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ and $\min a, b = \frac{1}{2}(a + b - |a - b|)$.
2. $\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$.

Problem 13 Let $S = \{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\}$. Find $\inf S$ and $\sup S$.

Problem 14 Let S be a nonempty bounded subset of $\mathbb{R}$. Prove that $\sup S = -\inf\{-s : s \in S\}$ and $\inf S = -\sup\{-s : s \in S\}$.

Problem 15 Let $A := \{a_n : n \in \mathbb{R}\}$ and $B = \{b_n : n \in \mathbb{N}\}$ be bounded sets. If $a_n \leq b_n$ for all n, then show that

1. $\sup A \leq \sup B$,
2. $\inf A \leq \inf B$.

Problem 16 Show that if A and B are bounded sets and $\sup A < \sup B$, then there exists an element $b \in B$ such that b is an upper bound for A.

Problem 17 Show that if A and B are bounded sets, then $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

Problem 18 Show that there exists a positive real number x such that $x^2 = 2$.

Problem 19 Let X be a nonempty set and let $f : X \to \mathbb{R}$ have bounded range in $\mathbb{R}$. If $a \in \mathbb{R}$, then show that

1. $\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}$, and
2. $\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}$.

Problem 20 Let I_n be a nested sequence of intervals in $\mathbb{R}$. If $\cap_{n=1}^{\infty} I_n$ contains more than one point, then $\cap_{n=1}^{\infty} I_n$ is uncountable.