NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 22, 2012

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 7 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
  The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
  The symbol I will denote the identity matrix of appropriate order.
  We denote by $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$), the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$).
  We denote by $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) the group (under matrix multiplication) of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and by $SL_n(\mathbb{R})$ (respectively, $SL_n(\mathbb{C})$), the subgroup of matrices with determinant equal to unity. The trace of a square matrix A will be denoted tr(A) and the determinant by det(A).
  The derivative of a function f will be denoted by f'.
  All logarithms, unless specified otherwise, are to the base e.
- **Calculators are not allowed.**

Section 1: Algebra

1.1 Solve the following equation, given that its roots are in arithmetic progression.

$$x^3 - 6x^2 + 13x - 10 \;=\; 0.$$

1.2 Evaluate:

$$\sum_{k=1}^{n} \frac{k}{n} \binom{n}{k} t^k (1-t)^{n-k}$$

where $\binom{n}{k}$ stands for the usual binomial coefficient giving the number of ways of choosing k objects from n objects.

1.3 Which of the following form a group under matrix multiplication?

a.

$$\left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \;:\; a \neq 0, a \in \mathbb{R} \right\}.$$

b.

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \;:\; |a| + |b| \neq 0, a, b \in \mathbb{R} \right\}.$$

c.

$$\left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \;:\; \theta \in [0, 2\pi[\right\}.$$

1.4 In each of the following, state whether the given set is a normal subgroup or, is a subgroup which is not normal or, is not a subgroup of $GL_n(\mathbb{C})$.
a. The set of matrices with determinant equal to unity.
b. The set of invertible upper triangular matrices.
c. The set of invertible matrices whose trace is zero.

1.5 Let S_5 denote the symmetric group of all permutations of the five symbols $\{1, 2, 3, 4, 5\}$. What is the highest possible order of an element in this group?

1.6 On $\mathbb{R}^2$, consider the linear transformation which maps the point (x, y) to the point $(2x + y, x - 2y)$. Write down the matrix of this transformation with respect to the basis

$$\{(1, 1), (1, -1)\}.$$

1.7 Let V be the subspace of $\mathbb{M}_2(\mathbb{R})$ consisting of matrices such that the entries of the first row add up to zero. Write down a basis for V.

1.8 Let $A \in \mathbb{M}_2(\mathbb{R})$ such that $\mathrm{tr}(A) = 2$ and $\det(A) = 3$. Write down the characteristic polynomial of A^{-1}.

1.9 A non-zero matrix $A \in \mathbb{M}_n(\mathbb{R})$ is said to be *nilpotent* if $A^k = 0$ for some positive integer $k \geq 2$. If A is nilpotent, which of the following statements are true?
a. Necessarily, $k \leq n$ for the smallest such k.
b. The matrix $I + A$ is invertible.
c. All the eigenvalues of A are zero.

1.10 Write down a necessary and sufficient condition, in terms of a, b, c and d (which are assumed to be real numbers), for the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

not to have a real eigenvalue.

Section 2: Analysis

2.1 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Pick out the cases which imply that the sequence is Cauchy.
a. $|x_n - x_{n+1}| \leq 1/n$ for all n.
b. $|x_n - x_{n+1}| \leq 1/n^2$ for all n.
c. $|x_n - x_{n+1}| \leq 1/2^n$ for all n.

2.2 Pick out the convergent series.
a.
$$\sum_{n=1}^{\infty} \left((n^3+1)^{\frac{1}{3}} - n \right).$$
b.
$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+\frac{3}{2}}}.$$
c.
$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}.$$

2.3 List the sets of points of discontinuity, if any, for the following functions.
a. $f : [-1,1] \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is irrational,} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$
b. $f : [-1,1] \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational,} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$
c. $f : [0,\infty[\to \mathbb{R}$ defined by
$$f(x) = \begin{cases} (x) & \text{if } [x] \text{ is even,} \\ 1-(x) & \text{if } [x] \text{ is odd} \end{cases}$$
where $[x]$ is the largest integer less than, or equal to x and $(x) = x - [x]$.

2.4 Let $\{f_n\}$ be a sequence of functions defined on $[0,1]$. Determine $f(x) = \lim_{n\to\infty} f_n(x)$, for each of the following.
a. $f_n(x) = n^2 x(1-x^2)^n$.
b. $f_n(x) = nx(1-x^2)^n$.
c. $f_n(x) = x(1-x^2)^n$.

2.5 For each of the cases (a), (b) and (c) of Question 2.4 above, determine if the following claim is true or false:
$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx.$$

2.6 Pick out the true statements:
a. $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
b. $|\sin 2x - \sin 2y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
c. $|\sin^2 x - \sin^2 y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

2.7 Let $x > 0$. Fill in the blanks with the correct sign $>, \geq, <$ or $\leq$:
a.
$$\tan^{-1} x \;\ldots\ldots\; \frac{x}{1+x^2}.$$
b.
$$\log(1+x) \;\ldots\ldots\; \frac{x}{1+x}.$$

2.8 Write down explicitly the expression for the n-th derivative of the function $f(x) = x^2 e^{3x}$.

2.9 Find all the square roots of the complex number $2i$.

2.10 Determine the points where $f'(z)$ exists and write down its value at those points in the following cases:
a. $f(z) = y(x + iy)$
b. $f(z) = x^2 + iy^2$
where $z = x + iy, x, y \in \mathbb{R}$.

Section 3: Geometry

3.1 Find the area of the pentagon whose vertices are the fifth roots of unity in the complex plane.

3.2 Let $a, b \in \mathbb{R}$. If P is the point in the plane whose coordinates are (x, y), define $f(P) = ax + by$.Let the line segment AB bisect the line segment CD. If $f(A) = 5, f(B) = 5$ and $f(C) = 10$, find $f(D)$.

3.3 Which of the following sets are bounded in the plane $\mathbb{R}^2$?
a. $\{(x, y) \ : \ 2x^2 + 2xy + 2y^2 = 1\}$.
b. $\{(x, y) \ : \ xy = 1\}$.
c. $\{(x, y) \ : \ y \geq 0, |x| = \sqrt{y}\}$.

3.4 Which of the sets described in Question 3.3 above are made up of two (or more) disjoint connected components?

3.5 Let $x_1 > 0$ and $y_1 > 0$. If the portion of a line intercepted between the coordinate axes is bisected at the point (x_1, y_1), write down the equation of the line.

3.6 Find λ such that the equation

$$x^2 + 5xy + 4y^2 + 3x + 2y + \lambda = 0$$

represents a pair of straight lines.

3.7 Write down the condition that the plane $\ell x + my + nz = p$ is tangent to the sphere $x^2 + y^2 + z^2 = r^2$.

3.8 Write down the equation of the plane parallel to $4x + 2y - 7z + 6 = 0$ which passes through the point $(2, -4, 5)$.

3.9 Write down the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

3.10 A plane moves so that its distance from the origin is a constant p. Write down the equation of the locus of the centroid of the triangle formed by its intersection with the three coordinate planes.