Mathematics-I

GENERALIZED MEAN VALUE THEOREM

(a) Taylor's Theorem.
(b) Maclaurin's Theorem.
(c) Taylor's Infinite Series.
(d) Maclaurin's Infinite Series.
(e) Power Series Expansion of Some Standard Functions.

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Generalized MVTh. : Taylor's Theorem

· $f(x)$ s.t.

(a) $(n-1)$-th derivative f^{n-1} is continuous in $[a, a+h]$

(b) n-th derivative f^{n} exists in $(a, a+h)$

Then $\exists$ at least one θ, $0<\theta<1$ s.t.

$$\boxed{\begin{aligned} f(a+h) &= f(a) + hf'(a) + \frac{h^2}{2!} f''(a) \\ &\quad + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n \\ R_n &= \frac{h^n}{n!} f^{n}(a+\theta h). \end{aligned}}$$

remainder after n terms.

· $\underline{n=1}$ $\rightarrow$ Lagrange's MVTh.

$\rightarrow$ Lagrange's form of remainder.

· Maclaurin's Theorem

$$[a, a+h] \rightarrow [0, x]$$

$$\boxed{\begin{aligned} f(x) &= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots \\ &\cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \underbrace{\boxed{\frac{x^n}{n!} f^n(\theta x)}}_{R_n} \\ & \qquad 0 < \theta < 1. \end{aligned}}$$

which holds when

i) f^{n-1} continuous in $[0, x]$

ii) f^n exists in $(0, x)$.

- $R_n \rightarrow$ error in approximating

$$f(x) \sim f(0) + x f'(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0).$$

- $\boxed{R_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty}$

<u>Example:</u> $\sin x \rightarrow$ Taylor's infinite series $\forall$ real x.

$\log(1+x) \rightarrow \quad -1 < x \leq 1$

Example:

$$f(x) = e^x.$$

By Maclaurin's ~~series expansion~~ Theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \underbrace{\frac{x^n}{n!} f^n(\theta x)}_{R_n},$$

$$0 < \theta < 1$$

$$R_n = \frac{x^n}{n!} \cancel{f^n} e^{\theta x}, \quad 0 < \theta < 1.$$

$$e^x = f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\theta x}.$$

$x = 1$: $\quad e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + R_n$

Error estimate in e

$$R_n = \frac{e^\theta}{n!} < \frac{e}{n!} < \frac{3}{n!} \qquad (2 < e < 3)$$

Find n to calculate e with an error at most 10^{-5}.

i.e. $R_n < \frac{3}{n!} < 10^{-5}$.

$\Rightarrow n = 10$.

$\therefore$ value of e correct to six decimal places is

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{9!} = 2.718281.$$

e is irrational

If not, let $e = \frac{p}{q}$, p, q two integers

Choose $n-1 > q$.

$$(n-1)!\, e = (n-1)!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \frac{e^\theta}{n!}\right)$$

$$= 2.(n-1)! + \frac{(n-1)!}{2!} + \frac{(n-1)!}{3!} + \cdots + \frac{(n-1)!}{(n-1)!} + \frac{e^\theta}{n}.$$

integer.

integer.

fraction.

$$0 < \theta < 1$$

$$e^{\theta} < e < 3.$$

$$\frac{e^{\theta}}{n} < \frac{e}{n} < \frac{3}{n} < 1$$

for sufficiently large n.

getting a contradiction

$\therefore$ e must be irrational.

Taylor's Infinite Series

- $f(x)$ possesses derivatives of all orders at any $x \in [a-h, a+h]$
- Taylor's Th. holds in $[a-h, a+h]$ with remainder $R_n \to 0$ as $n \to \infty$.

Then for each $x \in [a-h, a+h]$, we have

$$\boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \cdots} \quad (1)$$

Note: Change $x \rightarrow a+h$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

Maclaurin's Infinite Series

$a = 0$ in (1)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

Example: Expand $\sin x$ in a finite series in powers of x, with remainder (in Lagrange's form)

Soln. By Maclaurin's Theorem,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n.$$

$$R_n = \frac{x^n}{n!} f^n(\theta x), \quad 0 < \theta < 1$$

$f(x) = \sin x.$

$$\boxed{f^n(x) = \sin\left(\frac{n\pi}{2} + x\right).}$$

$$R_n = \frac{x^n}{n!} \sin\left(\frac{n\pi}{2} + \theta x\right).$$

$\boxed{\text{\underline{Claim}: } R_n \to 0 \text{ as } n \to \infty.}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

for real x.

Try to prove it.

$$\boxed{\frac{x^n}{n!} \to 0 \text{ as } n \to \infty \text{ for all real } x}$$

Exercise.

Example: Find the Maclaurin's Series of fn. $f(x) = \log\left[x + \sqrt{1+x^2}\right]$ upto x^7 term.

Soln. $y(x) = \log\left[x + \sqrt{1+x^2}\right]$.

$$y_1 = \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{1}{\cancel{2}}\frac{\cancel{2}x}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{\sqrt{1+x^2}}$$

$$(1+x^2)\,y_1^2 - 1 = 0 \;\longrightarrow\; y_1(0) = 1.$$

$$(1+x^2)\,2y_1y_2 + (2x)\,y_1^2 = 0.$$

$$\boxed{(1+x^2)\,y_2 + x\,y_1 = 0.} \quad \text{— } 1$$

Differentiating 1 w.r.to x n times.

$$\boxed{\begin{aligned}(1+x^2)\,y_{n+2} &+ n(2x)\,y_{n+1} + \frac{n(n-1)}{2!}(2)\,y_n \\ &+ x\,y_{n+1} + n(1)\,y_n = 0.\end{aligned}}$$

$u = u(x)$, $v = v(x)$

$(uv)_n \to$ Differentiation n-times

$\|$

$u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots$

$+ \binom{n}{n} u\, v_n$

$\frac{d}{dx}(uv)$.

$u'v + v'u$

$$\Rightarrow \boxed{y_{n+2}(0) = -n^2\, y_n(0).}$$

$$y = \log\left[x + \sqrt{1+x^2}\right] = y(0) + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \cdots$$

$$= x - \frac{x^3}{3!} 1^2 + \frac{x^5}{5!} 3^2 \cdot 1^2 - \frac{x^7}{7!} (5^2 \cdot 3^2 \cdot 1^2) + \cdots$$

<u>Exercise</u> $f(x) = \sin^{-1} x.$

↳ Maclaurin's Series expansion upto x^7 term.

$$\boxed{\text{Leibnitz's Rule for Successive differentiation}}$$

Exercise If $y = \sin^{-1} x$

$= a_0 + a_1 x + a_2 x^2 + \cdots$

then prove that

$$n(n+1)\, a_{n+1} = (n-1)^2\, a_{n-1}.$$

→ Maclaurin's series expansion

$$a_n = \frac{y_n(0)}{n!}.$$

- e^x
- $\sin x$
- $\cos x$

$\Big\}$ $\forall$ real x.

- $\log(1+x)$ $-1 < x \le 1$

- $(1+x)^m = \begin{cases} \text{finite} \to \text{if } m \text{ int. } \forall \text{ real } x. \\ \text{infinite series} \to m \text{ is not a +ve int.} \end{cases}$

$-1 < x < 1$

Example: Given $y = f(x) = \dfrac{1}{\sqrt{1+2x}}$

i) Prove that $(1+2x)\,y_{n+1} + (2n+1)\,y_n = 0$.

ii) Expand $f(x)$ by <u>Maclaurin's Theorem with Remainder</u> after n terms.

<u>Taylor's Series expansion about $x = a$ of $f(x)$</u>

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \cdots + \frac{(x-a)^n}{n!}\,f^n(a) + \cdots$$

<u>Maclaurin's series expansion</u>

$a = 0$.

<u>Sol^n.</u>

$$y = \frac{1}{\sqrt{1+2x}}.$$

$$y\sqrt{1+2x} = 1. \Rightarrow y_1\sqrt{1+2x} + y\,\frac{1\cdot\cancel{2}}{\cancel{2}\sqrt{1+2x}} = 0$$

or $y_1(1+2x) + y = 0$. —— 1

Using ~~Leibnitz~~ $\underline{\text{Leibnitz's}}$ Rule on ~~successive~~ derivatives

$$\boxed{(uv)_{\underline{\underline{n}}} = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} \times v_2 + \cdots + \binom{n}{n} u v_n}$$

differentiating n times, we get from eqn. 1.

$$y_{n+1}(1+2x) + \binom{n}{1} y_n (2) + \binom{n}{2} y_{n-1}(0) + y_n = 0.$$

$$\boxed{(1+2x)\, y_{n+1} + (2n+1)\, y_n = 0.}$$

$\underline{\underline{\text{proved}}}$

ii) $y = f(x) = (1+2x)^{-1/2}$

$$y_n = f^n(x) = \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}-1\right)\cdots\left(-\tfrac{1}{2}-n+1\right) \times (1+2x)^{-\frac{1}{2}-n}\, 2^n.$$

$$\boxed{\underset{\swarrow\, f^n(0)}{y_n(0)} = (-1)^n\, 1.3.5\cdots(2n-1)\,(1+2x)^{-\frac{(2n+1)}{2}}.}$$

$\therefore$ By Maclaurin's Theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x),$$

$$0 < \theta < 1.$$

$$= 1 - x + \frac{x^2}{2!}\, 1.3 - \frac{x^3}{3!}\, 1.3.5 + \cdots$$

Power Series expansion of ~~some~~ std. fu^{n}!

- $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots,$ (1)

  $a_i \rightarrow$ real csts. $\rightarrow$ Real power series

  $x \rightarrow$ real variable

- if $\lim_{n \to \infty} \underline{\underline{S_n(x)}} = S(x)$, then the series (1) converges; otherwise diverges.

- always converges at $x=0$.
- $\forall x$ in $]-R,R[$, the power series converges.

→ R is called the <u>radius of convergence</u>.

→ $]-R,R[$ is called the <u>interval convergence</u>.

- $R=\infty$ when the power series converges $\forall$ real x.

<u>Example:</u>

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ ($\forall$ real x).
- $\sin x = \frac{x}{\cancel{1!}} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots$

($\forall$ real x)

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots$

  ($\forall$ real x).

- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$

  ($-1 < x \leq 1$).

- $(1+x)^m = \begin{cases} 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + x^m, & (\forall \text{ real } x) \text{ if } m \text{ is a +ve integer.} \\ R_n \to 0 \text{ as } n \to \infty. & \\ 1 + mx + \binom{m}{2}x^2 + \cdots & (-1 < x \leq 1) \text{ if } m \text{ is not a +ve integer.} \end{cases}$

Example: $f(x) = e^x$.

Maclaurin's infinite series of $f(x) = e^x$ is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

holds if $\boxed{R_n = \frac{x^n}{n!} f^n(\theta x) \to 0 \text{ as } n \to \infty.}$

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{x^n}{n!} e^{\theta x}, \quad 0 < \theta < 1.$$

$\boxed{f^n(x) = e^x}$

Claim: $\lim_{n \to \infty} \frac{x^n}{n!} = 0$

let $a_n = \frac{x^n}{n!}$, $\quad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} = \frac{x}{n+1}$

$\to 0$ as $n \to \infty$ for all real x.

Theorem

$$\boxed{\text{if } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l, \quad 0 \le l < 1, \qquad \text{then } \lim_{n\to\infty} a_n = \cancel{l}\; 0}$$

$$\cancel{\lim_{n+1\to\infty} \frac{a_{n+1}}{a_{n+1}}} =$$

$$a_1, a_2, \cdots, \underline{a_n, a_{n+1},} \cdots$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{x^n}{n!} e^{\theta x}, \quad 0<\theta<1$$

$$= 0 \quad \forall \text{ real } x.$$

- $\left[\begin{array}{l} y = f(x), \quad \underline{\underline{f', f'' \rightarrow \text{exist}}} \\ \qquad\qquad\qquad\qquad\qquad \text{roots give} \\ y' = f'(x) = 0 \longrightarrow \text{critical pts.} \\ y'' = \underline{\underline{f''(x) \neq 0}} . \begin{array}{l} \nearrow \text{max if } f''(x) < \\ \searrow \text{min if } f''(x) > \end{array} \end{array}\right.$

- $f(x) = |x|$

$\rightarrow$ min. at $x = 0$

although f' does not exist at $x = 0$.

- f defined on an interval I, c be an interior pt. of I.

- If at c,

i) $f'(c) = f''(c) = \cdots = f^{n-1}(c) = 0.$

ii) $\underline{f^n(c) \text{ exists}}$, but $\neq 0$.

then at $x = c$, $f(x)$ has

a) no extrema if n is odd +ve integer.

b) an extreme value if n is an even +ve integer.

- max if $f^n(c) < 0$
- min if $f^n(c) > 0$.

Example:

$$f(x) = x^3, \quad \underline{x=0}$$

$f'(x) = 3x^2 = 0$ at $x=0$.

$f''(x) = 6x = 0$ at $x=0$.

$f'''(x) = 6 \neq 0 \rightarrow$ no extrema at $x=0$.

odd order.