Mathematics-I

FUNCTIONS OF SEVERAL VARIABLES

(a) Limit.
(b) Repeated/ Iterated Limit.
(c) Partial Derivatives of First Order.
(d) Partial Derivatives of Higher Order.

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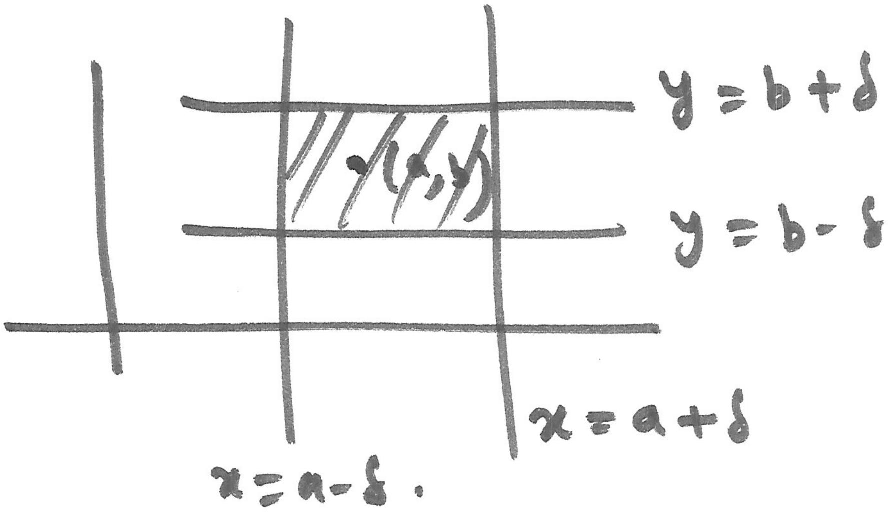
Mathematics I.

(Autumn 2015)

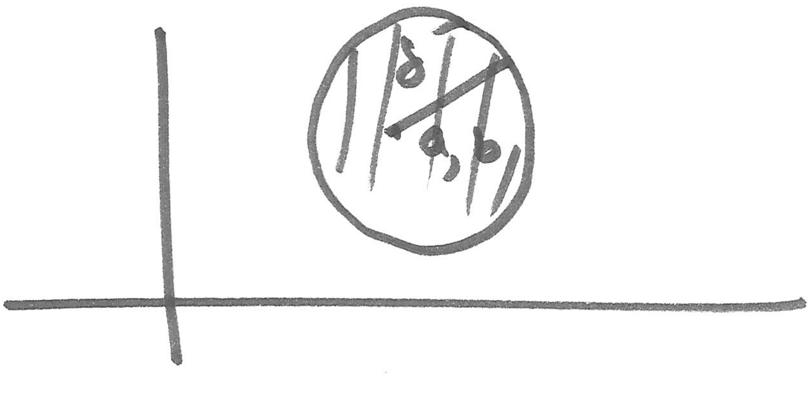
Lectures 10 - 12

Functions of Several Variables

- $z = f(x, y)$ $\qquad x^2 + y^2 + z^2 = 1$.
- $f(x_1, x_2, \ldots, x_n)$.
- <u>δ-nbd. of a pt.</u>



$$|x - a| < \delta, \quad |y - b| < \delta.$$



$$(x - a)^2 + (y - b)^2 < \delta^2$$

<u>Limit.</u>

- $f(x,y) \to$ defined over a certain domain $\underline{S}$.

- $(a,b) \Rightarrow$ a cluster pt. of S.

e.g. $z = \sqrt{x+y-1}$.

$\downarrow$

domain of def^n. of fn^n. z is $x+y-1 \geq 0$.

- $$\lim_{(x,y)\to(a,b)} f(x,y) = A.$$

or

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = A$$

$\to$ double limit.

<u>Def^n.</u> $\forall$ +ve ε, $\exists$ +ve. δ s.t.

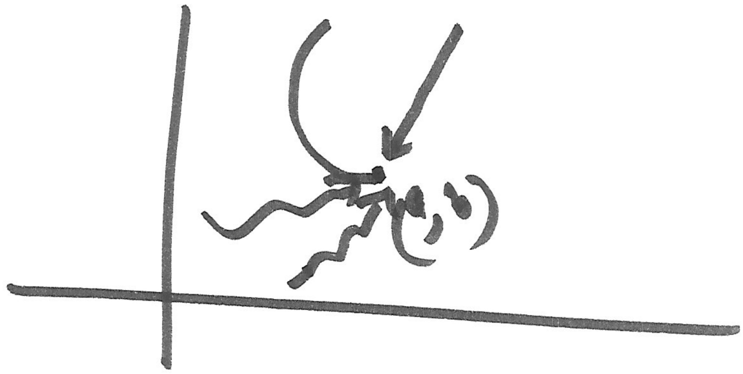
$|f(x,y) - A| < \varepsilon$ whenever

$(x-a)^2 + (y-b)^2 < \delta^2$.

or

$|x-a| < \delta$, $|y-b| < \delta$.

- The limit A must be <u>unique</u>, along <u>whatever path we may approach to</u> (a,b).



<u>Example</u>: $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2}{x^2+y^2}$.

$$= \lim_{\substack{x\to 0\\ y\to 0\\ y=mx}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{1-m^2}{1+m^2}$$

$$= \frac{1-m^2}{1+m^2}.$$

(different for different value of m).

$\Rightarrow$ the double limit does not exist.

- ## Repeated / Iterated limit.

$$\lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\}, \quad \lim_{x\to a}\left\{\lim_{y\to b} f(x,y)\right\}$$

Example: $f(x,y) = \dfrac{xy}{x^2+y^2}$.

* $\lim_{(x,y)\to(0,0)} f(x,y) = \dfrac{m}{1+m^2} \Rightarrow$ does not exist.
along $y = mx$.

* $$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\{0\} = 0.$$

$$\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{y\to 0}\{0\} = 0.$$

Note: If repeated limits exist & are unequal, then the double limit can not exist.

- Converse is not true.

Example: $f(x,y) = \dfrac{x+y}{x-y}$.

$$\lim_{x \to 0} \left\{ \lim_{y \to 0} f(x,y) \right\} = \lim_{x \to 0} 1 = 1$$

$$\lim_{y \to 0} \left\{ \lim_{x \to 0} f(x,y) \right\} = \lim_{y \to 0} (-1) = -1.$$

$\Rightarrow \lim_{(x,y) \to (0,0)} f(x,y)$ does not exist.

Example: $f(x,y) = \begin{cases} x \sin\left(\frac{1}{y}\right) + y \sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0 \end{cases}$

Repeated limits do not exist ✓

But double limit exists.

as $\lim_{x \to 0} f(x,y)$, $\lim_{y \to 0} f(x,y)$ do not exist.

prove it.

Claim $\lim_{(x,y) \to (0,0)} f(x,y) = 0$.

i.e. to prove that

$\forall \varepsilon > 0$ (however small may be chosen), $\exists$ a $\delta > 0$ (depending on ε) s.t

$|f(x,y) - 0| < \varepsilon$ whenever $|x-0| < \delta$, $|y-0| < \delta$.

<u>proof of the claim.</u>

$x = r\cos\theta$
$y = r\sin\theta$.

$$\left| x \sin\left(\frac{1}{y}\right) + y \sin\left(\frac{1}{x}\right) - 0 \right| \le |x| + |y| \le 2r < \varepsilon$$

$$\text{i.e. } \sqrt{x^2+y^2} < \frac{\varepsilon}{2}.$$

$$x^2+y^2 < \frac{\varepsilon^2}{4}.$$

$$\text{i.e. } x^2 < \frac{\varepsilon^2}{8}, \quad y^2 < \frac{\varepsilon^2}{8}.$$

$$\text{i.e. } |x-0| < \frac{\varepsilon}{2\sqrt{2}}, \quad |y-0| < \frac{\varepsilon}{2\sqrt{2}}.$$

δ. δ.

Example: $\lim_{(x,y)\to(1,2)} (x^2+2y) = 5$.

using ε-δ approach.

Sol^n: $\forall \varepsilon > 0$, arb. small, $\exists \delta > 0$ s.t.

$|x^2+2y-5| < \varepsilon$ whenever $|x-1| < \delta$ & $|y-2| < \delta$.

Let $|x-1| < \delta$, $|y-2| < \delta$. To prove.

$$|x^2+2y-5| = |x^2-1+2y-4| \le |x^2-1| + 2\underbrace{|y-2|}_{<\delta}. \quad — (1).$$

$|x-1| < \delta \Rightarrow 1-\delta < x < 1+\delta$.

$1-2\delta+\delta^2 < x^2 < 1+2\delta+\delta^2$

$$-3\delta < -2\delta+\delta^2 < x^2-1 < 2\delta+\delta^2 < 3\delta$$

if we choose $0 < \delta < 1$

$\delta < 1 \Rightarrow \delta^2 < \delta$ & $\delta^2 > -\delta$.

① $\Rightarrow$

$$|x^2+2y-5| \leq 3\delta + 2\delta = 5\delta = \varepsilon.$$

whenever $|x-1|<\delta$, $|y-2|<\delta$

$(0<\delta\leq 1)$.

<u>Example:</u>

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3} \text{ does not exist.}$$

along $x = my^2$.

<u>Example</u>

$$\lim_{(x,y)\to(0,0)} \frac{|x|}{y^2} e^{-|x|/y^2}.$$

Along $x = my \longrightarrow$ limit $= 0$.

Along $x = y^2 \longrightarrow$ limit $= \frac{1}{e}$.

Example:
$$f(x,y) = \begin{cases} \dfrac{2(x^3+y^3)}{x^2+2y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Show that f is not continuous at $(0,0)$

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}$$

e.g. $$f(x,y) = \begin{cases} x^2+2y, & (x,y) \neq (1,2) \\ 0 & (x,y) = (1,2) \end{cases}$$

$$\lim_{(x,y)\to(1,2)} f(x,y) = 5.$$

Not continuous at $(1,2)$.

Sol^n.

Along path 1: $y = mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{x\to 0} \frac{2x^3(1+m^3)}{x(x+2m)}$$

$$= 0$$

Along path 2: $y = -\frac{x^2}{2}e^x$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=-\frac{x^2}{2}e^x}} f(x,y) = -2.$$

$\Rightarrow$ ~~not~~ double limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

& so $f(x,y)$ is not continuous

Algebra of limits.

- $f(x,y)$, $g(x,y) \rightarrow$ defined on some nbd. of (a,b).
- $\lim_{(x,y)\to(a,b)} f(x,y) = l$, $\lim_{(x,y)\to(a,b)} g(x,y) = m$.

Thm i) $\lim_{(x,y)\to(a,b)} (f \pm g) = \lim f \pm \lim g$
$= l \pm m$

ii) $\lim (f \cdot g) = \lim f \cdot \lim g = lm$

iii) $\lim \left(\frac{f}{g}\right) = \dfrac{\lim f}{\lim g} = \frac{l}{m}$,

provided $m \neq 0$

Example: $\lim_{(x,y)\to(1,2)} (x^2+2y) \cancel{\neq} 5$.

$= \lim_{(x,y)\to(1,2)} (x^2) + \lim_{(x,y)\to(1,2)} (2y)$

$= 1 + 4 = 5$.

Exercise. $\lim_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2-x}$

Exercise: $\lim_{(x,y)\to(0,0)} f(x,y)$ where

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Do not exist.

Example: $$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} , & (x,y)\neq(0,0) \\ 0 , & (x,y)=(0,0). \end{cases}$$

Check continuity of $f(x,y)$ at $(0,0)$.

Sol^n. ~~Prove tha~~

Claim $$\boxed{\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.}$$

i.e. $\forall \varepsilon > 0$ (arb. small), $\exists \delta > 0$

s.t. $$\left| f(x,y) - 0 \right| < \varepsilon$$

whenever $|x - 0| < \delta$, $|y - 0| < \delta$.

<u>proof of the claim</u>

$$\left| f(x,y) - 0 \right| = \left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| .$$

let $x = r\cos\theta$
$y = r\sin\theta$.

$$= \left| \frac{r^2 \sin\theta\cos\theta}{r} \right|$$

$$= \left| \frac{r}{2}\sin 2\theta \right| < \frac{r}{2} < \varepsilon$$

whenever $x^2 + y^2 < (2\varepsilon)^2 = 4\varepsilon^2$

i.e. $x^2 < 2\varepsilon^2$, $y^2 < 2\varepsilon^2$

i.e. $|x - 0| < \sqrt{2}\,\varepsilon$, $|y - 0| < \sqrt{2}\,\varepsilon$.

δ.

<u>choose $\delta = \sqrt{2}\,\varepsilon$</u>

$\Rightarrow$ <u>establishes the claim.</u>

Example:

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \left[\frac{x^2y^2 \cancel{+1} \cancel{-1}}{(x^2+y^2)\left(\sqrt{x^2y^2+1}+1\right)} \right]$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{x^2y^2}{x^2+y^2}\right) \cdot \lim_{(x,y)\to(0,0)} \left(\frac{1}{\sqrt{x^2y^2+1}+1}\right)$$

$$= \lim_{r\to 0} \frac{\overset{r^2}{\cancel{r^4}} \sin^2\theta\cos^2\theta}{\cancel{r^2}}$$

$$\times \lim_{r\to 0} \frac{1}{\sqrt{r^4\sin^2\theta\cos^2\theta+1}+1}$$

$x = r\cos\theta$
$y = r\sin\theta$

$(x,y)\to(0,0) \Rightarrow r\to 0.$

$$= 0 \times \frac{1}{2} = 0$$

~~Exam~~ **Exercise.**

Prove that.

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} = 0$$

ε-δ approach

Example: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{5}{2}$.

Sol^n. $|x-0|<\delta,\quad |y-0|<\delta.$

$x^2+y^2+2 \geq 2$

$$\left| \frac{3x^2-y^2+5}{x^2+y^2+2} - \frac{5}{2} \right|$$

$$= \left| \frac{x^2-7y^2}{2\underbrace{(x^2+y^2+2)}_{\geq 2}} \right| < \frac{8\delta^2}{4} = 2\delta^2 = \varepsilon$$

whenever $|x-0|<\delta,\ |y-0|<\delta$

when $\delta = \sqrt{\frac{\varepsilon}{2}}$.

Example: $\displaystyle\lim_{(x,y)\to(2,1)} (x^2+2x-y^2) = 7.$

Sol^n. We $|x-2|<\delta,\quad |y-1|<\delta.$

$\delta>0$, arb. small

$$|x^2+2x-y^2-7| = |(x-2)^2-(y-1)^2 + 6(x-2) - 2(y-1)|$$

$$< 2\delta^2 + 8\delta < \varepsilon$$

Choose δ s.t.

$$2\delta^2+8\delta < \varepsilon$$

i.e. $$\delta^2+4\delta < \frac{\varepsilon}{2}$$

$$(\delta+2)^2 < \frac{\varepsilon}{2}+4.$$

or $$\delta < \sqrt{\frac{\varepsilon+8}{2}} - 2.$$

$\therefore$ $|x^2+2x-y^2-7| < \varepsilon$ whenever

$\cancel{\&}$ $|x-2| < \delta$, $|y-2| < \delta$

when $\delta < \sqrt{\frac{\varepsilon+8}{2}} - 2.$

$\Rightarrow$ ~~Proved~~. The double limit $= \cancel{6}\ 7$

Example: $$\lim_{(x,y)\to(0,0)} \left(1+x^2y^2\right)^{-\frac{1}{x^2+y^2}}$$

Evaluate

$$= 1.$$

Sol^n: $$u = \left(1+x^2y^2\right)^{-\frac{1}{x^2+y^2}}$$

$$\log u = -\frac{1}{x^2+y^2}\log\left(1+x^2y^2\right).$$

let $x = r\cos\theta,\ y = r\sin\theta.$

$$(x,y)\to(0,0) \implies r\to 0.$$

$$\lim_{r\to 0}\log u = -\lim_{r\to 0}\frac{1}{r^2}\log\left(1+r^4\sin^2\theta\cos^2\theta\right)$$

$$= -\lim_{r\to 0}\frac{1\times 4\cancel{r^3}^{\,r^2}\sin^2\theta\cos^2\theta}{\left(1+r^4\sin^2\theta\cos^2\theta\right)2\cancel{r}}\ \left(\frac{0}{0}\right).$$

$$= 0$$

$$\implies \lim_{r\to 0} u = e^0 = 1. \ \underline{\text{Ans.}}$$

Example:

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x^2+1}.$$

$$= \lim_{(x,y)\to(0,0)} e^{xy} \cdot \lim_{(x,y)\to(0,0)} \frac{1}{x^2+1}.$$

$x = r\cos\theta,\ y = r\sin\theta.$

$$\lim_{r\to 0} e^{r^2\sin\theta\cos\theta} = 1.$$

$$\frac{1}{1} = 1$$

$= 1 \cdot 1 = 1$ Ans.

Partial Derivatives of 1st. order

- $f(x,y)$ over a region R.

- $$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,\, y) - f(x,y)}{h}$$
$$\| $$
$$\frac{\partial f}{\partial x}(x,y)$$

- $$f_y(x,y) = \lim_{k\to 0} \frac{f(x,\, y+k) - f(x,y)}{k}$$
$$\|$$
$$\frac{\partial f}{\partial y}(x,y)$$

Easy generalization

- $u = f(x,y,z)$

$$\begin{array}{ccc} f_x, & f_y, & f_z \\ \| & \| & \| \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}. \\ \swarrow & \swarrow & \swarrow \\ u_x & u_y & u_z \end{array}$$

Example: $f(x,y) = \dfrac{x+y-1}{x+y+1}$

$f_x(2,1) = ?$ $f_y(2,1) = ?$

Sol^n. $f_x(2,1) = \lim\limits_{h \to 0} \dfrac{f(2+h,1) - f(2,1)}{h}$

$$\boxed{\begin{aligned} f_x(a,b) &= \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} \\ f_y(a,b) &= \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k} \end{aligned}}$$

$= 1/8$

Alternatively, $f_x(x,y) = \dfrac{2}{(x+y+1)^2}$

(differentiating $f(x,y)$ w.r.to. x, treating y as const.)

$f_x(2,1) = 1/8$.

Example:

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & x^2+y^2 \neq 0. \\ 0, & x^2+y^2 = 0. \end{cases}$$

$f_x(0,0) = ?$, $f_y(0,0) = ?$

$f_x(0,0) = 0$, $f_y(0,0) = 0$

$f(x,y) \rightarrow$ not continuous at $(0,0)$.

as $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist

$y = mx$.

observation: both f_x, f_y exist at $(0,0)$

although $f(x,y)$ is not continuous at $(0,0)$.

Continuity of $f(x,y)$ at (a,b)

- $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exist
- $f(a,b)$ must be defined.
- $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example:

$$f(x,y) = \begin{cases} \dfrac{2(x^3+y^3)}{x^2+2y}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0). \end{cases}$$

Show that $f_x(0,0)$, $f_y(0,0)$ exist.
Also show that f is not continuous at $(0,0)$.

- $f_x(0,0) = 2$
- $f_y(0,0) = 0$

Partial derivatives of higher order

- $f(x,y)$

1st. order $f_x(x,y)$, $f_y(x,y)$
$\longrightarrow$ themselves fns. of x, y

2nd. order

- $f_{xx}(x,y) = \dfrac{\partial}{\partial x}(f_x)$
$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

- $f_{yy}(x,y) = \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$.

- $f_{x\underline{y}}(x,y) = \dfrac{\partial}{\partial x}(f_y) = \dfrac{\partial^2 f}{\partial x \partial y}$

- $f_{yx}(x,y) = \dfrac{\partial}{\partial y}(f_x) = \dfrac{\partial^2 f}{\partial y \partial x}$.

3rd. order f_{xxx}, f_{xyx}, $\dots$, f_{yyy}

In general
$$\frac{\partial^n f}{\partial x^n} = \frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right)$$

$$\frac{\partial^n f}{\partial y\, \partial x^{n-1}} = \frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right)$$

$$\vdots$$

- $f_{xy}(a,b) \overset{?}{=} f_{yx}(a,b)$

$$\underline{f_{xy}(a,b)} = \lim_{h \to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

$$\left.\frac{\partial}{\partial x}(f_y)\right]_{(a,b)}$$

$$F(h,k)$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{\left[\begin{array}{l} f(a+h,b+k) - f(a+h,b) \\ \quad - f(a,b+k) + f(a,b) \end{array}\right]}{hk}$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{F(h,k)}{hk}$$

$$f_{yx}(a,b) = \lim_{k \to 0} \lim_{h \to 0} \frac{F(h,k)}{hk}.$$

repeated limits.

may differ in value.

- $f_{xy} \neq f_{yx}$ in general.

e.g. $f(x,y) = \dfrac{x+y}{x-y}$

$$\lim_{x \to 0} \lim_{y \to 0} f(x,y) = 1$$

$$\neq$$

$$\lim_{y \to 0} \lim_{x \to 0} f(x,y) = -1.$$

__Example:__ $f(x,y) = \begin{cases} +xy & \text{when } |x| \geq |y| \\ -xy & \text{when } |x| < |y|. \end{cases}$

Show that $f_{xy}(0,0) = 1$, $f_{yx}(0,0) = -1$.

__Solⁿ.__

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k \to 0} \frac{hk - 0}{k} = h$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= 0 .$$

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{h - 0}{h} = 1 .$$