Mathematics-I

MEAN VALUE THEOREM (MVT)

(a) Lagrange's Form.
(b) h~θ Form.
(c) Cauchy's Form.

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Mean - Value Theorem (Lagrange's form).

- If a fu^n. f is

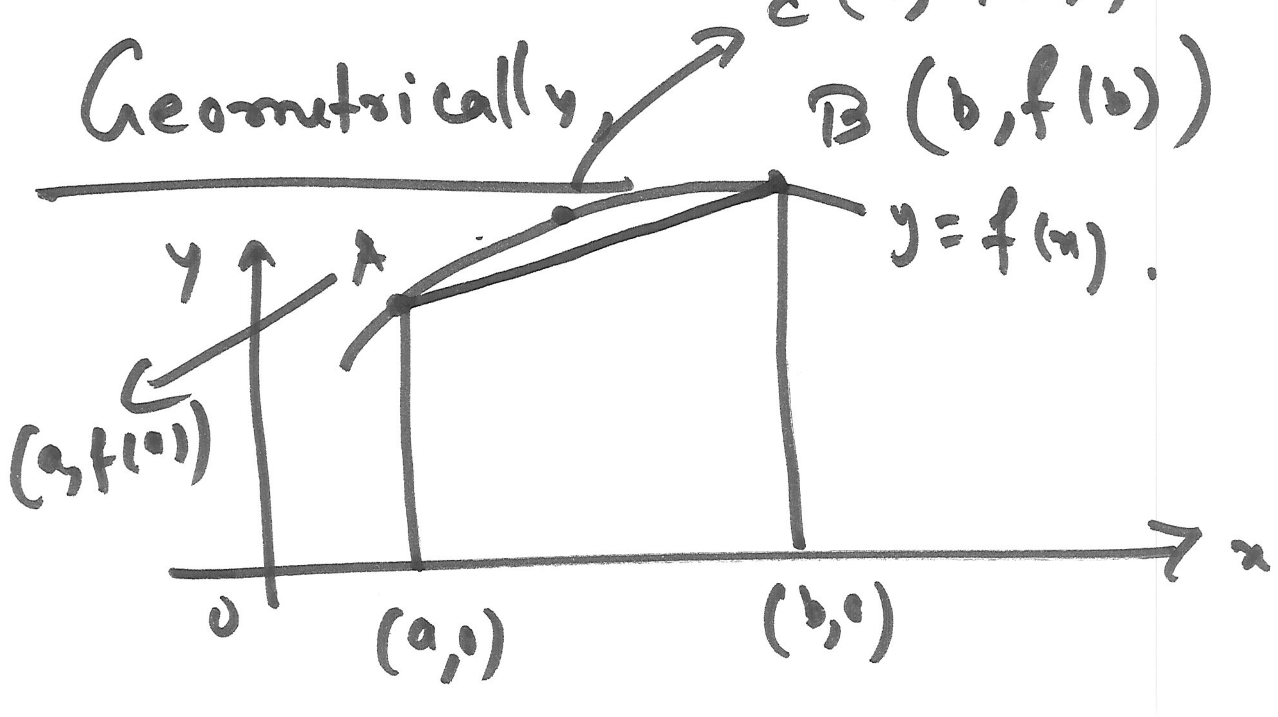
(i) Continuous in $[a, b]$, and

(ii) derivable in (a, b),

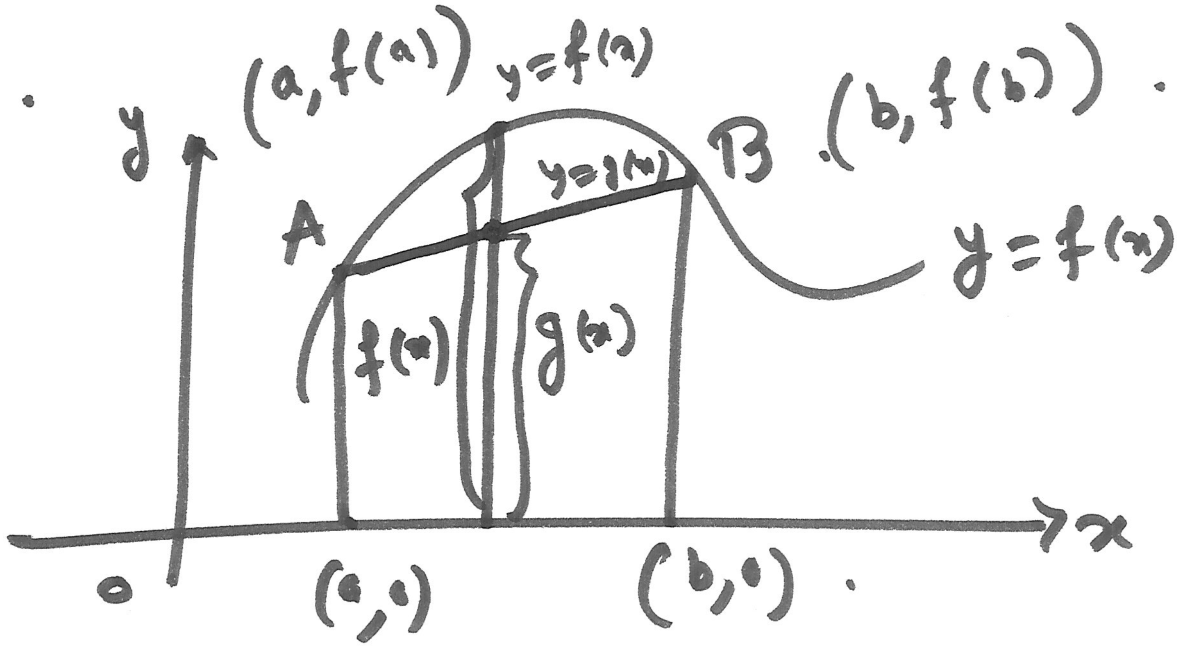
then $\exists$ <u>at least</u> one value of x, say c, $a < c < b$ s.t.

$$\boxed{\frac{f(b) - f(a)}{b - a} = f'(c)}$$

Geometrically,



proof.



AB. → $y = g(x)$.

↳ eqn.

$$\frac{g(x) - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

or $$g(x) = f(a) + \frac{f(b) - f(a)}{(b - a)}(x - a)$$

$$h(x) = f(x) - g(x).$$

$$= f(x) - f(a) - \frac{f(b) - f(a)}{(b - a)}(x - a)$$

→ Continuous in $[a, b]$

→ derivable in (a, b)

→ $h(a) = 0 = h(b)$

∴ By Rolle's Th., $\exists$ at least one $c \in (a,b)$ s.t.

~~$f'(c) = 0$.~~ $h'(c) = 0$.

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b-a} = 0$$

$$\Rightarrow \boxed{\frac{f(b) - f(a)}{b-a} = f'(c)},$$

$a < c < b$.

Example: $f(x) = \begin{cases} x \sin\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0. \end{cases}$

Applicability of MVTh? over $[-1, 1]$.

Sol^n. <u>Continuity</u> $x=0$

$$\lim_{x\to 0+} f(x) = 0$$

$$\lim_{x\to 0-} f(x) = 0$$

<u>Differentiability</u> $x=0$.

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{x\sin\frac{1}{x} - 0}{x}$$

$$= \lim_{x\to 0+} \sin\frac{1}{x}$$

does not exist.

$\therefore$ MVTh. can not be applied.

- However, the conclusion of MVTh. may/may not be true.
- Conditions of MVTh. $\rightarrow$ only sufficient
  $\rightarrow$ by no means necessary

Example:

$$f(x) = \frac{1}{|x|}, \quad [a, b],$$

when $a = -1$, $b \geq 1$

$\Rightarrow$ not defined at $x = 0$.

$f(0) =$ finite value.

- Condition of MVTh. are not satisfied
- So conclusion may / may not be true in $[-1, b]$.
- However, the conclusion of the MVTh. holds iff $b > 1 + \sqrt{2}$.

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proof.

$f(x) = \frac{1}{|x|}$, $[-1, b]$, $b \geq 1$.

$$\frac{f(b) - f(-1)}{b - (-1)} = \frac{\frac{1}{b} - 1}{b + 1}$$

$$= \frac{1 - b}{b(b+1)}.$$

To find c s.t.

$$\frac{f(b) - f(-1)}{b - (-1)} = f'(c), \quad -1 < c < b.$$

i.e. $$\frac{1 - b}{b(b+1)} = \begin{cases} \frac{1}{c^2} & \text{if } c < 0. \\ -\frac{1}{c^2} & \text{if } c > 0. \end{cases} \to \times$$

$$f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ -\frac{1}{x}, & x < 0. \\ \text{finite value}, & x = 0. \end{cases}$$

$$\frac{b-1}{b(b+1)} = \frac{1}{c^2} \qquad \oplus \; \underline{\underline{c < b.}}$$

$$\frac{b(b+1)}{b-1} = c^2 < b^2$$

$$b+1 < b(b-1) \qquad \frac{b \neq 0}{b \geq 1.}$$

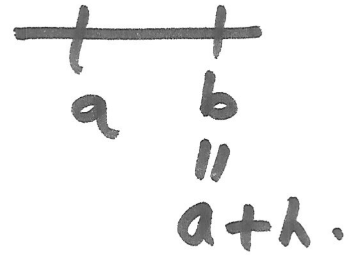
$$b^2 - 2b - 1 > 0$$

$$\text{or } (b-1)^2 > 2 \Rightarrow |b-1| > \sqrt{2}$$

$$\Rightarrow b > 1 + \sqrt{2} \text{ or } b < 1 - \sqrt{2}$$

Ans. $b > 1 + \sqrt{2}$.

h-θ form of MVTh.



- $b = a + h$
- a number lying between a & $b = a + h$ can be written as $a + \theta h$, $0 < \theta < 1$
- The MVTh. takes the form:

i) $f \rightarrow$ continuous in $[a, a+h]$

ii) $f \rightarrow$ derivable in $(a, a+h)$.

Then $\exists$ at least one no. θ, $0 < \theta < 1$ s.t.

$$f(a+h) = f(a) ~~+ h f'(\theta h)~~ + h f'(a + \theta h)$$

Maclaurin's form.

- put $h = x$, $a = 0$.

$$f(x) = f(0) + x f'(\theta h)$$

$0 < \theta < 1$

Example:

$$f(x) = px^2 + qx + r,$$

$$[a, a+h].$$

$\theta = ?$

$$f(a+h) = f(a) + hf'(a + \theta h).$$

Ans. Whatever p, q, r, h, a may be

$$\theta = \frac{1}{2}$$

Example: $\sqrt[3]{28} = ?$

Sol^n. Let $f(x) = \sqrt[3]{x}$ in $[27, 28]$ (27 = a, 28 = a+h)

→ Continuous in $[27, 28]$

→ Derivable in $]27, 28[$.

∴ By MVTh.,

$$f(28) = f(27) + (28-27) f'(c).$$

$27 < c < 28.$

$$\sqrt[3]{28} = 3 + \frac{1}{3c^{2/3}}$$

$$< 3 + \frac{1}{3 \cdot (27)^{2/3}}.$$

$$= 3 + \frac{1}{27}.$$

Deductions from MVTh.

1. Functions with zero derivatives are consts.

Let $f'(x) = 0$ when $a \le x \le b$

Then $f(x) = f(a)$ $\forall x \in [a, b]$.

proof. $a < x_1 \le b$. $[a, x_1]$

$f'(x)$ exists $\Rightarrow$
- f continuous in ~~$[x_1, b]$~~ $[a, x_1]$
- f derivable in ~~(x_1, b)~~ (a, x_1).

$\therefore$ By MVTh., $\exists$ at least one c,

$c \in (a, x_1)$ ~~$x_1 < c < b$~~ s.t.

$$\frac{f(b) - f(x_1)}{b - x_1} = f'(c) = 0$$

$\Rightarrow f(x_1)$

$c \in (a, x_1)$ s.t.

$$\frac{f(x_1) - f(a)}{x_1 - a} = f'(c) = 0$$

by given condition

$\Rightarrow f(x_1) = f(a)$ when

$x_1 \rightarrow$ any arb. pt. in $(a, b]$

2. Functions with the same derivative differ by a const.

proof: let $\frac{d}{dx}[f(x)] = \frac{d}{dx}[g(x)]$

$\forall x \in [a, b]$

$\Downarrow$

$$\frac{d}{dx}[f(x) - g(x)] = 0.$$

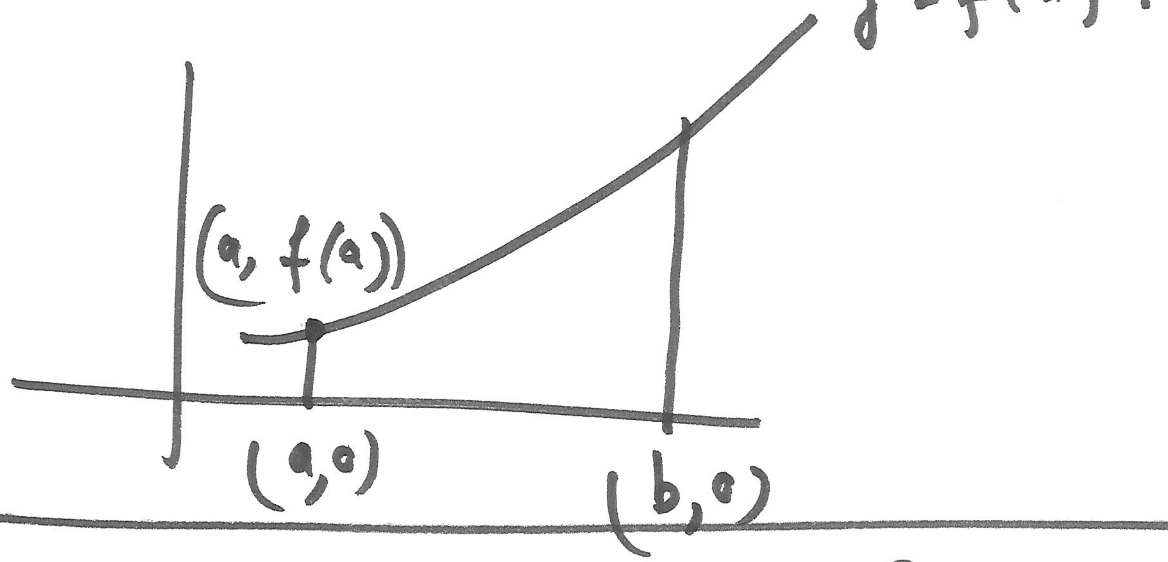
use the previous deduction.

$f(x) - g(x) = \text{const.}$

$\forall x \in [a, b]$.

3. If f is continuous in $[a,b]$, $f'(x) > 0$, then f is strictly (monotone) increasing fn^n.

Def^n. $f \uparrow$ over $[a,b]$.



- $f(x_1) < f(x_2)$ holds for every pair of pts. x_1, x_2 with $x_1 < x_2$ in $[a,b]$.

proof. Given $f \rightarrow$ continuous in $[a,b]$

$f'(x) > 0 \;\forall x \in [a,b]$

let $a \leq x_1 < x_2 \leq b$.

- f continuous in $[x_1, x_2]$
- derivable in $]x_1, x_2[$

By MVTh.,

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c),$$

for at least one $c \in]x_1, x_2[$

As $f'(c) > 0$ & $x_2 > x_1$

$$\Rightarrow f(x_2) - f(x_1) > 0$$

$$\Rightarrow f \uparrow \text{ in } [a,b].$$

- if $f'(x) < 0$ in $[a,b]$, f continuous in $[a,b]$

then $f \downarrow$ in $[a,b]$.

Example: Show that

$$\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$$

$0 < u < v$.

& hence deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Sol^n: Let $f(x) = \tan^{-1}x$, $0 < u < x < v$.

$$f'(x) = \frac{1}{1+x^2}.$$

By MVTh.,

$$\frac{f(v) - f(u)}{v-u} = f'(c)$$

for at least one c, $u < c < v$

$$\frac{\tan^{-1}v - \tan^{-1}u}{v-u} = \frac{1}{1+c^2} \quad \text{---} (1)$$

As $u < c < v$.

$$1+u^2 < 1+c^2 < 1+v^2$$

$$\frac{1}{1+v^2} < \frac{1}{1+c^2} < \frac{1}{1+u^2}$$

$$\frac{1}{1+v^2} < \frac{\tan^{-1} v - \tan^{-1} u}{v-u} < \frac{1}{1+u^2}$$

yielding the result.

Deduction $u = 1$, $v = \frac{4}{3}$.

Example: Show that

$$\frac{x}{1+x} < \log(1+x) < x, \quad \forall x > 0.$$

Sol^n. $f(x) = \log(1+x)$, $[0, x]$

↓ Apply MVTh.

$$f(x) = f(0) + x f'(\theta x), \quad 0 < \theta < 1$$

($f(0) = 0$)

i.e. $$\log(1+x) = \frac{x}{1+\theta x} \quad — (1)$$

Now $0 < \theta < 1 \Rightarrow 1 < 1+\theta x < 1+x$

$$\Rightarrow \frac{x}{1+x} < \frac{x}{1+\theta x} < x$$

$$\Rightarrow \frac{x}{1+x} < \log(1+x) < x$$

using ①

$$f(x) = \log(1+x)$$

$$f'(x) = \frac{1}{1+x}$$

$$f'(\theta x) = \frac{1}{1+\theta x}.$$

Example:

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)} \quad \forall x > 0.$$

Sol^n: Let $f(x) = \log(1+x) - x + \frac{x^2}{2}$,

$[0, x]$

$$\therefore f'(x) = \frac{1}{1+x} - (1 - x).$$

$$= \frac{1 - (1 - x^2)}{1+x}$$

$$= \frac{x^2}{1+x} > 0 \quad \forall x > 0$$

$\Rightarrow f \uparrow$ in $[0, x]$.

Also $f(0) = 0$

$\underline{f(x)} > f(0) = 0 \quad \forall x > 0.$

$\Rightarrow \log(1+x) > x - \frac{x^2}{2}$

Complete it

Exercise. Prove that $\left(1+\frac{1}{x}\right)^x$ & $\left(1-\frac{1}{x}\right)^x$ are both increasing fu^n. using MVTh.

4. Approximating fu^n. → use of MVTh.

$\sqrt[3]{28}$.

Exercise.

(Numerical approximation)

Prove that $\sin 46^\circ$ is approximately equal to $\frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$.

Mean Value Theorem : Cauchy's form

- If two fu^ns. f and g
  - i) are both continuous in $[a, b]$
  - ii) are both derivable in $]a, b[$
  - iii) $g'(x)$ does not vanish at any value of x in $a < x < b$.

then $\exists$ at least one value, say c, $a < c < b$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

proof. Consider

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \left(g(x) - g(a) \right)$$

↓ apply the Rolle's Th. on $[a, b]$.

- $g(x) = x \rightarrow$ Lagrange's MVTh.
- h-θ form of Cauchy's MVTh.

- replace b by $a+h$
- replace c by $a+\theta h$, $0 < \theta < 1$

Exercise.

→ $f(x) = e^{x}$, $g(x) = e^{-x}$

→ c is AM. bet. a, b.

→ $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$

→ c → G.M. bet. a, b

$$c = \sqrt{ab}$$

→ $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$

→ c = harmonic mean of a, b.

$$c = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

Example:

Show that the equⁿ.

$$3^x + 4^x = 5^x$$

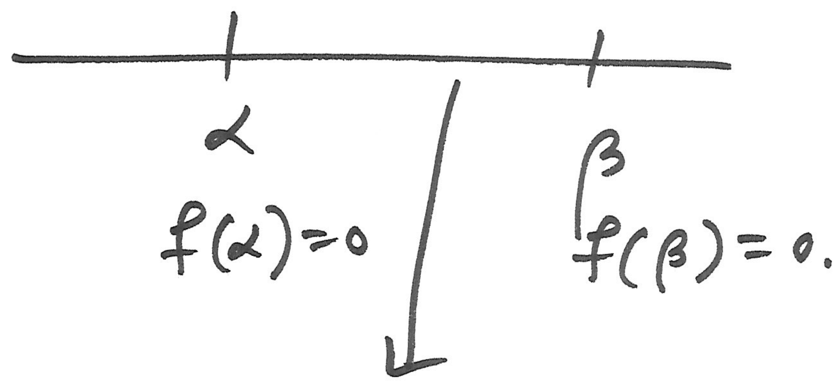
has **exactly** one real root.

Solⁿ.

$$f(x) = \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x - 1$$

$x = 2$ $\quad f(2) = 0$

if f has more than one real root, then by Rolle's Th., f' must have one root in between them.



$f'(c) = 0$ by Rolle's Th.

$$f'(x) = \left(\frac{3}{5}\right)^x \log\left(\frac{3}{5}\right) + \left(\frac{4}{5}\right)^x \log\left(\frac{4}{5}\right)$$

$$< 0 \quad \forall \text{ real } x.$$

$\Rightarrow f(x) = 0$ has exactly one root.

Proof of Rolle's Th.

f <u>continuous</u> in $[a,b]$
$\Rightarrow$ f is bdd. & the lub M & glb m must be <u>attained</u> in $[a,b]$.

$f(x)$ → continuous $[a,b]$
→ derivable (a,b)
→ $f(a)=f(b)$

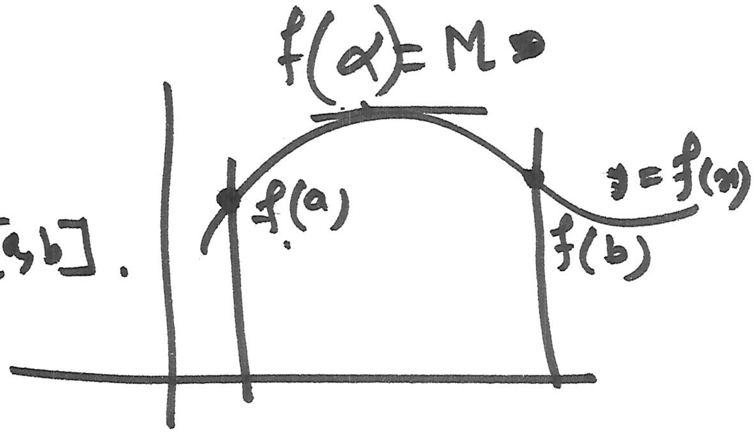
$\Rightarrow \exists$ at least one $c \in (a,b)$ s.t. $f'(c)=0$.

$$f(\beta) = \underset{(glb)}{m} \le f(x) \le \underset{(lub)}{M} = f(\alpha)$$

$$a \le \alpha, \beta \le b.$$

<u>Case 1</u> $m = M$.

$f(x) = M \ \forall x \in [a,b]$.



<u>Case 2</u> <u>$m \ne M$</u>.

As $f(a) = f(b)$, at least one of m or $M \ne f(a) = f(b)$

Let $M \neq f(a)\ (= f(b))$.

$M = f(\alpha) \neq f(a) \Rightarrow \alpha \neq a$.

$M = f(\alpha) \neq f(b) \Rightarrow \alpha \neq b$.

$\Rightarrow \quad a < \alpha < b$.

- $f'(x)$ exists $\Rightarrow f'(\alpha)$ exists.

<u>Claim</u> $\quad f'(\alpha) = 0$.

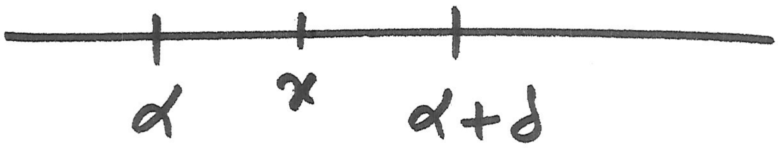
↳ <u>proof of claim</u>

if not then

i) either $f'(\alpha)$ is finite +ve no. or $+\infty$

ii) or $f'(\alpha)$ is finite −ve no. or $-\infty$

$$\rightarrow f'(\alpha) = \lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} > 0.$$



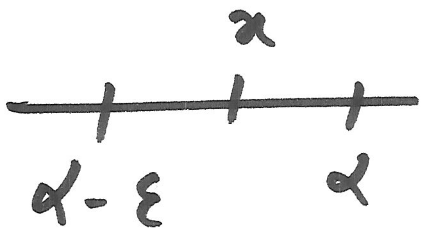
$\Rightarrow \exists$ an open interval $(\alpha, \alpha+\delta)$ at every pt. of which

$$f(x) > f(\alpha) = M$$

$(\rightarrow\leftarrow)$ to the fact that M is lub.

ii) $f'(\alpha) < 0$

Complete it.



Extra Slot for clearing doubts.

- Thursday - 5:30 pm - 7:00 pm.
- Maths. Dept. Class room.

Example:

Show that if $0 < p < q$, then

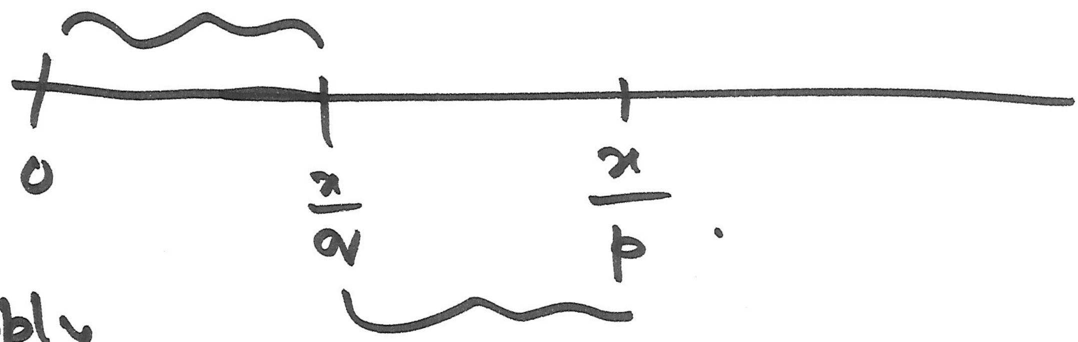
$$\left(1+\frac{x}{p}\right)^p < \left(1+\frac{x}{q}\right)^q \text{ for } x>0.$$

using MVTh.

Soln. let $f(x) = \log(1+x)$.

$$f'(x) = \frac{1}{1+x}$$

$$0<p<q, \quad x>0 \Rightarrow \frac{x}{p} > \frac{x}{q} > 0$$



$$\left(0, \frac{x}{q}\right) \xrightarrow[\text{on } f]{\text{apply MVTh.}}$$

$$\log\frac{x}{q} -$$

$$\frac{\log\left(1+\frac{x}{q}\right)-\log 1}{\frac{x}{q}-0} = f'(a_0) = \frac{1}{1+a_0},$$

$$\boxed{0 < a_0 < \frac{x}{q}} \quad \text{——— (1)}.$$

$\left(\frac{x}{q}, \frac{x}{p}\right) \xrightarrow[\text{MVTh. on } f]{\text{apply}}$

$$\frac{\log\left(1+\frac{x}{p}\right)-\log\left(1+\frac{x}{q}\right)}{\frac{x}{p}-\frac{x}{q}} = f'(a_1) = \frac{1}{1+a_1},$$

$$\boxed{\frac{x}{q} < a_1 < \frac{x}{p}.} \quad \text{——— (2)}$$

$a_0 < a_1 \Rightarrow \frac{1}{1+a_0} > \frac{1}{1+a_1}$

by (1), (2), we get

$$\frac{\log\left(1+\frac{x}{q}\right)}{\frac{x}{q}} > \frac{\log\left(1+\frac{x}{p}\right)-\log\left(1+\frac{x}{q}\right)}{\frac{x}{p}-\frac{x}{q}}.$$

simplify & obtain the result.

Example: Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$ for $0 < x < \frac{\pi}{2}$, using MVTh.

Sol^n: To prove that

$$\frac{\tan x \sin x - x^2}{x \sin x} > 0 \text{ for } 0 < x < \frac{\pi}{2}.$$

Let $f(x) = \tan x \sin x - x^2$.

$$f'(x) = \tan x \cos x + \sec^2 x \sin x - 2x.$$

$$= \sin x + \sec^2 x \sin x - 2x.$$

$$f''(x) = \cos x + \cos x \sec^2 x + 2 \sec x \sec x \tan x \sin x - 2$$

$$= \underline{\cos x + \sec x} + 2 \sin x \tan x \sec^2 x \underline{- 2}$$

$$= \left(\sqrt{\cos x} - \sqrt{\sec x}\right)^2 + 2 \sin x \tan x \sec^2 x.$$

> 0 for $0 < x < \frac{\pi}{2}$

$\Rightarrow f'(x) \uparrow$ for $0 < x < \frac{\pi}{2}$

Also $f'(0) = 0$

$\therefore f'(x) > 0 \quad \forall \quad 0 < x < \frac{\pi}{2}$.

$\Rightarrow f(x) \uparrow$ in $0 < x < \frac{\pi}{2}$.

Also $f(0) = 0$

$\Rightarrow f(x) > 0$; yielding the result.

<u>Example:</u> Use <u>Cauchy's MVTh. to</u> evaluate $\lim_{x \to 1} \left[\frac{\cos \frac{\pi x}{2}}{\log\left(\frac{1}{x}\right)} \right]$.

<u>Soln.</u>

$f(x) = \cos \frac{\pi x}{2}$

$g(x) = \log \frac{1}{x}$

↓

Apply Cauchy's MVTh. on ~~[0, x]~~ $[x, 1]$.

$f, g \quad [a, b]$

$g' \neq 0$ anywhere in $[a, b]$.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$c \in (a, b)$.

$$\frac{f(1) - f(x)}{g(1) - g(x)} = \frac{f'(c)}{g'(c)}, \quad x < c < 1$$

i.e. $$\frac{0 - \cos\frac{\pi x}{2}}{0 - \log\left(\frac{1}{x}\right)} = \frac{-\frac{\pi}{2}\sin\frac{\pi c}{2}}{-\frac{1}{c}}.$$

$$\boxed{\frac{-\cos\frac{\pi x}{2}}{\log\left(\frac{1}{x}\right)} = -\frac{\pi}{2} c \sin\frac{\pi c}{2}.}$$

~~$g(x) = -\log x$.~~

$$\lim_{x \to 1} \frac{\cos\frac{\pi x}{2}}{\log\left(\frac{1}{x}\right)} = \frac{\pi}{2}\sin\frac{\pi}{2} = \boxed{\frac{\pi}{2}.}$$

Ans.

$$\frac{x < c < 1}{x \to 1}$$

$\Rightarrow c \to 1$