Boundary Integral Methods and Applications to Fluid Mechanics

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1. **Difficulties**

2. **Numerical Finite part integration**

3. **Meth. of Kolm and Rokhlin**
   - Numerical tests
Computational Techniques

Numerical Methods in Mechanics

- Finite Element
- Boundary Element
- Finite Difference
Discretisation of a Domain

- Based on Variational methods
- Based on Boundary integral methods
Advantages of BEM

- Reduction of problem dimension by 1.
  - Less data preparation time.
  - Easier to change the applied mesh.
  - Useful for problems that require re-meshing.
- High Accuracy.
  - Stresses are accurate as there are no approximations imposed on the solution in interior domain points.
  - Suitable for modeling problems of rapidly changing stresses.
- Less computer time and storage.
  - For the same level of accuracy as other methods BEM uses less number of nodes and elements.
- Filter out unwanted information.
  - Internal points of the domain are optional.
  - Focus on particular internal region.
  - Further reduces computer time.

BEM is an attractive option.
Fundamental solution of Laplacian: Those functions $\phi$ that satisfy the Laplace equation $\nabla^2 \phi(x) = \delta(x)$ in a domain $D$ subject to homogeneous boundary conditions are known as Green’s functions for the Laplace equation.

**1 dim:**

$$\frac{d^2 \phi^*}{dx^2} = \delta(x - x_0)$$

$$\Rightarrow \phi^*(x, x_0) = \begin{cases} -\frac{1}{2}(x - x_0), & x > x_0 \\ \frac{1}{2}(x - x_0), & x < x_0 \end{cases}$$
2D Case:

\[ G(x - x_0) = \frac{1}{2\pi} \log r = \frac{1}{2\pi} \log |x - x_0| \]

3D Case:

\[ G(x - x_0) = \frac{1}{4\pi r} \]
Two-dimensional Potential Problem

\[ \nabla^2 \phi = 0 \quad \text{in} \quad \Omega \]

Can guess easily that "suitable boundary data" is required!
Given a data, solution can be attempted!
Divergence Theorem (flux conservation)

\[ \int_S F_i n_i \, ds = \int_\Omega \frac{\partial F_i}{\partial x_i} \, d\Omega \]  
(1)

\[ \int_S \frac{\partial \phi}{\partial x_i} n_i \, ds = \int_S \frac{\partial \phi}{\partial n} \, ds = \int_\Omega \nabla^2 \phi \, d\Omega \]  
(2)

Remark: \( \Omega \subset \mathbb{R}^n \) : bounded, simply connected; \( \Gamma \): sufficiently smooth (say, here \( C^2 \)). Looking for classical solutions.
Green’s identities

**Green’s first identity**

\[
\int_S \phi \frac{\partial \psi}{\partial x_i} n_i ds = \int_\Omega \frac{\partial}{\partial x_i} \left( \phi \frac{\partial \psi}{\partial x_i} \right) d\Omega \tag{3}
\]

\[
= \int_\Omega \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_i} d\Omega + \int_\Omega \phi \nabla^2 \psi d\Omega \tag{4}
\]

**Green’s second identity**

\[
\int_S \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds = \int_\Omega \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) d\Omega \tag{5}
\]
Consider the Green’s identity

\[ \int_S \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) \, ds = \int_{\Omega} \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) \, d\Omega \quad (6) \]

Choice: \( \phi \), the harmonic function that needs to be obtained
\( \phi^* \), the fundamental solution
\( x = (x_1, x_2); y = (y_1, y_2) \)

\[ \int_S \left( \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) \, ds = \int_{\Omega} \left( \phi(y) \nabla^2 \phi^*(x, y) ight. \]
\[ \left. - \phi^*(x, y) \nabla^2 \phi(y) \right) \, d\Omega_y \quad (7) \]
Exclusion of internal source point
Two - Dimensional Potential Problem contd.

\[
\int_S \left( \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) \, ds + \int_{S_\epsilon} \left( \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) \, ds_\epsilon = \\
\int_{\Omega - \Omega_\epsilon} \left( \phi(y) \nabla^2 \phi^*(x, y) - \phi^*(x, y) \nabla^2 \phi(y) \right) \, d\Omega_y
\]

- In \((\Omega - \Omega_\epsilon)\), \(\nabla^2 \phi = 0, \nabla^2 \phi^* = 0 \Rightarrow RHS = 0\).
\[ \int_S \left( \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) \, ds + \int_{S_\epsilon} \left( \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) \, ds_\epsilon = 0 \quad (BI) \]

- over \( S_\epsilon \) where \( ds_\epsilon(y) = \epsilon d\theta \)

\[ \frac{\partial \phi^*(x, y)}{\partial n_y} = \frac{\partial \phi^*(x, y)}{\partial r} \frac{\partial r}{\partial n_y} = \frac{1}{2\pi\epsilon} \]
Two - Dimensional Potential Problem contd.

Consider the second term on the LHS of (BI)

\[ \int_{S_\epsilon} \left( \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) \, ds_\epsilon(y) = \frac{1}{2\pi} \int_0^{\theta_x} \left( \phi(y) \frac{1}{\epsilon} + \ln \epsilon \frac{\partial \phi(x, y)}{\partial n_y} \right) \epsilon \, d\theta_x \]

\[ = \frac{1}{2\pi} \int_0^{\theta_x} \left( \phi(y) + \epsilon \ln \epsilon \frac{\partial \phi(x, y)}{\partial n_y} \right) \, d\theta_x \]

\[ \lim_{\epsilon \to 0} \int_{S_\epsilon} \left( \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) \, ds_\epsilon(y) = \frac{\theta_x}{2\pi} \phi(x) \quad (8) \]
The general integral representation is

$$\frac{\theta_x}{2\pi} \phi(x) = \int_S \phi^*(x, y) \frac{\partial \phi(y)}{\partial n_y} ds_y - \int_S \phi(y) \frac{\partial \phi^*(x, y)}{\partial n_y} ds_y$$

where

$$\theta_x = \begin{cases} 
2\pi, & x \in \Omega \\
\pi, & x \in \partial \Omega \\
0, & x \notin \Omega 
\end{cases}$$

analogy: general solution using separation of variables
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Layer Potentials

Consider the Integral Representation for $x \in \Omega$

$$\phi(x) = \int_S \left( \phi^*(x, y) \frac{\partial \phi(y)}{\partial n_y} - \phi(y) \frac{\partial \phi^*(x, y)}{\partial n_y} \right) ds_y$$

Here $\sigma = \frac{\partial \phi}{\partial n}, \tau = \phi$ are the Cauchy data on $\Gamma$.

$$V\sigma(x) = \int_S \phi^*(x, y) \sigma(y) ds_y \quad (Single \ - \ Layer)$$

$$W\tau(x) = \int_S \frac{\partial \phi^*(x, y)}{\partial n_y} \tau(y) ds_y \quad (Double \ - \ Layer)$$
Boundary potentials

for \( x \in \Gamma \)

\[
V(\sigma)(x) := V(\sigma)(x),
\]

\[
K(\tau)(x) := W(\tau)(x) + \frac{1}{2} \tau(x)
\]

\[
K^*(\sigma)(x) := \nabla_x V(\sigma)(x) \cdot \mathbf{n}_x - \frac{1}{2} \sigma(x)
\]

\[
D(\tau)(x) := -\nabla_x W(\tau)(x) \cdot \mathbf{n}_x;
\]
Consider the Integral Representation, when $\xi \in \partial \Omega$

$$
\frac{1}{2} \phi(\xi) = \int_{S} \phi^{*}(\xi, y) \frac{\partial \phi(y)}{\partial n_{y}} \, ds_{y} - \int_{S}^{pv} \phi(y) \frac{\partial \phi^{*}(\xi, y)}{\partial n_{y}} \, ds_{y}
$$

**Dirichlet Problem:** If the boundary value $\phi$ over $S$ is given, the missing data is $\sigma = \frac{\partial \phi}{\partial n} |_{\Gamma}$

$$
V \sigma(x) = \frac{1}{2} \phi(\xi) + K \tau(x)
$$

$$
\int_{S}^{pv} \phi^{*}(x, y) \sigma(y) \, ds_{y} = f(x)(\text{Fredholm I kind})
$$
Neumann Problem:

If the boundary value \( \frac{\partial \phi}{\partial n} \) over \( S \) is given, the missing data is \( \phi |_{\Gamma} \)

\[
\frac{1}{2} \sigma(x) - K^*(\sigma(x)) = D\tau(x)
\]

\[
\sigma(x) - 2 \int_{S}^{pv} \frac{\partial \phi^*(\xi, y)}{\partial n_y} \sigma(y) ds_y = g(x)
\]

(Fredholm II kind)
The challenges:

Discretizing the surface integrals (3D) or line integrals (2D)
Handling ”Singular Integrals”
approximate the boundary $\Gamma$ by line segments $\Gamma_i$ with endpoints $x_{i-1}, x_i$, with a nodal point at the midpoint of $\Gamma_i$.

values at the mid-points of each element $\Gamma_i$ are assumed equal to their values over the whole element.

$i \in 1, \ldots, N$ where $N$ is the no. of nodal points.
Discretization

Figure: BEM with constant elements.
W T Ang, A beginner’s course in BEMs, Universals-Publishers, 2007
\[
\frac{1}{2} \phi(x) = \int_S \phi^*(x, y) \frac{\partial \phi(y)}{\partial n_y} \, ds_y - \int_S \phi(y) \frac{\partial \phi^*(x, y)}{\partial n_y} \, ds_y, \quad x \in \partial \Omega
\]

\[
\frac{1}{2} \phi^j = -\sum_{j=1}^N \int_{\Gamma_j} \phi^*(x_i, y) \frac{\partial \phi(y)}{\partial n_y} \, ds_y + \sum_{j=1}^N \int_{\Gamma_j} \phi(y) \frac{\partial \phi^*(x_i, y)}{\partial n_y} \, ds_y
\]

\[
\phi = \phi^j; \quad \frac{\partial \phi}{\partial n} = \partial \phi^j \text{ on the } j^{th} \text{ element}
\]
\[-\frac{1}{2} \phi^i + \sum_{j=1}^{N} \left( \int_{\Gamma_j} \frac{\partial \phi^*}{\partial n} dS \right) \phi^j = \sum_{j=1}^{N} \left( \int_{\Gamma_j} \phi^* dS \right) \partial \phi^j \]

\[\hat{H}_{ij} = \int_{\Gamma_j} \frac{\partial \phi^*(x_i, y)}{\partial n_y} dS, \quad G_{ij}^* = \int_{\Gamma_j} \phi^*(x_i, y) dS\]
\[-\frac{1}{2} \phi*^i + \sum_{j=1}^{N} \hat{H}_{ij} \phi*^j = \sum_{j=1}^{N} G_{ij} \partial \phi*^j\]

Let, \( H_{ij} = \hat{H}_{ij} - \frac{1}{2} \delta_{ij} \), we have

\[
\sum_{j=1}^{N} H_{ij} \phi*^j = \sum_{j=1}^{N} G_{ij} \partial \phi*^j
\]

\[
[H]_{N \times N} \phi^*_{N \times 1} = [G]_{N \times N} \partial \phi^*_{N \times 1}
\]
Example:

\[ \nabla^2 \phi = 0, \quad 0 < x < 1, \quad 0 < y < 1 \]

\[ \phi(0, y) = 0; \quad \phi(1, y) = \cos \pi y \]

\[ \frac{\partial \phi}{\partial n}(x, 0) = 0; \quad \frac{\partial \phi}{\partial n}(x, 1) = 0 \]

\[ \phi(x, y) = \frac{\sinh \pi x \cos \pi x}{\sinh \pi} \]
Laplacian

Figure: Boundary nodes (o) and midpoints of boundary elements (x).
Laplacean

Figure: (a) Surface plot of solution $\phi$, (b) BEM solution Vs exact solution of along $x=0.5$ and $x=0.8$. 
Helmholtz equation-application to Acoustics

Figure: The geometry of the circular cylinder
Figure: Distribution of solid angles $C(P)$ and surface pressure (real and imaginary part) along the boundary nodes
Figure: Distribution of field pressure $p_f$ away from the surface of the cylinder
Stokes flow - Example

\[- \nabla p + \nabla^2 \mathbf{v} = 0 \quad \text{in } \Omega, \quad (9)\]
\[\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \quad (10)\]

\[v_{\theta}(a, \theta) = f(\theta) = 4a^3 \cos 2\theta - 3a^2(1 - \sin \theta) \cos \theta\]
\[p(a, \theta) = g(\theta) = -2a(1 - 12 \sin \theta) \cos \theta\]

Stokes flow

**Figure:** The normal components of velocity along the boundary with 12 constant elements.
Stokes flow

Figure: tangential components of velocity along the boundary with 12 constant elements
Difficulties as per M.Sc. project by Aftab Yusuf Patel

- Off diagonal coefficients are regular. Standard integration methods used.
- Diagonal coefficients involve singular, hypersingular integrals, for which singular terms cancel (see [Kohr, Raja Sekhar, 2008]). Standard methods do not work due to complexity of kernels (esp. corresponding to Brinkman eq.).
- Standard method of taking distorted paths of integration around singularities is too slow to be of practical utility. Program written in MATHEMATICA proved to be too inefficient to run on available hardware.
Principal value and finite part integrals

- Attacking the subproblems

\[
\int_{a}^{b} \frac{\phi(x)}{x - y} \, dx
\]  

(11)

where \( y \in (a, b) \) do not exist in the classical Riemann or Lebesgue sense.

- Cauchy Principal Value:

\[
\lim_{\epsilon \to 0} \left( \int_{a}^{y-\epsilon} \frac{\phi(x)}{x - y} \, dx + \int_{y+\epsilon}^{b} \frac{\phi(x)}{x - y} \, dx \right)
\]  

(12)
Strongly singular or hypersingular

\[
\int_a^b \frac{\phi(x)}{(x - y)^2} \, dx
\]

with \( y \in (a, b) \) are divergent in the classical sense.

- Hadamard Finite Part interpretation:

\[
\lim_{\epsilon \to 0} \left( \int_a^{y-\epsilon} \frac{\phi(x)}{(x - y)^2} \, dx + \int_{y+\epsilon}^b \frac{\phi(x)}{(x - y)^2} \, dx - \frac{2\phi(y)}{\epsilon} \right)
\]
In [Kolm, Rokhlin, 2001] a method was developed to numerically integrate functions of the form,

\[ f(x) = A(x) + B(x) \log|x| + \frac{C(x)}{x} + \frac{D(x)}{x^2} \quad (15) \]

- 0 belongs to the interior of the interval of integration
- the functions \( A, B, C, D \) are not available separately
- Method was implemented in C for maximum efficiency and is the first such publicly available implementation to the best of our knowledge
The implementation of the method was tested on the test cases:

\[ \int_{-1}^{1} \frac{\sin(x + \frac{\pi}{3}) + \cos(x)}{x} \, dx \]

has a singularity of the $1/x$ type.

\[ \int_{-1}^{1} K_2(|x|) \, dx \]

has singularities of $\log|x|$ and $1/x^2$ type. The method was found to be accurate to 10 decimal places. The degree $M$ was taken to be $M = \lceil N/4 \rceil + 1$, \( \lceil \cdot \rceil \) being the greatest integer function.
Test case 1

**Figure:** Error in computing integral \( \int_{-1}^{1} \left( \frac{\sin(x + \frac{\pi}{3}) + \cos(x)}{x} \right) dx \) from \(-1\) to 1 with \( M = \lceil N/4 \rceil + 1 \), vs. number of quadrature points \( N \).
Test case 2

Figure: Error in computing integral \[ \int_{-1}^{1} K_2(|x|) \, dx \] from \(-1\) to \(1\) with \(M = \lceil N/4 \rceil + 1\), vs. number of quadrature points \(N\).
Summary

try to avoid handling the entire domain via boundary integrals
Green’s function is required
friendly to linear PDEs
Boundary and Domain integral methods for Non-linear PDEs possible
Rich theory of existence and uniqueness for the Integral Operators: Lipschitz domains, Sobolev spaces
Numerical methods: Boundary Elements
handling singular integrals is the bottle neck when we succeed, the expenses are going to be very less compared to domain methods
Numerical tests


Numerical tests


ThAnKs FoR yOuR aTTEnTiOn!