MA60053 - Computational Linear Algebra
Problem Sheet 1

Problem 1. *Let* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{n\times n}$*. Prove that the non-zero eigenvalues of the matrices* BA *and* AB *are the same.*

A square matrix A is said to be

(a) involutory if $A^2 = I$,

(b) idempotent if $A^2 = A$,

(c) nilpotent if $A^k = 0$ for positive integer k.

Problem 2. *Let* $x \in \mathbb{C}^{n\times 1}$, $y \in \mathbb{C}^{1\times n}$*. When* xy *is idempotent? When* xy *is nilpotent?*

Problem 3. *Let* $x \in \mathbb{C}^n$ *and* $x^*x = 1$*. Show that* $I - xx^*$ *is idempotent.*

Problem 4. *Let* $x \in \mathbb{C}^n$ *and* $x^*x = 1$*. Show that* $I - 2xx^*$ *is involutory.*

Problem 5. *Prove that an idempotent matrix is either the identity or else is singular.*

Problem 6. *Prove that square root of a symmetric positive (semi)definite matrix is unique.*

Problem 7. *Let* $A \in \mathbb{R}^{n\times n}$ *be symmetric positive definite, and let* $B \in \mathbb{R}^{n\times n}$*. Then show the following:*

1. *the matrix* B^TAB *is positive semi-definite,*
2. $rank(B^TAB) = rank(B)$,
3. *the matrix* B^TAB *is positive definite if and only if* $rank(B) = m$.

Problem 8. *Let* A *be an orthogonal matrix. Prove that* $det(A) = \pm 1$*. If* B *is an orthogonal matrix such that* $det(B) = -det(A)$*, then show that* $A + B$ *is singular.*