Partial Differential Equations (MA20103)

Assignment -1

I- Power Series Solution

1. Find power series solution of $y'' + xy' + x^2 y = 0$ about $x = 0$

$$\left[Ans: y = c_0\left(1 - \frac{1}{12}x^4 + \frac{1}{90}x^6 - ...\right) + c_1\left(x - \frac{1}{6}x^3 + \frac{1}{40}x^5 - ...\right)\right]$$

2. Find the power series solution of $y'' + (x-3)y' + y = 0$ in powers of $(x-2)$

$$\left[Ans: y = c_0\left(1 - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{12}(x-2)^4 ...\right) + c_1\left((x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{6}(x-2)^4 ...\right)\right]$$

3. Solve $y'' - 2x^2 y' + 4xy = x^2 + 2x + 4$ in powers of x

$$\left[Ans: y = c_0\left(1 - \frac{2}{3}x^3 - \frac{2}{45}x^6 - ...\right) + c_1\left(x - \frac{1}{6}x^4 - \frac{1}{63}x^7 - ...\right) + 2x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{1}{126}x^7 + ...\right]$$

4. Apply Frobenius method to solve the following differential equation in series about x = 0

$$2x^2 y'' - xy' + (1 - x^2) y = 0$$

$$\left[Ans: y = ax\left(1 + \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 9} + ...\right) + bx^{\frac{1}{2}}\left(1 + \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 7} + ...\right)\right]$$

5. Show that $x = 0$ is a regular singular point of $(2x + x^3) y'' - y' - 6xy = 0$ and find its series solution about $x = 0$.

$$\left[Ans: y = a\left(1 + 3x^2 + \frac{3}{5}x^4 + ...\right) + bx^{\frac{3}{2}}\left(1 + \frac{3}{8}x^2 - \frac{3}{128}x^4 + ...\right)\right]$$

6. Solve $x^2 y'' + xy' + (x^2 - 1) y = 0$ in series near $x = 0$.

$$\left[Ans: y = ax^{-1}\left(-\frac{x^2}{2} + \frac{x^4}{2^2\cdot 4} + \frac{x^6}{2^2\cdot 4^2\cdot 6} + ...\right) + bx^{-1}\left(1 + \frac{x^2}{2^2} - \frac{1}{2^2\cdot 4}\left(\frac{2}{2} + \frac{1}{4}\right)x^4 + ...\right) + bx^{-1}\ln x\left(-\frac{x^2}{2} + \frac{x^4}{2^2\cdot 4} + \frac{x^6}{2^2\cdot 4^2\cdot 6} + ...\right)\right]$$

II Legendre equation and Legendre polynomial

7. Show that

$$P_0(x)=1, P_1(x)=x, P_2(x)=\frac{1}{2}\left(3x^2-1\right), P_3(x)=\frac{1}{2}\left(5x^3-3x\right), P_4(x)=\left(35\ x^4-30\ x^2+3\right)\Big/8$$

where $P_n(x)$ is the Legendre function of 1^{st} kind.

8. Express $P(x)=x^4+2x^3+2x^2-x-3$ in terms of Legendre's polynomials.

9. Show that
(i) $P_n\ (1)=1$
(ii) $P_n(-x)=(-1)^n P_n(x)$

10. Prove that when n is odd $P_n(0)=0$

11. Prove that $C+\int P_n\,dx=\dfrac{P_{n+1}-P_{n-1}}{2n+1}$ where C is a constant.

III Bessel equation and Bessel function

12. Show that $J_{-\frac{1}{2}}(x)=\sqrt{\left(\dfrac{2}{\pi x}\right)}\cos x$ and $J_{\frac{1}{2}}(x)=\sqrt{\left(\dfrac{2}{\pi x}\right)}\sin x$ where $J_n(x)$ is the Bessel function of 1^{st} kind.

13. Prove that

$$J_0{}'=-J_1$$

$$J_2-J_0=2J_0{}''$$

14. Establish the differential formula

$$x^2 J_n{}''(x)=\left(n^2-n-x^2\right)J_n(x)+xJ_{n+1}(x),\ n=0,1,2,....$$

15. Prove that

$$\frac{d}{dx}\left(xJ_nJ_{n+1}\right)=x\left(J_n^2-J_{n+1}^2\right)$$