Mathematics-I

CURVATURE

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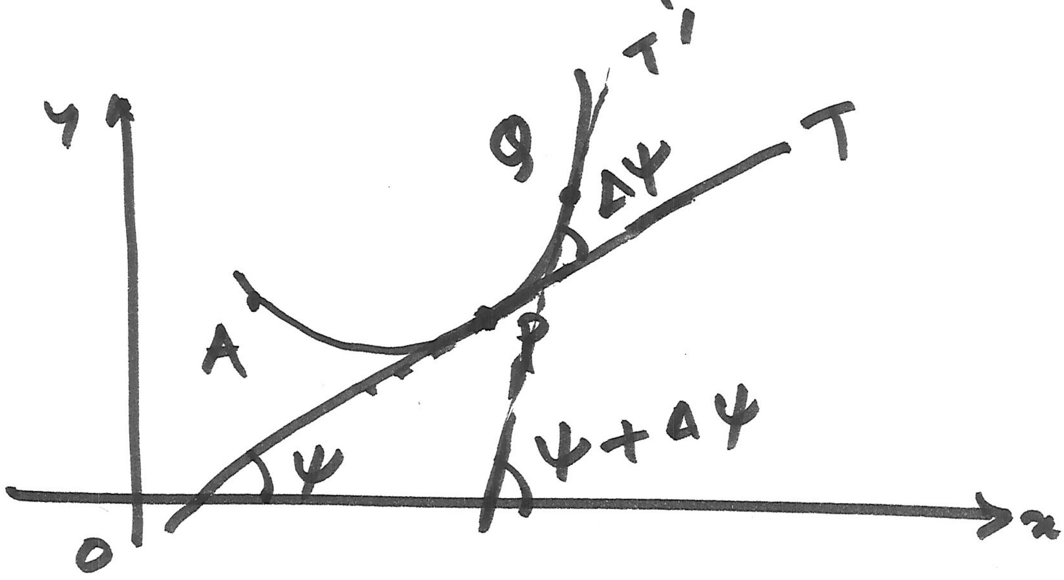
Curvature

- Curvature → measure of bending of a curve.



Arc $AP = s$.

Arc $AQ = s + \Delta s$.

i.e. arc $PQ = \Delta s$.

- Total curvature or total bending of the arc PQ → $\Delta\psi$.

- av. curvature → $\dfrac{\Delta\psi}{\Delta s}$

- Curvature at P

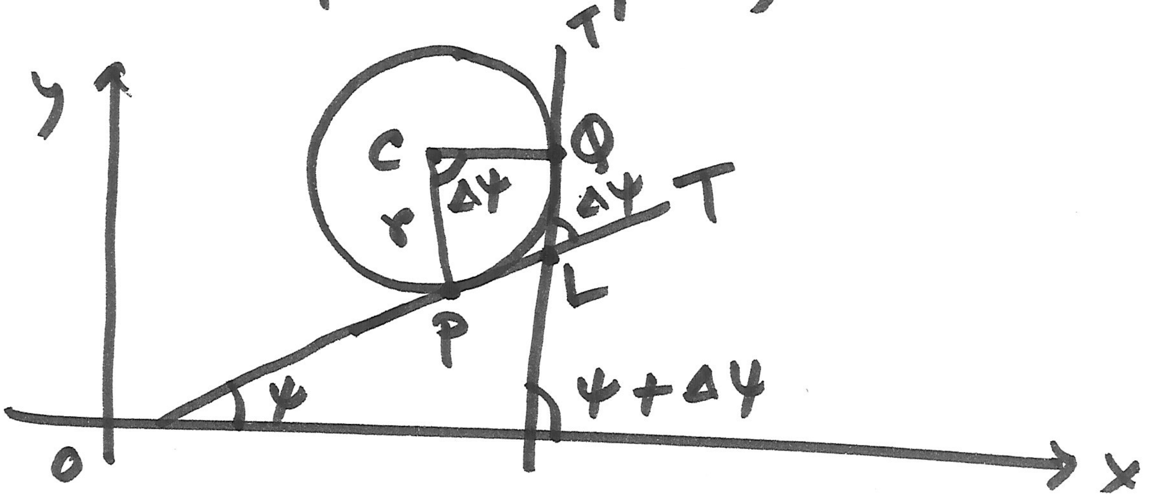
letting $P \to Q$

i.e. $\Delta s \to 0$

$$\boxed{K = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds}}$$

Example: Curvature of a circle.

- circle of radius r, centre at C.



$\angle PCQ = \Delta\psi$.

Now $\underline{\text{arc } PQ} = r.\Delta\psi$

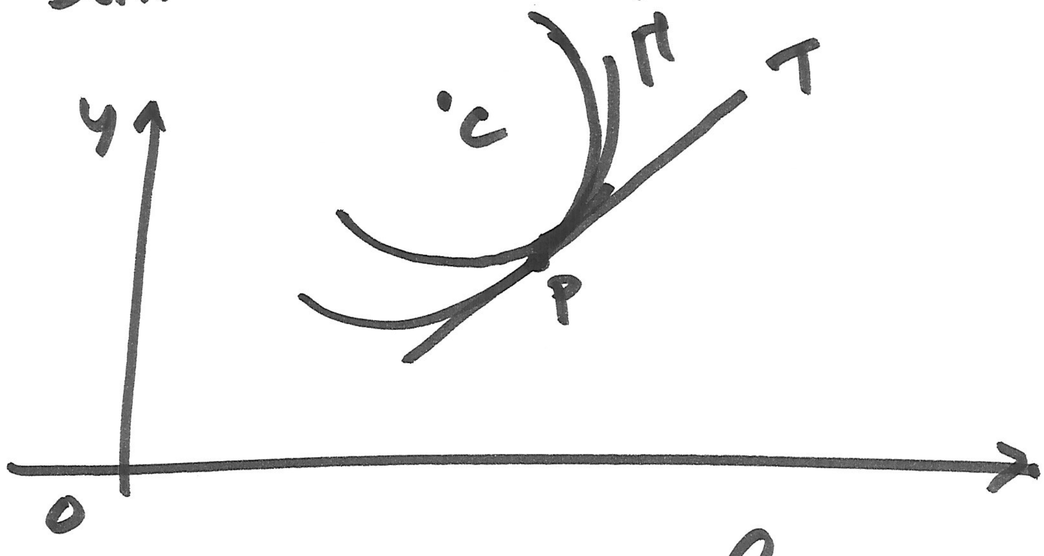
$$\frac{\Delta s}{\Delta\psi} = r.$$

letting $\Delta s \to 0$

$$\boxed{\lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds} = \frac{1}{r}.}$$

Curvature at any pt. of a circle is always const. & is equal to the reciprocal of the radius of the circle.

- Curvature $K \neq 0$ at a pt. P on a curve Γ.
- Consider $\rho = \frac{1}{K}$ & centre at C so that the circle & the curve Γ both have the same tangent at P



- This circle & Γ lie on the same side of the tangent.

- $K \rightarrow$ curvature at P.
  $= \frac{d\psi}{ds}$
- $\rho \rightarrow$ radius of curvature at P.
  $= \frac{ds}{d\psi} = \frac{1}{K}$.
- $C \rightarrow$ centre of curvature at P.
- This circle $\rightarrow$ circle of curvature.

Formulae for the radius of curvature

1. Intrinsic eqn: $s = f(\psi)$

$$\rho = \frac{ds}{d\psi}.$$

Example: $s = c\tan\psi$ (catenary).

$$\rho = c\sec^2\psi.$$

2. Cartesian eqn. (explicit fn.).

$$y = f(x) \quad \text{or} \quad x = f(y)$$

We have, $\dfrac{dy}{dx} = \tan\psi.$

$$\boxed{\cos\psi = \frac{dx}{ds}, \quad \sin\psi = \frac{dy}{ds}.}$$

Diff. w.r.to. s,

$$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \sec^2\psi\,\frac{d\psi}{ds}.$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{ds} = \sec^2\psi\cdot\frac{d\psi}{ds}$$

$$\frac{d^2y}{dx^2}\cdot\cos\psi = \sec^2\psi\,\frac{d\psi}{ds}$$

$$\rho = \frac{ds}{d\psi} = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}}$$

$$= \frac{(1+\tan^2\psi)^{3/2}}{y_2}, \quad y_2 = \frac{d^2y}{dx^2}$$

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}$$

$y_2 \neq 0$

+ve if $y_2 > 0$

−ve if $y_2 < 0$.

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|}}$$

Example:

Folium of Descartes:

$$x^3+y^3=3axy \quad \text{—} ①$$

radius of curvature at $\left(\frac{3}{2}a, \frac{3}{2}a\right)$.

$\underline{\underline{\text{Sol}^n.}}$ Diff. ① w.r.to x,

$$3x^2+3y^2y_1=3a(y+xy_1) \quad \text{—} ②$$

$$\Rightarrow \left. y_1 \right]_{\left(\frac{3}{2}a, \frac{3}{2}a\right)} = -1.$$

Diff. ② w.r.to x,

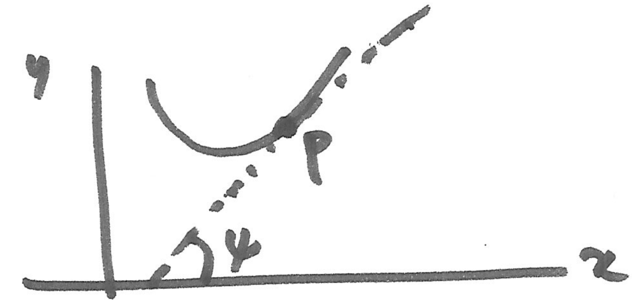
$$\left. y_2 \right]_{\left(\frac{3}{2}a, \frac{3}{2}a\right)} = -\frac{32}{3a}.$$

$$\rho = \frac{\left[1+y_1^2\right]^{3/2}}{|y_2|} = \boxed{\frac{3}{16}\sqrt{2}\,a}$$

Ans.

- Curvature at P of a plane curve

$$\boxed{\kappa = \frac{d\psi}{ds}}\,, \qquad y_1 = \tan\psi$$



- radius of curvature at P

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}.$$

- Intrinsic eqn. : $s = f(\psi)$

$$\rho = \frac{ds}{d\psi}$$

- Cartesian eqn. : $y = f(x)$ or $x = \phi(y)$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{|y_2|}$$

- Parametric eqn. : $x = f(t),\ y = \phi(t).$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}\,, \quad x'y'' - y'x'' \neq 0.$$

$$x' = \frac{dx}{dt},\ y' = \frac{dy}{dt}.$$

proof: $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$.

$$y_1 = \frac{dy/dt}{dx/dt} = \frac{y'}{x'}.$$

$$\frac{d}{dx}(y_1) = y_2 = \frac{y''x' - x''y'}{x'^2} \cdot \frac{1}{x'}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \frac{y'^2}{x'^2}\right)^{3/2}}{\frac{y''x' - x''y'}{x^3}}.$$

- Polar equ^n. : $r = f(\theta)$.

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}.}$$, $r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2 r}{d\theta^2}$.

proof. $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{dx}{d\theta} = r_1\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = r_1\sin\theta + r\cos\theta$$

$$y_1 = \frac{dy/d\theta}{dx/d\theta} = \frac{r^2 + r_1^2}{(r_1\cos\theta - r\sin\theta)^2}$$

$$1+y_1^2$$

$$y_2 = \frac{r^2+2r_1^2 - rr_2}{(r_1\cos\theta - r\sin\theta)^3}.$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \text{reqd. result}.$$

Exercise: Polar eqn : $u = f(\theta)$, $u = \frac{1}{r}$.

$$\rho = \frac{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]^{3/2}}{u^3\left[u + \frac{d^2u}{d\theta^2}\right]}.$$

Example: Prove that

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

Soln: $\tan\psi = \frac{dy}{dx} = \frac{dy/ds}{dx/ds}$

$\Rightarrow \sin\psi = \frac{dy}{ds}, \ \cos\psi = \frac{dx}{ds}.$

$$\frac{d^2x}{ds^2} = \frac{d}{d\psi}\left(\frac{dx}{ds}\right) \cdot \frac{d\psi}{ds} \rightarrow \frac{1}{\rho}.$$

$$= \frac{d}{d\psi}(\cos\psi)\,\frac{1}{\rho} = -\frac{\sin\psi}{\rho} \quad \text{— (1)}$$

$$\frac{d^2y}{ds^2} = \frac{d}{d\psi}\left(\frac{dy}{ds}\right) \cdot \frac{d\psi}{ds} \rightarrow \frac{1}{\rho}.$$

$$\frac{dy}{ds} \rightarrow \sin\psi.$$

$$= \frac{\cos\psi}{\rho}. \quad \text{— (2)}$$

(1) (2) give

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 = \frac{(-\sin\psi)^2 + \cos^2\psi}{\rho^2}$$

$$= \frac{1}{\rho^2}. \quad (\underline{\underline{\text{proved}}})$$

Example: Show that for the curve

$\underline{\underline{s^2 = 8ay}}$, $\rho = 4a\sqrt{1-\frac{y}{2a}}$.

Sol^n. diff. w.r.to s.

$$2s = 8a\frac{dy}{ds}$$

$$= 8a\sin\psi \quad \text{--- (1)}$$

$\tan\psi = \frac{dy}{dx}$

$\Rightarrow \sin\psi = \frac{dy}{ds}$

$\cos\psi = \frac{dx}{ds}$.

$$\boxed{\sin\psi = \frac{2s}{8a} = \frac{s}{4a}}.$$

Diff. (1) w.r.to ψ

$$2\frac{ds}{d\psi} = 8a\cos\psi.$$

ρ

$$\rho = 4a\cos\psi.$$

$$= 4a\sqrt{1-\sin^2\psi}.$$

$$= 4a\sqrt{1-\left(\frac{s}{4a}\right)^2} = 4a\sqrt{1-\frac{8ay}{16a^2}}$$

$$\cancel{= 4a\sqrt{1-\frac{8ay}{16a^2}}} = 4a\sqrt{1-\frac{y}{2a}}.$$

Proved.