ASYMPTOTES

(a) Oblique Asymptotes.
(b) Horizontal Asymptotes.
(c) Asymptotes Similar the axis for a rational curve.
(d) Vertical Asymptotes

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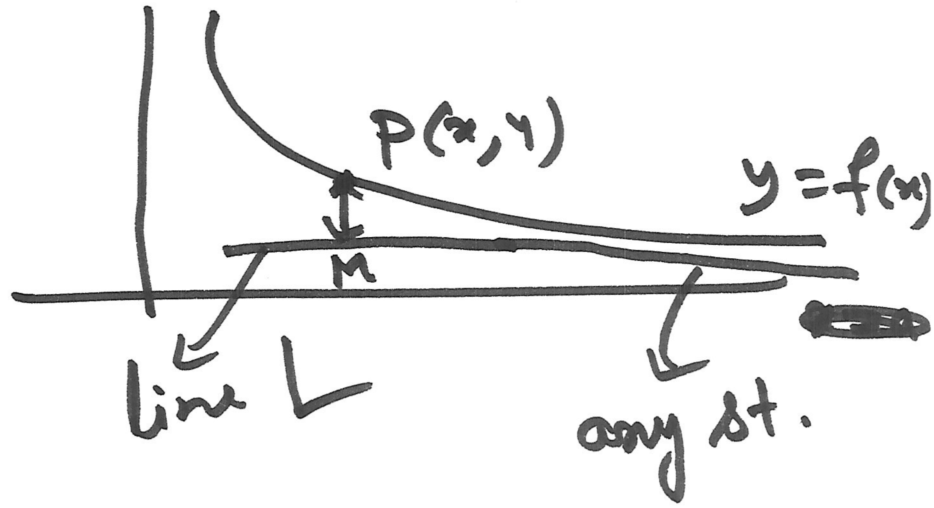
Asymptotes

- Point $P(x,y)$ on an <u>infinite branch of a curve</u> is said to tend to ∞ ($P \to \infty$) along the curve if either $x \to \infty$, or $y \to \infty$, or both as P traverses along the branch.

$P \to \infty$



Asymptotes

$PM \to 0$ as $P \to \infty$.

A st. line is said to be an asymptote of an infinite branch of a curve <u>if</u> as a pt. $P \to \infty$ along the branch, the ($\perp$r distance of P from from the st. line) $\to 0$.

Examples:

circle, ellipse $\rightarrow$ no asymptotes as no infinite branches.

parabola $\rightarrow$ no asymptotes although it is extending to infinity.

- Horizonatal, Vertical, Oblique Asymptotes.

Horizonatal $\downarrow$ $y = b$.

Vertical $\downarrow$ $x = b$.

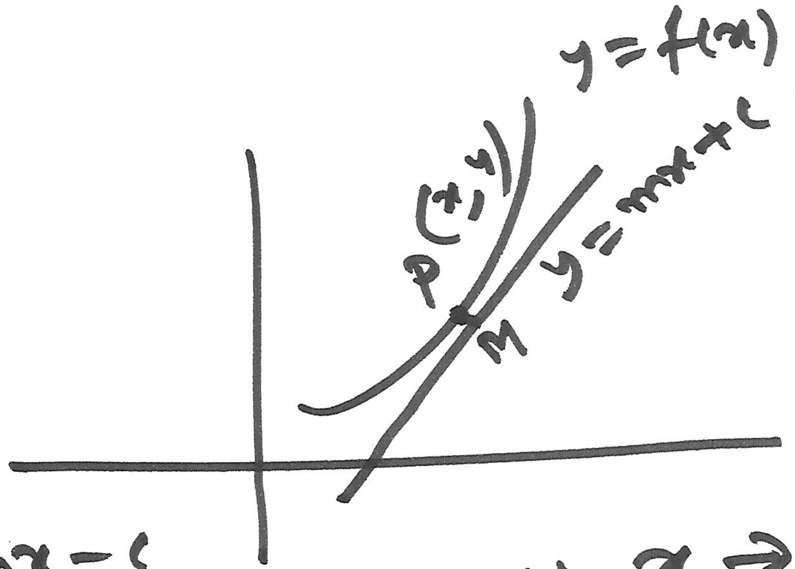
Oblique Asymptotes $\downarrow$ $y = mx + c$, $m, c \rightarrow$ finite.

Obligue asymptotes (not parallel to y-axis).

- $y = mx + C$, m, c finite.

$$\boxed{m = \lim_{x \to \infty} \frac{y}{x}, \quad c = \lim_{x \to \infty} (y - mx)}.$$

(proof. sketch)



$$|PM| = \frac{y - mx - c}{\sqrt{1+m^2}} \to 0 \text{ as } x \to \infty.$$

$$\Rightarrow \boxed{c = \lim_{x \to \infty} (y - mx).}$$

$$\frac{y}{x} - m = (y - mx) \cdot \frac{1}{x}$$

$$\to c \cdot 0 \text{ as } x \to \infty.$$

$$\Rightarrow \boxed{\lim_{x \to \infty} \frac{y}{x} = m}$$

<u>Example:</u> Discuss the asymptotes of the curve

$$y = \frac{3x}{2} \log \left(e - \frac{1}{3x}\right).$$

Sol^n: $\rightarrow$ y is infinite when $x = 0$ or $x = \frac{1}{3e}$.

$x = 0$?

$$\lim_{x \to 0} y = \text{infinite or not}$$

$\downarrow$

Not

$x = \frac{1}{3e}$?

$$\lim_{x \to \frac{1}{3e}} y = \text{infinite or not.}$$

$\Downarrow$

YES.

Oblique asymptotes $\quad y = mx + c$

$$m = \lim_{x \to \infty} \frac{y}{x} = \lim_{x \to \infty} \frac{3}{2} \log\left(e - \frac{1}{3x}\right) = \frac{3}{2}.$$

$$c = \lim_{x \to \infty} (y - mx) = \lim_{x \to \infty} \left[\frac{3x}{2} \log\left(e - \frac{1}{3x}\right) - \frac{3}{2}x \right].$$

$$= -\frac{1}{2e}. \qquad \boxed{y = \frac{3x}{2} - \frac{1}{2e}}$$

<u>Example:</u> $y^2 = 9x$ parabola.

Oblique asymptotes. $\rightarrow$ not $\parallel$ to y-axis.

$y = mx + c.$

$$\boxed{m = \lim_{x\to\infty} \frac{y}{x}, \quad c = \lim_{x\to\infty}(y - mx)}.$$

Note: To determine asymptotes $\parallel$ to y-axis, start with $x = my + d.$

Compute
$$\boxed{\begin{aligned} m &= \lim_{y\to\infty} \frac{x}{y} \\ d &= \lim_{y\to\infty} x - my \end{aligned}}$$

Theorem In order that the line $x = a$ may be an asymptote. (i.e. vertical asymptote) to a curve $y = f(x)$, it is NASC that

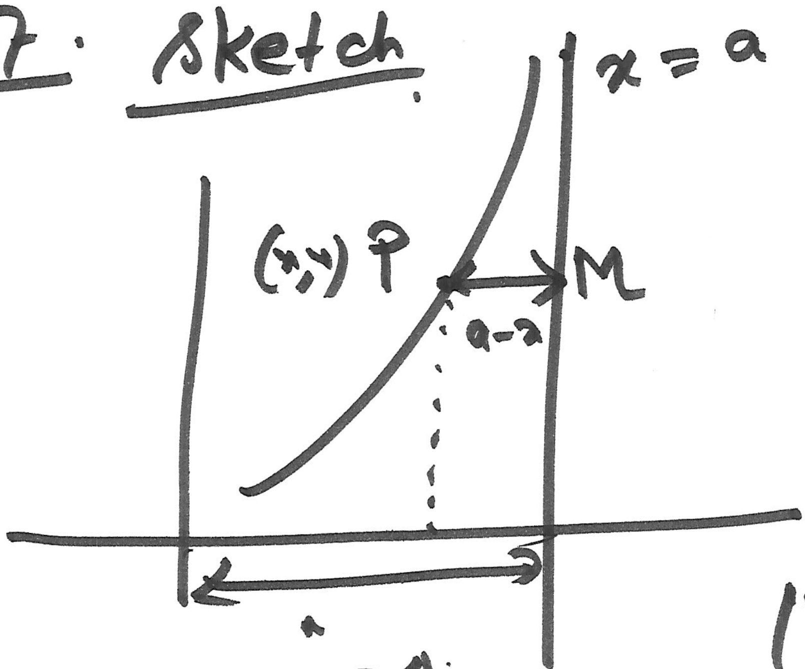
$|y| \to \infty$ as $x \to a$.

- **Horizontal asymptotes for $\boxed{x = \phi(y)}$**

$y = b$.

NASC $|x| \to \infty$ as $y \to b$.

proof. Sketch



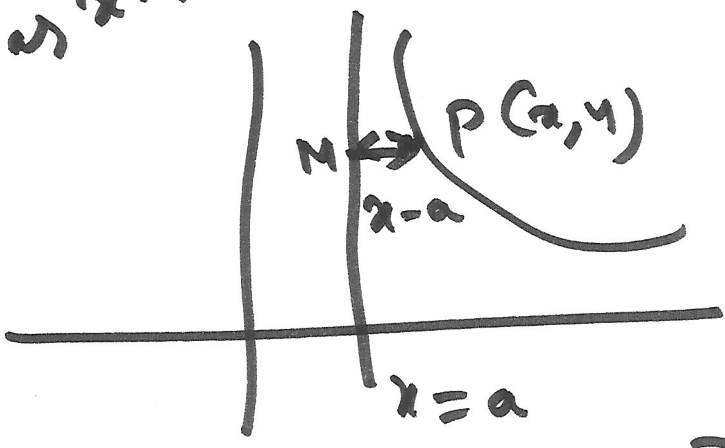
$|PM| \to 0$

as ~~$x \to a$~~

$P \to \infty$.

$|PM| = |a - x|$

let $|y| \to \infty$ as $x \to \infty$.



$|a - x| \to 0$ as $x \to a$

i.e. $P \to \infty$.

Example:

$$y = \frac{3x}{2} \log \left(e - \frac{1}{3x}\right).$$

Vertical asymptotes

$|y| \to \infty$ as $x \to a$.

$x = 0, \frac{1}{3e}$.

$x \to 0$ $\quad |y| \not\to \infty$.

$\downarrow$ Not an asymptote.

$x \to \frac{1}{3e}$ $\quad |y| \to \infty$.

$\downarrow$ a vertical asymptote.

Asymptotes $\parallel$ to the axes for a rational curve.

- $f(x,y) = y^m \phi(x) + y^{m-1} \phi_1(x) + y^{m-2} \phi_2(x) + \cdots + \phi_m(x) = 0.$

$$\phi(x) + \underbrace{\frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \cdots + \frac{1}{y^m}\phi_m}_{\text{}} = 0.$$

let $y \to \alpha$

$$k = \lim_{y \to \alpha} x$$

$$\to \phi(k) = 0. \cancel{\text{i.e.}}$$

let $\phi(x) = (x - k_1)(x - k_2) \cdots (x - k_l).$

$x = k_1,\ x = k_2,\ \ldots,\ x = k_l$

$\longrightarrow$ vertical asymptotes.

<u>Example:</u>

$$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0.$$

<u>Vertical asymptotes.</u>

co-efficient of $y^2 \to x^2 - a^2 = (x - a)(x + a).$

$$\underline{x = \pm a.}$$

<u>Horizontal asymptotes</u> $\quad y = \pm a.$

Example:

$$x^2y^2 - 9x^2 + 2 = 0.$$

Horizontal asymptotes co-eff. of x^2

$$y = \pm 3.$$

Vertical asymptotes

co-eff. of y^2

$$x = 0.$$

$$y = \frac{\sqrt{9x^2 - 2}}{x}$$

as $x \to 0$, one get $y \to$ imaginary.

No vertical asymptote.