Mathematics-I

INDETERMINATE FORM: L'HOSPITAL RULE

Dr. Ratna Dutta

Department of Mathematics

Indian Institute of Technology

Kharagpur 721302

Indeterminate Forms: L' Hospital's Rule.

A> Form $\frac{0}{0}$

- $f(x) \to 0$ & $g(x) \to 0$ as $x \to a$.
- To evaluate $\lim_{x \to a} \frac{f(x)}{g(x)}$.

L' Hospital's Rule for $\frac{0}{0}$ form

- $f, g \to$ two fun^{ns}. s.t.

i) $\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0$

ii) f', g' exists & $g'(x) \neq 0$ ~~for~~ $\forall x \in (a-\delta, a+\delta)$, $\delta > 0$, except possibly at a, &

iii) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists

then $\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$

Example: $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x - \sin x} \left(\dfrac{0}{0}\right)$

$= 2$

$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1 ?$

Example: Find the values of a and b in order that

$$\lim_{x \to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3}$$

may be equal to 1.

Soln. $\lim\limits_{x \to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3} \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x \to 0} \dfrac{(1 + a\cos x) + x(-a\sin x) - b\cos x}{3x^2}$

$\left(\dfrac{0}{0}\right)$ if $1 + a - b = 0$ —— (1).

Ans. $\boxed{a = -\dfrac{5}{2},\ b = -\dfrac{3}{2}}$

$$= \lim_{x \to 0} \frac{-a\sin x - a\sin x + x a\cos x + b\sin x}{6x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-2a\cos x + a\cos x - x a\sin x + b\cos x}{6}$$

$$= 1$$

$$-3a + b - 6 = 0 \quad \text{——— } (2).$$

B) **Form** $\left(\frac{\infty}{\infty}\right)$.

Example: $\displaystyle \lim_{x \to \infty} \frac{x^n}{e^x} \quad \left(\frac{\infty}{\infty}\right)$.

$$= \lim_{x \to \infty} \frac{n x^{n-1}}{e^x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \cdots$$

$$= \lim_{x \to \infty} \frac{n!}{e^x} = 0.$$

c> <u>Other Indeterminate forms.</u>

i) $0 \times \infty \Rightarrow$ reducible to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$f(x)\, g(x)$

ii) $\infty - \infty \Rightarrow$ reducible to $0 \times \infty$

$f(x) - g(x) \quad (\infty - \infty)$

$= f(x) g(x) \left[\frac{1}{g(x)} - \frac{1}{f(x)} \right]. \quad (\infty \times 0)$

iii) $0^0, \quad \infty^0, \quad 1^{\pm\infty}$

$(f(x))^{g(x)} \longrightarrow 0^0$

↓ take log

$g(x) \log(f(x)) \quad (0 \times \infty)$

Example:

$$\lim_{x \to 0} \cot x \cdot \log \frac{1+x}{1-x} \quad (\infty \times 0)$$

$$= 2$$

Example:

$$\lim_{x \to 0} \left(\sin x\right)^{2\tan x} = 1.$$

Example:

let $$L = \lim_{x \to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4},$$

$a > 0$.

If L is finite, then prove that

$L = 1/64$.

Sol^n.

$\underline{\underline{a = 2}}$

Example: Evaluate the limit

$$\lim_{x \to \infty} \left[x - \sqrt[n]{(x-a_1)(x-a_2) \cdots (x-a_n)} \right]$$

Sol^n: $y = \frac{1}{x}$.

$$\lim_{y \to 0} \left[\frac{1}{y} - \frac{\sqrt[n]{(1-a_1 y)(1-a_2 y) \cdots (1-a_n y)}}{y} \right]$$

$$= \lim_{y \to 0} \frac{1 - f(y)}{y}, \quad \left(\frac{0}{0}\right).$$

$$f(y) = \sqrt[n]{(1-a_1 y) \cdots (1-a_n y)}$$

$$= \lim_{y \to 0} \frac{-f'(y)}{1} = -f'(0)$$

find $f'(0)$

$$= \frac{a_1 + a_2 + \cdots + a_n}{n}$$

Example: If $\lim_{x \to 0} \left[1 + x \ln(1+b^2)\right]^{\frac{1}{x}}$

$= 2b\sin^2\theta, \quad b > 0$

and $\theta \in (-\pi, \pi)$, then prove that the value of θ is $\pm \frac{\pi}{2}$.

Sol^n: Let $u = \left[1 + x \ln(1+b^2)\right]^{\frac{1}{x}}$.

$$\ln u = \frac{1}{x} \ln\left[1 + x\ln(1+b^2)\right].$$

$$\lim_{x \to 0} \ln u = \lim_{x \to 0} \frac{\ln\left[1 + x\ln(1+b^2)\right]}{x} \qquad \left(\frac{0}{0}\right)$$

i.e. $$\ln \lim_{x \to 0} u = \lim_{x \to 0} \frac{1 \times \ln(1+b^2)}{1 + x\ln(1+b^2)}.$$

$$= \ln(1+b^2).$$

$$\Rightarrow \lim_{x \to 0} v = \underline{1 + b^2 = 2b\sin^2\theta}$$

$$\Rightarrow \sin^2\theta = \frac{1+b^2}{2b} \geq 1$$

$1+b^2-2b = (1-b)^2 \geq 0.$

$1+b^2 \geq 2b$

But $\sin^2\theta \leq 1$

$$\Rightarrow \sin^2\theta = 1$$

$$\Rightarrow \sin\theta = \pm 1 \Rightarrow \theta = \pm\frac{\pi}{2}.$$

as $\theta \in (-\pi, \pi)$.

<u>Exercise</u>:

i) $\lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{x\tan x}\right] = \frac{1}{3}.$

ii) $\lim_{x\to 0}\left[(\sin x)^{\frac{1}{x}} + \left(\frac{1}{x}\right)^{\sin x}\right] = 1.$

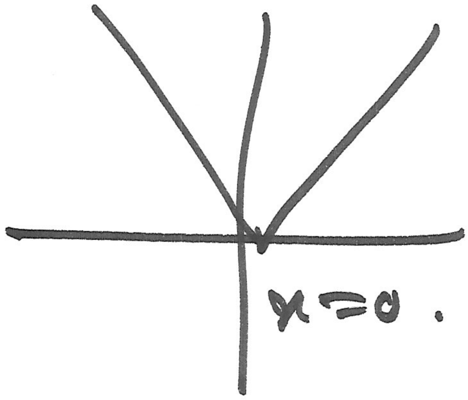
iii) $p(x) \to$ a poly. of deg. 4 having extremum at $x = 1, 2$ &

$$\lim_{x\to 0}\left[1 + \frac{p(x)}{x^2}\right] = 2.$$

Prove that $P(2) = 0$

Example:

$$f(x) = |x|.$$



At $\underline{x=0}$

$$f(x) = x^4.$$

Example:

$$\Rightarrow f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x} + \sin\frac{1}{x}.$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1/x}{1/x + \sin\frac{1}{x}} = \lim_{x\to 0} \frac{1}{1+x\sin\frac{1}{x}}$$

$$= 1.$$

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{1}{1+\cos\frac{1}{x}}$$

does not exist.

Example:

$$\lim_{x \to \infty} \frac{e^{-2x}(\cos x + 2\sin x)}{e^{-x}(\cos x + \sin x)} \quad \begin{matrix} = f(x) \\ = g(x). \end{matrix}$$

does not exist.

l'Hospital's rules.

i), ii) are satisfied.

Condition (iii) is violated.