Problem set 6

Spring 2018

MATHEMATICS-II (MA10002)(Integral Calculus)

1. Discuss the convergence of the following integrals using definition:

   i. $\int_0^1 \frac{1}{1-x} dx$

   ii. $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

   iii. $\int_0^2 \frac{1}{\sqrt{x(2-x)}} dx$

   iv. $\int_1^\infty \frac{1}{x \log x} dx$

   v. $\int_1^\infty \frac{1}{(1+x)\sqrt{x}} dx$

   vi. $\int_1^3 \frac{10x}{(x^2-9)^{\frac{1}{3}}} dx$

2. Discuss the convergence of the following integrals:

   i. $\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}} dx$

   ii. $\int_0^{\frac{\pi}{2}} \frac{1}{e^x - \cos x} dx$

   iii. $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x} dx$

   iv. $\int_0^{\frac{\pi}{2}} \log(\sin x) dx$

   v. $\int_0^\infty \frac{1-\cos x}{x^2} dx$

   vi. $\int_0^\infty \frac{\cos x}{\sqrt{x^3+x}} dx$

   vii. $\int_0^\infty (\frac{1}{x^2} - \frac{1}{x \sinh x}) dx$

   viii. $\int_0^1 x^{n-1} \log x \; dx$

   ix. $\int_1^\infty \frac{e^x}{\sqrt{x^2 - \frac{1}{2}}} dx$

   x. $\int_0^1 \frac{1}{1-x^4} dx$

3. Show that the improper integral $\int_0^\infty \mid \frac{\sin x}{x} \mid dx$ is not convergent.

4. Examine the convergence of $\int_0^\infty \frac{1}{e^x - x} dx$.

5. Show that the improper integral $\int_0^\infty \frac{\sin x(1-\cos x)}{x^n} dx$ is convergent if $0 < n < 4$.

6. Evaluate $\int_0^\infty \frac{5\sin(4x) - 4\sin(5x)}{x^2} dx$.

7. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if $m,n>0$. Find the value of the integral for $m=\frac{5}{2}, n=\frac{7}{2}$.

8. Show that $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if $m>0$. Find the value of the integral for $m=2019$.

9. Evaluate $\int_0^1 x^4(1-\sqrt{x})^5dx$.

10. Express the following integral in terms of Gamma function: $\int_0^\infty \frac{x^a}{a^x}dx,\ (a>1)$.

11. Prove that $\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}, 0<n<1$(Using $\int_0^\infty \frac{x^{n-1}}{1+x}dx=\frac{\pi}{\sin(n\pi)}$).

12. Prove that

    i. $\int_0^1(\log\frac{1}{y})^{n-1}dy=\Gamma(n)$

    ii. $\int_0^{\frac{\pi}{2}}\tan^p\theta d\theta=\frac{\pi}{2}\sec\frac{p\pi}{2}$ and indicate the restriction on the values of p.

    iii. $\int_a^b(x-a)^{m-1}(b-x)^{n-1}dx=(b-a)^{m+n-1}B(m,n), m>0, n>0$.

    iv. $\int_0^1\frac{1}{(1-x^6)^{\frac{1}{6}}}dx=\pi/3$.

    v. $\int_0^1\frac{x^2dx}{\sqrt{1-x^4}}\times\int_0^1\frac{dx}{\sqrt{1+x^4}}=\frac{\pi}{4\sqrt{2}}$

13. Using Beta and Gamma functions, evaluate the integral

$$I=\int_{-1}^{1}(1-x^2)^n dx, \text{ where n is a positive integer.}$$

14. Show that

$$\Gamma(2n)=\frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n+\frac{1}{2})\Gamma(n)$$

and

$$\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})=\pi\sqrt{2}.$$