MA60053 - Computational Linear Algebra
Problem Sheet 2

Problem 1. *Prove that* $||A||_2 \leq \sqrt{||A||_1||A||_\infty}$

Problem 2. *For the matrix* $A = uv^T$, *where* $u \in \mathbb{R}^m$ *and* $v \in \mathbb{R}^n$. *Show that* $||A||_2 = ||u||_2||v||_2$.

Problem 3. *If* O *is an orthogonal matrix, then prove the following:*

1. $||O||_2 = 1$,
2. $||AO||_2 = ||A||_2$,
3. $||AO||_F = ||A||_F$.

Problem 4. *If* P *and* Q *are orthogonal matrices, then prove that*

1. $||PAQ||_F = ||A||_F$,
2. $||PAQ||_2 = ||A||_2$.

Problem 5. *Prove that* $||A||_\infty = max_{1\leq i,j\leq n}|a_{ij}|$ *is a norm, but not a matrix norm.*

Problem 6. *Show that for any induced matrix norm* $||.||$, $\rho(A) \leq ||A||$, *where* $\rho(A)$ *is the spectral radius of the matrix*A.

Problem 7. *Show that for any induced matrix norm* $||.||$, *if* $||E|| < 1$, *then* $(I - E)$ *is non-singular, and* $||(I - E)^{-1}|| \leq (1 - ||E||)^{-1}$.

Problem 8. *The Hilbert matrix* $H_n = (\frac{1}{i+j-1})$ *is positive definite.*

Problem 9. *For an* $n \times n$ *matrix* A, *if* $\kappa_\alpha(A)$ *and* $\kappa_\beta(A)$ *are condition numbers with respect to different matrix norms, then prove that there* $c_1\kappa_\alpha(A) \leq \kappa_\beta(A) \leq c_2\kappa_\alpha(A)$, *for some constants* $c_1 > 0,\ c_2 > 0$.

Problem 10. *Let* A *be an* $n \times n$ *nonsingular matrix. Then, prove that*

$$\min\left\{||E||_2 : A + E \text{ is singular}\right\} = \frac{1}{||A^{-1}||_2}.$$

Problem 11. *Let* A *be an* $n \times n$ *nonsingular matrix. Then, prove that*

$$\min\left\{\frac{||E||_2}{||A||_2} : A + E \text{ is singular}\right\} = \frac{1}{\kappa_2(A)},$$

where $\kappa_2(A) = ||A||_2||A^{-1}||_2$