Mathematics-I

FUNCTIONS OF SEVERAL VARIABLES-II

(a) Schwarz's Theorem.

(b) Total Differential: Concept of Differentiability.

(c) Function of three Variables: Notion of Differentiability.

(d) Homogeneous Function: Euler's Theorem.

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<u>Example</u>:

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2=0. \end{cases}$$

Prove that $\quad f_{xy}(0,0) \neq f_{yx}(0,0)$

<u>Sol^n</u>.

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{hk\,\dfrac{h^2-k^2}{h^2+k^2} - 0}{k}$$

$$= h$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0.$$

$$\cdot\; f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

$$f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{1}{k} \left\{ \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} - \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \right\}$$

$$= \lim_{k \to 0} \lim_{h \to 0} \frac{\left\{ f(h,k) - f(0,k) - f(h,0) + f(0,0) \right\}}{hk}$$

$$= -1.$$

$$f_{xy}(0,0) = 1 \neq f_{yx}(0,0) = -1$$

Example: $f(x,y) = \log \dfrac{x^2+y^2}{xy}$

Show that $f_{xy} = f_{yx}$ at pts. where the fun^n. is defined.

Sol^n: $f_{xy} = -\dfrac{4xy}{(x^2+y^2)^2} = f_{yx}$.

- ## Sufficient conditions for $f_{xy} = f_{yx}$

1. **Schwarz's Theorem**

$(a,b) \in S$, $S \rightarrow$ domain of defn. of $f(x,y)$

(i) f_x exists in some nbd. of (a,b)

(ii) f_{xy} is continuous at (a,b).

Then f_{yx} at (a,b) exists &

$$f_{xy}(a,b) = f_{yx}(a,b).$$

2. **Young's Theorem.**

- f_x, f_y exist in some nbd. of (a,b) & they are $\boxed{\text{differentiable}}$ at (a,b).

Then $f_{xy} = f_{yx}$ at (a,b).

?

The total differential : Concept of differentiability

- $z = f(x, y)$.
- $x \uparrow \Delta x, \quad y \uparrow \Delta y$ (given)

$\Rightarrow z \uparrow \Delta z$. (obtained).

Then $\quad z + \Delta z = f(x+\Delta x,\ y+\Delta y)$.

$$\Delta z = f(x+\Delta x,\ y+\Delta y) - f(x, y).$$

$$= \left[f(x+\Delta x,\ y+\Delta y) - f(x+\Delta x,\ y) \right] + \left[f(x+\Delta x,\ y) - f(x, y) \right].$$

$$= \Delta y\ f_y(x+\Delta x,\ y+\theta_1 \Delta y) + \Delta x\ f_x(x+\theta_2 \Delta x,\ y), \quad \text{—} ①$$

(by MVTh.)

$(0 < \theta_1, \theta_2 < 1.)$

let $f_x(x+\theta_2 \Delta x, y) - f_x(x,y) = \varepsilon_1$

& $f_y(x+\Delta x, y+\theta_1 \Delta y) - f_y(x,y) = \varepsilon_2$.

(1) $\Rightarrow$

$$\Delta z = \Delta y \left[\varepsilon_2 + f_y(x,y) \right] + \Delta x \left[\varepsilon_1 + f_x(x,y) \right].$$

$$= \Delta x\, f_x(x,y) + \Delta y\, f_y(x,y) + \Delta x \cdot \varepsilon_1 + \Delta y \cdot \varepsilon_2.$$

$$= \underbrace{\Delta x \frac{\partial z}{\partial x} + \Delta y \frac{\partial z}{\partial y}}_{\text{total differential of } z,\text{ denoted by } dz.} + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$$\boxed{\begin{array}{l} \Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y. \\ \text{where } dz = \Delta x \dfrac{\partial z}{\partial x} + \Delta y \dfrac{\partial z}{\partial y} \end{array}}$$

$z = x \Rightarrow z_x = 1, \quad z_y = 0.$

$$\Rightarrow dz = dx = \Delta x \cdot 1 + \Delta y \cdot 0$$
$$= \Delta x.$$

i.e. $dx = \Delta x$

Similarly, $dy = \Delta y$.

Total differential of $z = f(x,y)$

$$\boxed{df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy}$$

$$\Delta z = dz + \underbrace{(\varepsilon_1 \Delta x + \varepsilon_2 \Delta y)}_{\text{error}}$$

for differentiable fn. $z = f(x,y)$ this error must tend to 0 as $(\Delta x, \Delta y) \to (0,0)$.

$\underline{\underline{\text{Def}^n}}$ (Differentiability) $\underline{z = f(x,y)}$

f is said to be differentiable at a pt. (x,y) if

$$\boxed{\begin{aligned} \Delta z &= f(x+\Delta x,\ y+\Delta y) - f(x,y) \\ &= A.\Delta x + B.\Delta y + \varepsilon_1.\Delta x + \varepsilon_2.\Delta y \end{aligned}}$$

when $A, B \rightarrow$ ind. of $\Delta x, \Delta y$.

& $\varepsilon_1, \varepsilon_2 \rightarrow$ are fun^{ns}. of $\Delta x, \Delta y$,

$\varepsilon_1, \varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0,0)$ in any manner

· Equivalently

$$\Delta z = \overbrace{A\cdot\Delta x + B.\Delta y}^{\to dz} + \varepsilon\rho$$

when $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$\Delta x = \rho\cos\theta$
$\Delta y = \rho\sin\theta$.
$(\Delta x, \Delta y) \rightarrow (0,0) \Rightarrow \rho \rightarrow 0$

& ε is a fn. of $\Delta x, \Delta y$ s.t.

$\varepsilon \to 0$ as $\rho \to 0$.

$$\boxed{\text{i.e. } \frac{\Delta z - dz}{\rho} \to 0 \text{ as } \rho \to 0.}$$

$$\boxed{\text{i.e. } \frac{\Delta z - dz}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \to 0 \text{ as } (\Delta x, \Delta y) \to (0,0).}$$

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i.e. $\Delta z = dz + \varepsilon_1 . \Delta x + \varepsilon_2 \Delta y$.

$\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

Example: $f(x,y) = xy$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - df}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$\Delta f = f(x+\Delta x,\ y+\Delta y) - f(x,y)$$

$$= (x+\Delta y)(y+\Delta y) - xy$$

$$= x\,\Delta y + y\,\Delta y + \underline{\Delta x\,\Delta y}$$

$$df = \Delta x\, f_x + \Delta y\, f_y$$

$$= y\,\Delta x + x\,\Delta y$$

$$\Delta x \le \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Delta y \le \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\therefore \frac{\Delta f - df}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{\Delta x\,\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$\le \sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$$

as $(\Delta x, \Delta y) \to (0,0)$

$$\Rightarrow \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - df}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.$$

$\therefore$ $f(x,y)$ is differentiable.

Example:
$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$$

- ✓ f continuous at $(0,0)$
- ✓ f_x, f_y exist at $(0,0)$.
- not <u>differentiable</u> at $(0,0)$.

Soln.
$$f_x(0,0) = 0 = f_y(0,0).$$

$$\mathcal{E} = \frac{\Delta f - \overset{=0}{df}}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \quad \text{at } (0,0).$$

~~$df = f_x^{(0,0)} \Delta x + f_y^{(0,0)} \Delta y$~~

$$df = \Delta x\, f_x(0,0) + \Delta y\, f_y(0,0) = 0.$$

$$\boxed{\mathcal{E} = \frac{\Delta x \Delta y}{(\Delta x)^2 + (\Delta y)^2}}$$

put $\Delta x = \rho \cos\theta$
$\Delta y = \rho \sin\theta$.

$\underset{\text{at }(0,0)}{\Delta f} = f(\Delta x, \Delta y) - f(0,0)$
$$= \frac{\Delta x \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$(\Delta x, \Delta y) \to (0,0) \implies \rho \to 0$

$\therefore \lim\limits_{(\Delta x, \Delta y) \to (0,0)} \varepsilon = \lim\limits_{\rho \to 0} (\sin\theta \cos\theta)$

$\neq 0$

The fn: f is not differentiable at (0,0).

<u>Exercise</u> $f(x,y) = \sqrt{|xy|}$

Differentiable at (0,0) ??

<u>Example</u>: $f(x,y) = |x|(1+y)$ at (0,0)

<u>Differentiability at (0,0)</u>

$$= \begin{cases} x(1+y), & x > 0 \\ -x(1+y), & x < 0 \end{cases}$$

$f_x(0,0) =$ does not exist.

f is Not diff. at (0,0).

- $z = f(x,y)$ differentiable
  $\Rightarrow f_x, f_y$ exist.
- Converse is not true.

- $\Delta x = h, \ \Delta y = k$.
- $\Delta z = dz + \varepsilon \rho, \quad \rho = \sqrt{h^2 + k^2}$

$$\boxed{\varepsilon = \frac{\Delta z - dz}{\rho} \to 0 \text{ as } \rho \to 0}$$

error term.

- Sufficient condition for differentiability of $z = f(x,y)$ at a pt.

Theorem. f_x, f_y exist & one of them is continuous at a pt.
$\Rightarrow f$ differentiable at that pt.

~~Example:~~
Exercise

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$$

- f ~~cont~~ continuous at $(0,0)$
- f_x, f_y exist at $(0,0)$
- f not differentiable at $(0,0)$.

Example: Test the differentiability at (0,0)

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x - y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Sol^n: $f_x(0,0) = 1$, $f_y(0,0) = 1$

- Check the continuity of f_x

$$f(x,y) = \begin{cases} x+y, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$f_x(x,y) = \begin{cases} 1, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

$\Rightarrow$ f_x is continuous at $(0,0)$.

Then both f_x, f_y exist at $(0,0)$ & f_x is continuous at $(0,0)$

$\Rightarrow$ f is differentiable at $(0,0)$

→ only sufficient conditions,
not necessary

i.e. if f_x, f_y exist, but none of them are continuous, then no conclusion about the diff. of f.

Exercise.
~~Example:~~

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$$

Example:

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y}, & \text{neither } x=0, \text{ nor } y=0 \\ x^2 \sin\frac{1}{x}, & y=0, \text{ but } x\neq 0 \\ y^2 \sin\frac{1}{y}, & x=0, \text{ but } y\neq 0 \\ 0, & x=0, \,\&\, y=0. \end{cases}$$

. neither f_x, nor f_y continuous at $(0,0)$, but $f(x,y)$ is differentiable at $(0,0)$.

$\underline{\underline{\text{Sol}^n}}$: $f_x(x,y) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \quad , x \neq 0$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{h^2\sin\frac{1}{h} + \cancel{y^2\sin\frac{1}{y}} - \cancel{y^2\sin\frac{1}{y}}}{h}$$

$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0.$$

$$f_y(x,y) = \begin{cases} 2y\sin\frac{1}{y} - \cos\frac{1}{x}, & y \neq 0. \\ 0, & y = 0. \end{cases}$$

<u>Continuity</u> of f_x, f_y at $(0,0)$

$$\lim_{(x,y)\to(0,0)} \underline{f_x(x,y)}, \quad \lim_{(x,y)\to(0,0)} f_y(x,y)$$

donot exist

Differentiability of $f(x,y)$ at $(0,0)$.

claim $\dfrac{\Delta f - df}{\sqrt{h^2+k^2}} \to 0$ as $(h,k) \to (0,0)$.

$$\Delta f = f(0+h, 0+k) - f(0,0)$$
$$= h^2 \sin\frac{1}{h} + k^2 \sin\frac{1}{k} - 0$$

$$df = h f_x(0,0) + k f_y(0,0) = h.0 + k.0 = 0.$$

$$\Delta f = \underset{0}{\underset{\shortparallel}{df}} + h.\underset{\varepsilon_1}{\underset{\shortparallel}{\left(h \sin\frac{1}{h}\right)}} + k.\underset{\varepsilon_2}{\underset{\shortparallel}{\left(k \sin\frac{1}{k}\right)}}$$

when $\left.\begin{array}{l} \varepsilon_1 = h\sin\frac{1}{h} \to 0 \\ \& \ \varepsilon_2 = k \sin\frac{1}{k} \to 0 \end{array}\right\}$ as $(h,k) \to (0,0)$.

$\Rightarrow$ f is diff. at $(0,0)$.

$$df = \underset{0}{\underset{\shortparallel}{f_x(0,0)}}\, h + \underset{0}{\underset{\shortparallel}{f_y(0,0)}}\, k$$

• This example shows that

- f diff. at $(a,b) \not\Rightarrow f_x, f_y$ continuous at (a,b).
- f continuous at $(a,b) \not\Rightarrow f_x, f_y$ bdd. $\forall x, y$.

$\boxed{\text{• differentiability } \Rightarrow \text{ Continuity of } f \text{ at } (a,b) \text{ of } f \text{ at } (a,b)}$

and

• f bdd. $\Rightarrow$ $|f(x,y)| \leq M$ $\forall (x,y) \in$ the domain of f.

proof $f(x,y)$ diff. at (x,y)

$$\Rightarrow \frac{\Delta f - df}{\sqrt{h^2+k^2}} \to 0 \text{ as } (h,k) \to (0,0)$$

Now $\Delta f = f(x+h, y+k) - f(x,y)$

$$= \underbrace{df}_{\text{total differential.}} + \epsilon_1 h + \epsilon_2 k.$$

$df = h f_x + k f_y \to 0$ as $(h,k) \to (0,0)$

as f_x, f_y are ind. of (h,k).

$\therefore \Delta f \to df$ as $(h,k) \to (0,0)$.

$\Rightarrow \Delta f \to 0$ as $(h,k) \to (0,0)$.

i.e. $f(x+h, y+k) \to f(x,y)$
as $(h,k) \to (0,0)$.

$\Rightarrow f(x,y)$ is continuous at (x,y).

$$\lim_{(h,k)\to(0,0)} f(x+h, y+k) = f(x,y).$$

$\Delta f - df \to 0$ as $(h,k) \to (0,0)$

$$\frac{\Delta f - df}{\sqrt{h^2+k^2}} \to 0 \text{ as } (h,k) \to (0,0).$$

• Sufficient condition for continuity

f_x, f_y exist & one of them is $\boxed{\text{bdd.}}$ at a pt.

Then f continuous at that pt

Example:

$$f(x,y) = \begin{cases} \dfrac{x^6 - 2y^4}{x^2+y^2}, & x^2+y^2 \neq 0. \\ 0, & (x,y) = (0,0). \end{cases}$$

Test the differentiability of $f(x,y)$ at $(0,0)$.

Soln:

$$\lim_{(h,k)\to(0,0)} \frac{\Delta f - df}{\sqrt{h^2+k^2}}$$

$$\Delta f = f(0+h, 0+k) - f(0,0) = \frac{h^6 - 2k^4}{h^2+k^2}$$

$$df = h f_x(0,0) + k f_y(0,0) = 0.$$

$$f_x(0,0) = 0, \quad f_y(0,0) = 0$$

$$\Delta f = df + h \cdot \underbrace{\frac{h^5}{h^2+k^2}}_{\varepsilon_1} + k \cdot \underbrace{\frac{(-2k^3)}{h^2+k^2}}_{\varepsilon_2}$$

$$= df + h\cdot\varepsilon_1 + k\cdot\varepsilon_2$$

when $\varepsilon_1 \to 0$ as $(h,k) \to (0,0)$

& $\varepsilon_2 \to 0$ as $(h,k) \to (0,0)$.

Example: $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$

- Continuity at $(0,0)$
- $f_x(0,0)$, $f_y(0,0)$
- Differentiability at $(0,0)$.

Sol^n

i) Continuity at $(0,0)$.

Use ε-δ approach

$x = r\cos\theta$
$y = r\sin\theta$

$$\left| \frac{x^3-y^3}{x^2+y^2} - 0 \right| = \left| \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2} \right|$$

$$\leq 2r < \varepsilon \qquad \Big| \; \varepsilon > 0 \text{ arb. small}$$

whenever $x^2+y^2 < \dfrac{\varepsilon^2}{4}$

i.e. $|x| < \frac{1}{2}\sqrt{\frac{\varepsilon^2}{4}}$, $|y| < \frac{1}{2}\sqrt{\frac{\varepsilon^2}{4}}$.

choosing $\delta = \frac{1}{2}\sqrt{\frac{\varepsilon^2}{4}}$.

$\Rightarrow$ f is continuous at $(0,0)$.

ii) $$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 1$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = -1$$

iii) <u>Differentiability at $(0,0)$.</u>

$$\Delta f \text{ at } (0,0) = f(h,k) - f(0,0)$$
$$= \frac{h^3 - k^3}{h^2+k^2}$$

$$df \text{ at } (0,0) = h f_x(0,0) + k f_y(0,0)$$
$$= h - k.$$

$$\frac{\Delta f - df}{\sqrt{h^2+k^2}} = \frac{\frac{h^3-k^3}{h^2+k^2} - (h-k)}{\sqrt{h^2+k^2}}$$

$$= \frac{hk(h-k)}{(h^2+k^2)^{3/2}}$$

$$= \cos\theta \sin\theta \, (\cos\theta - \sin\theta)$$

$$\not\to 0 \text{ as } r \to 0$$

$h = r\cos\theta$

$k = r\sin\theta$

$(h,k) \to (0,0)$

$\Rightarrow r \to 0$

$\therefore$ f is not differentiable at $(0,0)$.

Example: $f(x,y) = \begin{cases} xy \dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$

is differentiable at $(0,0)$.

Sol^n. $f_x(0,0) = 0 = f_y(0,0)$.

. Both f_x, f_y exist at $(0,0)$.

$$|f_x(x,y) - f_x(0,0)|$$

$$= \left| \frac{y(x^4 - y^4 + 4x^2y^2)}{(x^2+y^2)^2} \right|$$

put $x = r\cos\theta$, $y = r\sin\theta$

$= \left| r\sin\theta \left(\cos^4\theta - \sin^4\theta + 4\sin^2\cos^2\theta \right| \right.$

$\leq 6r < \varepsilon$, ($\varepsilon > 0$ arb. small)

whenever $|x| < \delta$, $|y| < \delta$

Choosing $\delta = \frac{1}{2}\sqrt{\frac{\varepsilon^2}{36}}$.

$\Rightarrow f_x$ is continuous at $(0,0)$.

f_x, f_y exists at $(0,0)$ one of them (f_x) is continuous at $(0,0)$

$\Rightarrow f$ is differentiable at $(0,0)$.

functions of three variables : Notion of Differentiability

- $f(x, y, z)$ over $D \subseteq \mathbb{R}^3$.
- $x \uparrow \Delta x$, $y \uparrow \Delta y$, $z \uparrow \Delta z$

$$\Delta f = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x, y, z).$$

- f is differentiable at $(x, y, z) \in D$ if Δf having the form

$$\Delta f = \cancel{df}\ \overbrace{(A \cdot \Delta x + B \cdot \Delta y + C \cdot \Delta z)}^{df} + \eta_1 \cdot \Delta x + \eta_2 \cdot \Delta y + \Delta\eta_3 \cdot \Delta z.$$

where A, B, C are consts., ind. of $\Delta x, \Delta y, \Delta z$.

& $\eta_1, \eta_2, \eta_3 \rightarrow$ depends on $\Delta x, \Delta y, \Delta z$

s.t.

$\eta_1, \eta_2, \eta_3 \rightarrow 0$ as $(\Delta x, \Delta y, \Delta z) \rightarrow (0,0,0)$.

or equivalently,

$$\boxed{\begin{aligned}\Delta f &= f_x \cdot \Delta x + f_y \cdot \Delta y + f_z \cdot \Delta z \\ &\quad + \eta \rho, \\ &\text{when } \eta \to 0 \text{ as } \rho \to 0.\end{aligned}}$$

[put ~~[illegible]~~

$$\Delta x = \rho \sin\theta \cos\phi$$

$$\Delta y = \rho \sin\theta \sin\phi$$

$$\Delta z = \rho \cos\theta$$

$(\Delta x, \Delta y, \Delta z) \to (0,0,0)$

$\Rightarrow \rho \to 0.$].

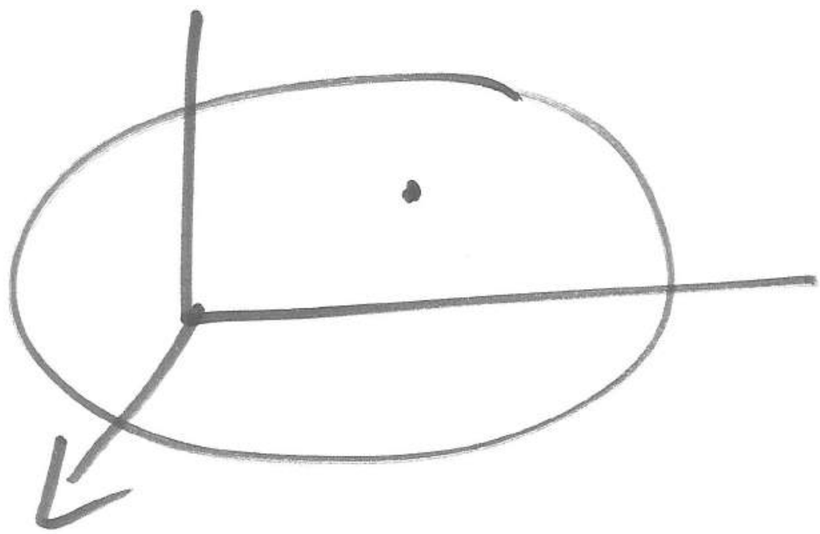
- $f_x(a,b)$, $f_y(a,b)$

$\longrightarrow$ Geometrical interpretation ?

$\underline{z = f(x,y)}$ $\quad\Big|\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

- $P(a, b, f(a,b))$

$\underline{f_x(a,b)} = \tan\varphi$



$\psi \rightarrow$ angle between +ve x-axis & the curve of intersection of $z = f(x, y)$ with plane $y = b$.

$f_y(a, b) = \tan\phi$

ϕ = angle between +ve y axis & the curve of intersection of $z = f(x, y)$ with plane $x = a$

Example: Find the slope of the curve of intersection of the ellipsoid $\frac{x^2}{24} + \frac{y^2}{12} + \frac{z^2}{6} = 1$ made by the plane $y = 1$ at the pt. $\left(4, 1, \sqrt{\frac{3}{2}}\right)$

Solⁿ: $f_x(4, 1) = \left.\frac{\partial z}{\partial x}\right]_{(4,1)}$.

· Homogeneous fu^n. : Euler's Theorem.

$f(x_1, x_2, \ldots, x_m)$

$\rightarrow$ homogeneous in $x_1, x_2, \ldots, x_m$ of ~~order~~ degree n

if

$$f(tx_1, tx_2, \ldots, tx_m) = t^n f(x_1, x_2, \ldots, x_m).$$

e.g. i) $f(x, y) = \dfrac{x^2}{y} + \dfrac{y^2}{x}$

$\rightarrow$ degree 1

ii) $f(x, y) = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$

$\rightarrow$ deg $\frac{1}{2}$

Euler's Theorem

· $u = f(x, y) \rightarrow$ hom fu^n. of deg. n

Then $\boxed{x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = n u .}$

$f(x,y)$ hom. in x, y of deg n

$$\Rightarrow \quad f(x,y) = x^n \phi\left(\frac{y}{x}\right)$$

or

$$f(x,y) = y^n \phi\left(\frac{x}{y}\right)$$

<u>proof.</u>

$$u = x^n \phi\left(\frac{y}{x}\right).$$

$$\frac{\partial u}{\partial x} = n x^{n-1} \phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$\therefore \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial x} = n x^n \phi\left(\frac{y}{x}\right) \cancel{- x^{n-2+1} y\, \phi'\left(\frac{y}{x}\right)}$$

$$\cancel{+ x^{n-1} y\, \phi'\left(\frac{y}{x}\right)}$$

$$= n u$$

<u>A more general result:</u>

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1) u.$$

Try to prove it.

Example: $u = \sin^{-1} \dfrac{x^2+y^2}{x+y}$.

Prove that i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

ii) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \tan^3 u$.

Sol^n

i) $\sin u = \dfrac{x^2+y^2}{x+y} \rightarrow$ hom. of x, y of deg. 1

Euler's Th. on $\sin u$

$$x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) = 1 \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \sin u$$

$\Rightarrow$ result (i).