

# Energy of Signed Graphs

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# Outline of Talk

- 1-Basic Definitions related to signed graphs.
- 2-Adjacency Spectrum and Coefficient Theorem.
- 3-Energy of a signed graph
- 4-Integral Representations.
- 5-Quasi-order .
- 6-Characterization of unicyclic signed graphs with minimal energy.

## Definition

A signed graph (sigraph in short) is an ordered pair  $S = (G, \sigma)$  where  $G = (V, E)$  is the graph called the underlying graph of  $S$  and  $\sigma : E \rightarrow \{+1, -1\}$ , called a signing (also called signature), is a function from edge set  $E$  of  $G$  into the set  $\{+1, -1\}$  of signs.

Throughout we assume  $S$  is simple and loop-free.

The sign of a cycle in a signed graph is defined to be the product of signs of its edges.

A cycle is said to be positive if its sign is positive and negative if its sign is negative.

### Definition

A signed graph is said to be balanced if each of its cycle is positive and unbalanced, otherwise.

## Switching Equivalence

Let  $S$  be a signed graph with vertex set  $V$ .

Switching  $S$  by set  $X \subset V$  means reversing the signs of all edges between  $X$  and its complement.

Two signed graphs are said to be switching equivalent if one can be obtained from the other by switching.

Switching equivalence is an equivalence relation on the signings of a fixed graph. An equivalence class is called a switching class.

A switching class of  $S$  is denoted by  $[S]$ .

## Adjacency matrix of a signed graph.

The adjacency matrix of a signed graph  $S$  whose vertices are  $v_1, v_2, \dots, v_n$  is the  $n \times n$  matrix  $A(S) = (a_{ij})$ , where

$$a_{ij} = \begin{cases} \sigma(v_i, v_j), & \text{if there is an edge from } v_i \text{ to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix of a signed graph  $S$  is  $(-1, 0, 1)$  symmetric matrix and so all its eigenvalues are real. The set of distinct eigenvalues of  $S$  together with multiplicities is known as spectrum of  $S$ . Two signed graph are said to be co-spectral if they have same spectrum.

**Elementary Figure:** A signed graph which is either a signed edge or a signed cycle.

**Basic Figure:** A signed subgraph is whose components are elementary figures.

The following is the coefficient theorem for signed graphs [1].

**Theorem 1** If  $S$  is a signed graph with characteristic polynomial

$$\phi_S(x) = x^n + a_1(S)x^{n-1} + \cdots + a_{n-1}(S)x + a_n(S),$$

then

$$a_j(S) = \sum_{L \in \mathcal{L}_j} (-1)^{p(L)} 2^{|c(L)|} \prod_{Z \in c(L)} s(Z),$$

for all  $j = 1, 2, \dots, n$ , where  $\mathcal{L}_j$  is the set of all basic figures  $L$  of  $S$  of order  $j$ ,  $p(L)$  denotes number of components of  $L$ ,  $c(L)$  denotes the set of all cycles of  $L$  and  $s(Z)$  the sign of cycle  $Z$ .

## Spectral Criterion for Balance

**Theorem 2** A signed graph  $S$  is balanced if and only if  $S$  and its underlying graph  $G$  are Co-spectral.

The concept of energy of a graph was given by Gutman [4].

Gutman defined the energy of a graph as the sum of absolute values of eigenvalues of adjacency matrix and also obtained integral representation for the energy. Germina and Hameed [3] generalized the concept of energy to Signed graphs.

### Definition

Let  $S$  be a signed graph with  $n$  vertices and let  $x_1, x_2, \dots, x_n$ , be the eigenvalues of  $A(S)$ . Then the energy of Signed graph  $S$  is defined as

$$E(S) = \sum_{i=1}^n |x_i|.$$

# Examples

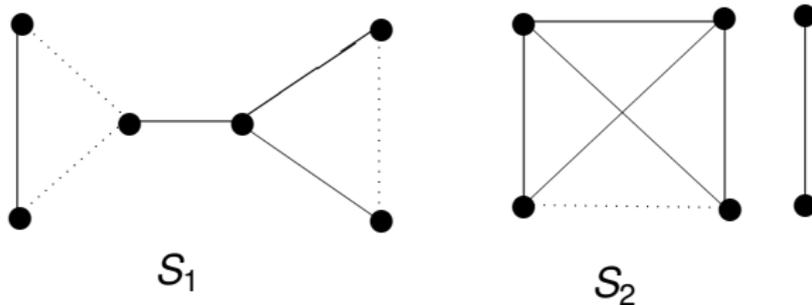


Fig. 1 A pair of co-spectral signed graphs.

Consider the signed graphs  $S_1$  and  $S_2$  as shown in Figure 1.

Their spectrum is given by

$$\text{spec}(S_1) = \text{spec}(S_2) = \{-\sqrt{5}, -1^{(2)}, 1^{(2)}, \sqrt{5}\}$$

and

$$E(S_1) = E(S_2) = 4 + 2\sqrt{5}.$$

## Coulson's Integral formula for the energy

Next, we state Coulson type integral formula for the energy of a signed graph.

**Theorem 3** Let  $S$  be a signed graph with  $n$  vertices having characteristic polynomial  $\phi_S(x)$ . Then

$$E(S) = \sum_{j=1}^n |x_j| = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( n - \frac{\iota x \phi_S'(\iota x)}{\phi_S(\iota x)} \right) dx,$$

where  $x_1, x_2, \dots, x_n$  are the eigenvalues of signed graph  $S$ ,  $\iota = \sqrt{-1}$  and  $\int_{-\infty}^{\infty} F(x) dx$  denotes the principal value of the respective integral.

## Another form of Coulson's Integral formula for the energy

An immediate Consequence of Coulson's integral formula is the following.

If  $S$  is a signed graph on  $n$  vertices, then

$$E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log |x^n \phi_S(\frac{t}{x})| dx.$$

Following is an immediate consequence of above integral representation.

**Theorem 4.** If  $S$  is a signed graph on  $n$  vertices with characteristic polynomial

$\phi_S(x) = x^n + a_1(S)x^{n-1} + \dots + a_{n-1}(S)x + a_n(S)$ , then

$$E(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log \left[ \left( \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j a_{2j}(S) x^{2j} \right)^2 + \left( \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j a_{2j+1}(S) x^{2j+1} \right)^2 \right] dx$$

We say that a signed graph has the pairing property if its spectrum is symmetric with respect to origin.

Bipartite signed graphs obviously have this property. There exist non bipartite signed graphs with spectrum symmetric about origin e.g., Signed graphs  $S_1$  and  $S_2$  in Figure 1.

We denote by  $\Delta_n$ , the set of all signed graphs on  $n$  vertices with pairing property.

Deepa Sinha et al. [3] characterized signed graphs with pairing property. These are precisely the signed graphs in which  $S$  and  $-S$  are switching equivalent.

Put  $b_j(S) = |a_j(S)|$ ,  $j = 1, 2, \dots, n$ . Note  $b_1(S) = 0$ ,  $b_2(S) =$  the number of edges of the signed graph  $S$  and so on. The integral formula given in Theorem 4 expresses energy of a signed graph in terms of the coefficients of the characteristic polynomial. It becomes helpful when even and odd coefficients respectively alternate in sign. Fortunately, we have this property in unicyclic signed graphs.

## Alternating property of coefficients

A connected signed graph with  $n$  vertices and  $m$  edges is said to be unicyclic if  $m = n$ . Let  $S(n, g)$  the set of unicyclic signed graphs with  $n$  vertices and having cycle of length  $g$ .

**Theorem 5.** Let  $S \in S(n, g)$ . Then (i)  $(-1)^j a_{2j}(S) \geq 0$  for all  $j \geq 0$  irrespective of  $S$  is balanced or unbalanced and  $g$  is odd or even. Also  $a_{2j+1} = 0$  if  $g$  is even.

(ii)  $(-1)^j a_{2j+1}(S) \geq 0$  for all  $j \geq 0$  if either  $g \equiv 3 \pmod{4}$  and  $S$  balanced or  $g \equiv 1 \pmod{4}$  and  $S$  is unbalanced.

(iii)  $(-1)^j a_{2j+1}(S) \leq 0$  for all  $j \geq 0$  if either  $g \equiv 1 \pmod{4}$  and  $S$  is balanced or  $g \equiv 3 \pmod{4}$  and  $S$  is unbalanced.

For unicyclic signed graphs, Theorem 4 takes the following form.

**Theorem 6.** Let  $S$  be a unicyclic signed graph of order  $n$ . Then

$$E(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log \left[ \left( \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} b_{2j}(S) x^{2j} \right)^2 + \left( \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} b_{2j+1}(S) x^{2j+1} \right)^2 \right] dx.$$

## Quasi-order Comparison.

Let  $S_1$  and  $S_2$  be two unicyclic signed graphs.

If  $b_j(S_1) \leq b_j(S_2)$  for all  $j \geq 0$ , we define  $S_1 \preceq S_2$ . If in addition  $b_j(S_1) < b_j(S_2)$  for some  $j$ , then we write  $S_1 \prec S_2$ . It is clear from Theorem 6 that  $S_1 \preceq S_2$  implies  $E(S_1) \leq E(S_2)$  and  $S_1 \prec S_2$  implies  $E(S_1) < E(S_2)$ .

For a signed graph  $S \in \Delta_n$ ,

$$\phi_S(x) = x^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k}(S) x^{n-2k}, \text{ where } b_{2k}(S) = |a_{2k}(S)|$$

for all  $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$

For a signed graph  $S \in \Delta_n$ , Theorem 6 takes the form

**Theorem 7.** If  $S \in \Delta_n$ , then

$$E(S) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2} \log[1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} b_{2j}(S) x^{2j}] dx.$$

In particular, if  $S_1, S_2 \in \Delta_n$  and  $S_1 \prec S_2$ , then  $E(S_1) < E(S_2)$ .

Two signed graphs need not be always quasi-order comparable. If it is the case, then their energy cannot be compared using Theorem 6.

In this case, we compare the energy by directly solving integrals. (This process usually becomes complicated and we do have any other efficient method and many energy comparison problems wait for solution e.g., characterizing bicyclic signed graphs with maximal energy etc.) We will come across this case at the end of the talk. Fortunately, in unicyclic case it becomes easy to handle integrals.

## Switching and cospectral classes.

There is just one switching class in trees. All signed trees with same underlying tree are switching equivalent.

There are two switching classes on the signings of a unicyclic graph; One class containing unicyclic signed graphs with positive cycle and other class containing unicyclic signed graphs with a negative cycle. In view of spectral criterion, there are two cospectral classes in Unicyclic signed graphs with same underlying graph.

## More Notations and Symbols

we denote a positive and a negative cycle of length  $n$  by  $C_n$  and  $\mathbf{C}_n$  respectively.

Let  $S_n^g$  (respectively,  $\mathbf{S}_n^g$ ) denote the balanced (respectively, unbalanced) unicyclic signed graph of order  $n$  obtained by identifying the center of the signed star on  $n - g + 1$  vertices with a vertex of a positive (respectively, negative) cycle of order  $g$ , where  $n \geq g \geq 3$  (see Fig. 2).

$m(S, j)$  denote number of matchings of  $S$  of size  $j$ .

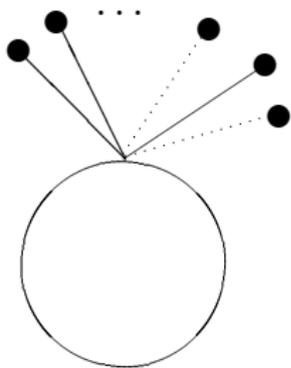


Fig.2 Signed graph  $S_n^g$  ( or  $\mathbf{S}_n^g$  ).

## Energy comparison in unicyclic signed graphs

**Theorem 7** Let  $G$  be a unicyclic graph of odd girth. Then any two signed graphs on  $G$  have the same energy.

**Proof.** Let  $G$  be a unicyclic graph of order  $n$  and odd girth  $g$ . Let  $S$  be any balanced signed graph on  $G$  and  $T$  be any unbalanced signed graph on  $G$ . Then  $S$  and  $T$  are non-cospectral by spectral criterion. The coefficients of signed graphs  $S$  and  $T$  are related as follows

$$a_{2j+1}(S) = -a_{2j+1}(T), \quad \text{for all } j = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$$

Next we use Theorem 4 to compare the energies of signed graphs obtained from a unicyclic bipartite graph.

**Theorem 8.** Let  $G$  be a unicyclic graph of order  $n$  and even girth  $g$ , i.e., the bipartite unicyclic graph and let  $S$  be any balanced signed graph on  $G$  and  $T$  be any unbalanced one. Then

- (i)  $E(S) < E(T)$  if  $g \equiv 0 \pmod{4}$ ,
- (ii)  $E(S) > E(T)$  if  $g \equiv 2 \pmod{4}$ .

**Proof.** Let  $G$  be a unicyclic graph of order  $n$  and even girth  $g \geq 4$  and let  $S$  and  $T$  respectively be any balanced signed graph and any unbalanced signed graph on  $G$ .

The coefficients of  $S$  are given by

$$a_{2j}(S) = (-1)^j m(S, j) \quad \text{for all } j = 1, 2, \dots, \frac{g}{2} - 1,$$

$$a_{g+2j}(S) = -2(-1)^j m(S - C_g, j) + (-1)^{\frac{g}{2}+j} m(S, \frac{g}{2} + j)$$

for all  $j = 0, 1, \dots, \lfloor \frac{n-g}{2} \rfloor$  and

$$a_{2j+1}(S) = 0 \quad \text{for all } j = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor,$$

whereas coefficients of  $T$  are given by

$$a_{2j}(T) = (-1)^j m(T, j) \quad \text{for all } j = 1, 2, \dots, \frac{g}{2} - 1,$$

$$a_{g+2j}(T) = 2(-1)^j m(T - \mathbf{C}_g, j) + (-1)^{\frac{g}{2}+j} m(T, \frac{g}{2} + j)$$

for all  $j = 0, 1, \dots, \lfloor \frac{n-g}{2} \rfloor$  and

$$a_{2j+1}(T) = 0 \quad \text{for all } j = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor.$$

Two cases arise, Case(i)  $g \equiv 0 \pmod{4}$  and Case(ii)  $g \equiv 2 \pmod{4}$ .

**Case (i).**  $g \equiv 0 \pmod{4}$ . We have

$$b_{2j+1}(S) = b_{2j+1}(T) = 0 \quad \text{for all } j = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor,$$

$$b_{2j}(S) = b_{2j}(T) \quad \text{for all } j = 1, 2, \dots, \frac{g}{2} - 1,$$

$$b_{g+2j}(S) = | -2m(S - C_g, j) + m(S, \frac{g}{2} + j) |$$

for all  $j = 0, 1, \dots, \lfloor \frac{n-g}{2} \rfloor$

and

$$b_{g+2j}(T) = |2m(T - \mathbf{C}_{g,j}) + m(T, \frac{g}{2} + j)| \text{ for all } j = 0, 1, \dots, \lfloor \frac{n-g}{2} \rfloor.$$

Clearly,  $b_j(S) \leq b_j(T)$  for all  $j = 1, 2, \dots, n$ . In particular  $b_g(S) < b_g(T)$ . Therefore,  $S \prec T$  and by Theorem 6,  $E(S) < E(T)$ .

The proof of **Case(ii)** follows on similar lines.

## Proof Continued...

and

$$a_{2j}(S) = a_{2j}(T), \quad \text{for all } j = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor.$$

Thus  $b_j(S) = b_j(T)$  for all  $j = 1, 2, \dots, n$ . By Theorem 6,  
 $E(S) = E(T)$ .

Similar to unsigned graphs, the following result holds for unbalanced unicyclic signed graphs

**Theorem 8** Let  $S \in S(n, g)$  be unbalanced and  $S \neq \mathbf{S}_n^g$ . Then  $\mathbf{S}_n^g \prec S$  and  $E(\mathbf{S}_n^g) < E(S)$ .

**Theorem 9.** Let  $n \geq g$ , where  $n \geq 6$  and  $g \geq 5$ . Then  $\mathbf{S}_n^4 \prec \mathbf{S}_n^g$  and  $E(\mathbf{S}_n^4) < E(\mathbf{S}_n^g)$ .

## Unicyclic signed graphs with minimal energy

**Theorem 10** Let  $S$  be an unbalanced unicyclic signed graph with  $n \geq 6$  vertices and  $S \neq \mathbf{S}_n^3$ . Then  $E(\mathbf{S}_n^3) < E(S)$ .

**Proof.** In view of Theorems 8 and 9, it suffices to prove that  $E(\mathbf{S}_n^3) < E(\mathbf{S}_n^4)$  for all  $n \geq 6$ . By Coefficient Theorem, we have

$$\phi_{\mathbf{S}_n^3}(x) = x^{n-4} \{x^4 - nx^2 + 2x + (n-3)\} \quad (1.1)$$

The coefficients of the characteristic polynomials of  $\mathbf{S}_n^3$  and  $\mathbf{S}_n^4$  are quasi-order in comparable, so we cannot use the previous quasi-order comparison. In this case, we compare the energy by directly solving the integrals.

$$E(\mathbf{S}_n^4) - E(\mathbf{S}_n^3) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{x^2} \log \frac{[(1 + nx^2 + 2(n-2)x^4)]^2}{[(1 + nx^2 + (n-3)x^4)^2 + 4x^6]} dx.$$

Let  $f(x) = [1 + nx^2 + 2(n-2)x^4]^2$  and

$g(x) = [(1 + nx^2 + (n-3)x^4)^2 + 4x^6]$ . Then

$$\begin{aligned} f(x) - g(x) &= [1 + nx^2 + 2(n-2)x^4]^2 \\ &- [(1 + nx^2 + (n-3)x^4)^2 + 4x^6] \\ &= 2(n-1)x^4 + 2(n-2)(n+1)x^6 \\ &+ (3n-7)(n-1)x^8 \\ &> 0 \end{aligned}$$

for all  $n \geq 6$ . Therefore,  $E(\mathbf{S}_n^3) < E(\mathbf{S}_n^4)$ , for all  $n \geq 6$ .



## Conclusion

Among all unicyclic signed graphs with  $n \geq 6$  vertices, all signed graphs in  $[S_n^3]$  and  $[\mathbf{S}_n^3]$  have minimal energy. Moreover, for  $n = 3, 4$  and  $5$  all signed graphs in  $[S]$  have minimal energy, where  $S$  is one of the signed graphs shown in Fig. 3.

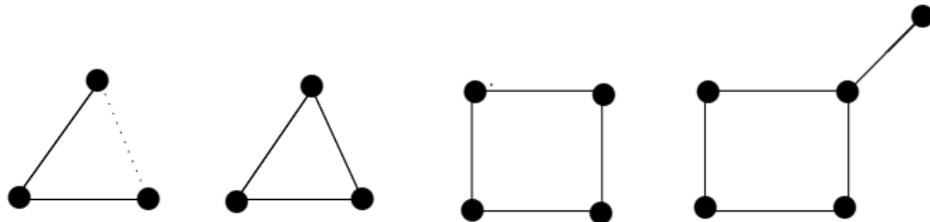


Fig. 3 Unicyclic signed graphs of order  $\leq 5$  with minimal energy.

This characterization verifies a signed analogue of the following conjecture by Caprossi et al.

**Conjecture** Among all connected graphs  $G$  with  $n \geq 6$  vertices and  $n - 1 \leq m \leq 2(n - 2)$  edges, the graph with minimum energy are stars with  $m - n + 1$  additional edges all connected to the same vertex for  $m \leq n + \lfloor \frac{(n-7)}{2} \rfloor$ , and bipartite graphs with two vertices on one side, one of which is connected to all vertices on the other side, otherwise.

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Spectral Analysis of t path signed graphs

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***Thank you for your  
attention***