MA30003/MA41003 - Linear Algebra

Problem Sheet 5 *

October 31, 2017

1 Linear operators on inner product spaces

Problem 1.1. *Give an example of a linear operator T on an inner product space V such that $N(T) \neq N(T^*)$.*

Problem 1.2. *Prove that if $V = W \oplus W^{\perp}$ and T is the projection on W along $W^{\perp}$, then $T = T^*$.* Hint: *Recall that $N(T) = W^{\perp}$.*

Problem 1.3. *Let T be a linear operator on an inner product space V. Prove that $\|T(x)\| = \|x\|$ for all $x \in V$ if and only if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$.*

Problem 1.4. *Let V be an inner product space, and let T be a linear operator on V. Prove the following results.*

(a) $R(T^*)^{\perp} = N(T)$.

(b) *If V is finite-dimensional, the $R(T^*)^{\perp} = N(T)$.*

Problem 1.5. *Let V be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y\rangle z$ for all $x \in V$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.*

Problem 1.6. *Let V be a complex inner product space, and let T be a linear operator on V. Define*

$$T_1 = \frac{1}{2}(T + T^*) \text{ and } T_2 = \frac{1}{2i}(T - T^*).$$

(a) *Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.*

(b) *Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.*

(c) *Prove that T is normal if and only if $T_1T_2 = T_2T_1$.*

Problem 1.7. *Let T be a linear operator on an inner product space V, and let W be a T - invariant subspace of V. Prove the following results.*

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(a) *If T is self-adjoint, then T_W is self-adjoint.*

(b) *$W^\perp$ is T^* -invariant.*

(c) *If W is both T-invariant and T^* -invariant, then $(T_W)^* = (T^*)_W$.*

(d) *If W is both T-invariant and T^*-invariant and T is normal, then T_W is normal.*

Problem 1.8. *Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of . Prove that if W is T-invariant, then W is also T^* -invariant.*

Problem 1.9. *Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(T) = N(T^*)$ and $R(T) = R(T^*)$.*

Problem 1.10. *Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results*

(a) *If T is self-adjoint , then $\langle T(x), x\rangle$ is real for all $x \in V$.*

(b) *If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = 0$.* Hint: *Replace x by $x + y$ and the by $x + iy$, and expand the resulting inner products.*

(c) *If $\langle T(x), x\rangle$ real for all $x \in V$, then $T = T^*$.*

Problem 1.11. *For $z \in \mathbb{C}$ define $T_z : \mathbb{C} \to \mathbb{C}$ by $T_z(u) = zu$. Characterize those z for which T_z is normal, self-adjoint, or unitary.*

Problem 1.12. *Let T be a self-adjoint linear operator on a finite-dimensional inner product space. Prove that $(T + iI)(T - iI)^{-1}$ is unitary.*

Problem 1.13. *Let U be a linear operator on a finite-dimensional inner product space V. If $\|U(x)\| = \|x\|$ for all x in some orthonormal basis for V , must U be unitary ?. Justify your answer with a proof or a counterexample.*

Problem 1.14. *Suppose that if A and B are diagonalizable matrices. Prove or disprove that A is similar to B if and only if A and B are unitarily equivalent.*

Problem 1.15. *Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that*

(a) *$U(W) = W$.*

(b) *$W^\perp$ is U-invariant.*

Problem 1.16. *Find an example of a unitary operator U on an inner product space and a U-invariant subspace W such that $W^\perp$ is not U-invariant.*