Problem Set - 3

SPRING 2018

MATHEMATICS-II (MA1002)

1. (a) Prove that if λ be an eigen value of a non-singular matrix A, then λ^{-1} is an eigen value of A^{-1}.
   (b) Prove that if A and P be both $n \times n$ matrices and P be non-singular, then A and $P^{-1}AP$ have same eigen values.
   (c) Prove that if λ be an r-fold eigen value of A, 0 is an r-fold eigen value of the matrix $A - \lambda I_n$.

2. For each of the following matrices, find all the eigenvalues and the corresponding eigen vectors.
   (a) $\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 0 & -2i \\ 0 & 2 & 0 \\ 2i & 0 & -1 \end{pmatrix}$.

3. $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$. Use Cayley Hamilton theorem to express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.

4. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Find A^9 and A^{-1} by using Cayley Hamilton theorem.

5. Let, $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$. If $A = P^{-1}DP$ then find the matrix D.

6. The square matrix A is defined as $A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$. Find the diagonal matrix D and the model matrix P such that $A = P^{-1}DP$.

7. The linear operator $L(x)$ is defined by the product $L(x) = bx$, where $b = [0\ 1\ 0]^T$ and $x = [x_1\ x_2\ x_3]^T$ are three dimensional vector. The 3×3 matrix M of this operation satisfies $L(x) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Find the eigen values of M.

8. If A is 4×4 matrix with real entries such that -1, 1, 2, -2 are its eigen values. If $B = A^4 - 5A^2 + 5I$ where I denotes the 4×4 identity matrix, then find trace of $(A + B)$.

9. (a) Show that $M = \begin{bmatrix} 1 & 1+i & 2i & 9 \\ 1-i & 3 & 4 & 7-i \\ -2i & 4 & 5 & i \\ 9 & 7+i & -i & 7 \end{bmatrix}$ is Hermitian.

   (b) Find real numbers x, y, z such that A is Hermitian, where $A = \begin{bmatrix} 3 & x+2i & yi \\ 3-2i & 0 & 1+zi \\ yi & 1-xi & -1 \end{bmatrix}$.

10. The eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. Then what is $a + b$?

11. (a) Let, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be such that A has real eigen values then find the values of θ.

(b) If $x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $x_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ are given vectors and $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and if $P = [x_1 \; x_2]$ then find the value $P^{-1}AP$.

12. Examine whether A is similar to B, where

(a) $A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

13. Let, A be an $n \times n$ matrix which is both Hermitian and unitary. Prove that $A^2 = I$.

14. Express the matrix $A = \begin{bmatrix} 1 & 2+i & 1-i \\ 2-i & 1+2i & 3 \\ 2+i & 2 & 1+i \end{bmatrix}$ as the sum of a Hermitian and skew Hermitian matrix.

15. Show that the matrix $\begin{bmatrix} \alpha + i\nu & -\beta + i\delta \\ \beta + i\delta & \alpha - i\nu \end{bmatrix}$ is unitary matrix if $\alpha^2 + \beta^2 + \nu^2 + \delta^2 = 1$.

16. If $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix}$ where $a = e^{\frac{2i\pi}{3}}$, then prove that $M^{-1} = \frac{1}{3}\bar{M}$.