NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 25, 2010

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 7 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
  The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
  The symbol I will denote the identity matrix of appropriate order.
  The derivative of a function f will be denoted by f'.
  All logarithms, unless specified otherwise, are to the base e.
- **Calculators are not allowed.**

Section 1: Algebra

1.1 Let $a, b \in \mathbb{R}$ and assume that $x = 1$ is a root of the polynomial

$$p(x) \;=\; x^4 + ax^3 + bx^2 + ax + 1.$$

Find the range of values of a for which p has a complex root which is not real.

1.2 Let $GL_n(\mathbb{R})$ denote the group of all $n \times n$ matrices with real entries (with respect to matrix multiplication) which are invertible. Pick out the normal subgroups from the following:
a. The subgroup of all real orthogonal matrices.
b. The subgroup of all invertible diagonal matrices.
c. The subgroup of all matrices with determinant equal to unity.

1.3 Pick out the true statements:
a. The set

$$\left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0 \right\}$$

is a group with respect to matrix multiplication.
b. The set

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

is a commutative ring with identity with respect to matrix addition and matrix multiplication.
c. The set

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

is a field with respect to matrix addition and matrix multiplication.

1.4 Let $\mathcal{C}[0,1]$ denote the ring of all continuous real-valued functions on $[0,1]$ with respect to pointwise addition and pointwise multiplication. Pick out the true statements:
a. $\mathcal{C}[0,1]$ is an integral domain.
b. Let $a \in [0,1]$. Set

$$\mathcal{I} \;=\; \{f \in \mathcal{C}[0,1] \mid f(a) = 0\}.$$

Then $\mathcal{I}$ is an ideal in $\mathcal{C}[0,1]$.
c. If $\mathcal{I}$ is any proper ideal in $\mathcal{C}[0,1]$, then there exists at least one point $a \in [0,1]$ such that $f(a) = 0$ for all $f \in \mathcal{I}$.

1.5 Let V be the real vector space of all polynomials in one variable with real coefficients and of degree less than, or equal to, 3, provided with the standard basis $\{1, x, x^2, x^3\}$. If

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

define

$$T(p)(x) = a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3.$$

Write down the matrix representing the linear transformation T with respect to this basis.

1.6 Let V be the real vector space of all polynomials in one variable with real coefficients and of degree less than, or equal to, 5. Let W be the subspace defined by

$$W = \{p \in V \mid p(1) = p'(2) = 0\}.$$

What is the dimension of W?

1.7 Let A be a non-zero 2×2 matrix with real entries. Pick out the true statements:
a. If $A^2 = A$, then A is diagonalizable.
b. If $A^2 = 0$, then A is diagonalizable.
c. If A is invertible, then

$$A = (\mathrm{tr}(A))I - (\det(A))A^{-1}$$

where $\mathrm{tr}(A)$ and $\det(A)$ denote the trace and determinant of A respectively.

1.8 Let A be an $n \times n$ matrix with real entries. Pick out the true statements:
a. There exists a real symmetric $n \times n$ matrix B such that $B^2 = A^*A$.
b. If A is symmetric, there exists a real symmetric $n \times n$ matrix B such that $B^2 = A$.
c. If A is symmetric, there exists a real symmetric $n \times n$ matrix B such that $B^3 = A$.

1.9 Let $S = \{\lambda_1, \cdots, \lambda_n\}$ be an ordered set of n real numbers, not all equal, but not all necessarily distinct. Pick out the true statements:
a. There exists an $n \times n$ matrix with complex entries, which is *not* self-adjoint, whose set of eigenvalues is given by S.
b. There exists an $n \times n$ self-adjoint, non-diagonal matrix with complex entries whose set of eigenvalues is given by S.
c. There exists an $n \times n$ symmetric, non-diagonal matrix with real entries whose set of eigenvalues is given by S.

1.10 Let p be a prime number and let $\mathbb{Z}_p$ denote the field of integers modulo p. Find the number of 2×2 invertible matrices with entries from this field.

Section 2: Analysis

2.1 Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Evaluate:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f'\left(\frac{k}{n}\right).$$

2.2 In each of the following verify whether the series is absolutely convergent, conditionally convergent or divergent:

a.

$$\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{n}{n+1}}$$

b.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \sin \frac{1}{n}$$

c.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{(n+1)(n+2)}$$

2.3 Pick out the uniformly continuous functions over the interval $]0,1[$:
a. $f(x) = \sin \frac{1}{x}$
b. $f(x) = x \sin \frac{1}{\sqrt{x}}$
c. $f(x) = exp(-\frac{1}{x^2})$

2.4 In each of the following, verify if the given function $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, differentiable but not continuously differentiable or not differentiable at the origin:
a. $f(x) = x \sin \frac{1}{\sqrt{x}}$, if $x \neq 0$ and $f(0) = 0$.
b. $f(x) = |x|^{\frac{3}{2}}$
c. $f(x) = x \sin |x|$

2.5 Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0) = 0$ and $f'(x) = 1/\sqrt{1+x^2}$. Write down the coefficient of x^7 in the Taylor series expansion of f about the origin.

2.6 Pick out the true statements:
a. $|\cos^2 x - \cos^2 y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
b. If $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \;\leq\; |x - y|^{\sqrt{2}}$$

for all $x, y \in \mathbb{R}$, then f must be a constant function.
c. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and such that $|f'(x)| \leq 4/5$ for all $x \in \mathbb{R}$. Then, there exists a unique $x \in \mathbb{R}$ such that $f(x) = x$.

2.7 Sum the following infinite series:

$$\frac{1}{6} + \frac{5}{6.12} + \frac{5.8}{6.12.18} + \frac{5.8.11}{6.12.18.24} + \cdots$$

2.8 Let C be the semicircle $z = 2e^{i\theta}$ where θ varies from 0 to π, in the complex plane. Evaluate:

$$\int_C \frac{z+2}{z}\, dz.$$

2.9 Find the order of the pole and its residue at $z = 0$ of the function

$$f(z) = \frac{\sinh z}{z^4}.$$

2.10 Let C denote the circle $|z| = 3$ in the complex plane, described in the positive (*i.e.* anti-clockwise) sense. Evaluate:

$$\int_C \frac{2z^2 - z - 2}{z-2}\, dz.$$

Section 3: Geometry

3.1 Let $f(x, y) = ax + by + c$ where $a, b, c \in \mathbb{R}$ and $c > 0$. Find the largest value of r such that $f(x, y) > 0$ for all pairs (x, y) satisfying $x^2 + y^2 < r^2$.

3.2 Find the value of a such that the lines $3x + y + 2 = 0$, $2x - y + 3 = 0$ and $x + ay - 3 = 0$ are concurrent.

3.3 Find the condition that amongst the pair of lines represented by the equation $ax^2 + 2hxy + by^2 = 0$, the slope of one is twice that of the other.

3.4 Find the condition that the straight line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$.

3.5 Let (x_1, y_1) lie on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the slope of the normal to the ellipse at this point.

3.6 Find the lengths of the semi-axes of the ellipse

$$5x^2 - 6xy + 5y^2 = 8.$$

3.7 Let L_n denote the perimeter and A_n the area of a regular polygon of n sides, each of whose vertices is at unit distance from its centroid. Evaluate:

$$\lim_{n \to \infty} \frac{L_n^2}{A_n}.$$

3.8 Find the coordinates of the reflection of the point $(1, -2, 3)$ with respect to the plane $2x - 3y + 2z + 3 = 0$.

3.9 Find the area of the circle formed by the intersection of the plane $x + 2y + 2z - 20 = 0$ with the sphere

$$x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0.$$

3.10 A sphere of radius r passes through the origin and the other points where it meets the coordinate axes are A, B and C. Find the distance of the centroid of the triangle ABC from the origin.