Mathematics-I

CONCAVITY And CONVEXITY

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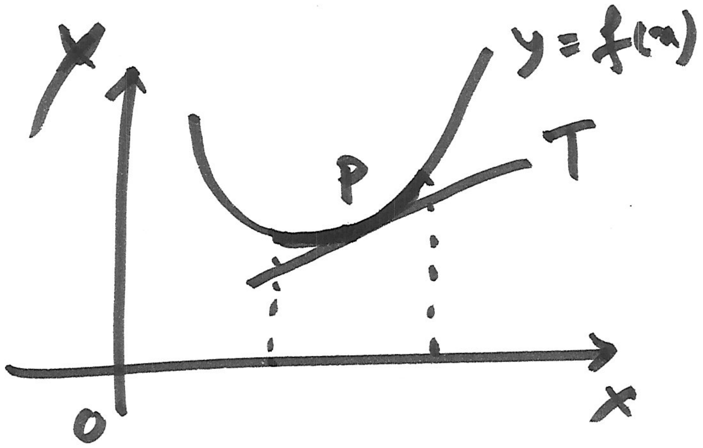
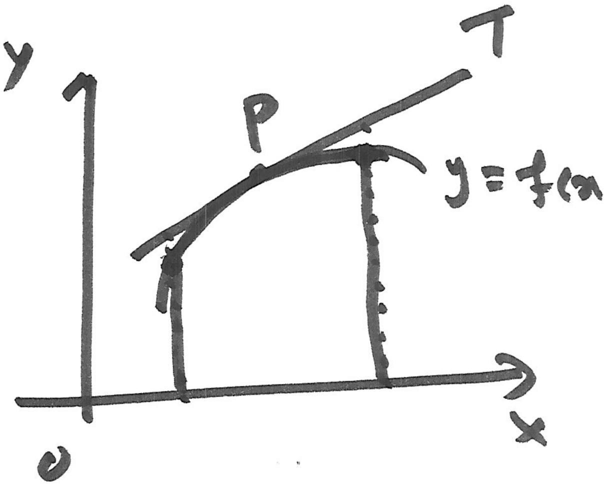
Concavity & Convexity of a curve.

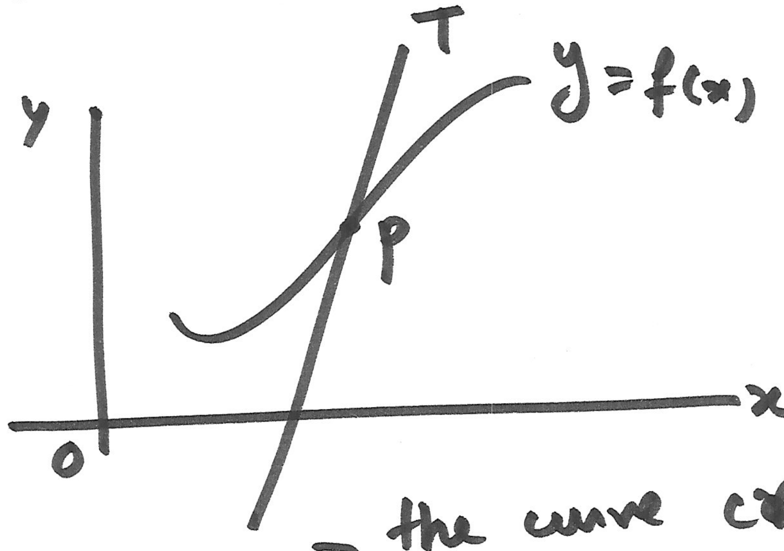
- $y = f(x) \rightarrow$ a plane curve.
- $P \rightarrow$ a pt. of that curve.
- $PT \rightarrow$ tangent to $y = f(x)$ at P, not parallel to y-axis.





→ Concave up or Convex down.

→ Concave down or Convex up.



→ the curve crosses the tangent at P.

→ point of inflexion.

- $y = f(x) \rightarrow$ differentiable f^n.
  - Concave up on an open interval I if $f' \uparrow$ in I.
  - Concave down on an open interval I if $f' \downarrow$ in I.

- Second derivative test for concavity.

- $y = f(x) \rightarrow$ twice-diff. f^n. on I.

1. if $f''(x) > 0$ on I, f is concave up.

2. if $f''(x) < 0$ on I, f is concave down

3. if $f''(c) = 0$. & $f''(x)$ changes sign as x passes through c for some $c \in I$, then the curve has a pt. of inflexion at $(c, f(c))$

$\longrightarrow f'''(c) \neq 0.$

<u>Example:</u> $y = x^3$. —— ①

Find the range of values of x for which ① is concave upwards or downwards. ~~Also d~~

Also determine the pt. of inflexion (if any).

<u>Sol^n:</u> $y' = 3x^2$

$$y'' = 6x \begin{cases} > 0 & \forall\ \underline{x > 0} \rightarrow \text{Concave up} \\ < 0 & \forall\ x < 0. \rightarrow \text{Concave down.} \end{cases}$$

<u>$x = 0$</u>

$y'' = 0$

$y''' = 6 \neq 0$ at $x = 0$.

$(0,0) \rightarrow$ a pt. of inflexion.

Note: Any root of $f''(x)=0$ may not give a pt. of inflexion.

i.e. an inflexion pt. may not exist when $f''(x)=0$.

$\rightarrow$ $f''(x)$ must change sign as x passes through a pt. of inflex

Example:

$y = x^4$. at $x = 0$

$\hookrightarrow (0,0) \rightarrow$ a pt. of inflexion or not?

$y' = 4x^3$.

$y'' = 12x^2$

$y'''(0) = 0$.

Ans. No conclussion.

$\boxed{\text{Ans. Not a pt. of inflexion.}}$

$\hookrightarrow$ explain later.

Example: $y = x^{1/3}$, at $x = 0$

y'' does not exist

$\rightarrow$ no conclusion.

Ans. $(0,0)$ is a pt. of inflexion.

Example: find the pts. of ~~inflexion~~ inflexions of $y = (\log x)^3$.

Soln.

$$y'' = 0 \Rightarrow x = 1, e^2.$$

Check

$$y'''\Big]_{x=1} \neq 0$$

$$y'''\Big]_{x=e^2} \neq 0.$$

$(1,0), (e^2, 8) \rightarrow$ two pts. of inflexions

More generally, if $f''(x), f'''(x), \ldots, f^{n-1}(x)$ vanish for a value of x, & $f^n(x)$ exists, $f^n(x) \neq 0$, $\boxed{n \text{ even}}$

- $y = f(x)$ is concave up if $f^n(x) >$
- $y = f(x)$ is concave down if $f^n(x) < 0$.

· roots of $f^n(x) = 0$ may give a pt. of inflexion if $f^m(x) \neq 0$ at that pt. when $m =$ odd & $f^{n+1}(x), f^{n+2}(x), \ldots, f^{m-1}(x)$ all vanish at that pt.

$f^n(x) = 0$ $\quad f^{n+1}(x) = 0 \quad \ldots \quad f^{m-1}(x) = 0 \quad f^m(x) \neq 0$

$\downarrow$

$m =$ odd.