COMPLEX ANALYSIS-III

Mathematics-I

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(a) Power Series.

(b) Residue and Residue Theorem.

(c) Line Integral Evaluation by Residue Theorem.

Example: let $f(z) = \ln(1+z)$, where we consider the branch which has value 0 when $z=0$.

a) Expand $f(z)$ in a Taylor's series about $z=0$.

b) Determine the region of convergence for the series in (a)

c) Expand $\ln\left(\frac{1+z}{1-z}\right)$ in a Taylor's series about $z=0$.

- $f(z) \sim \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $\boxed{a_n = \frac{f^n(x)}{n!}}$

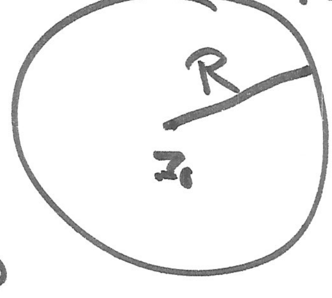
  $\hookrightarrow$ power series expansion of $f(z)$ about $z=z_0$.

- $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ or $\lim \sqrt[n]{|a_n|} = L$.

- $\boxed{\text{Radius of convergence}}$ $\longrightarrow$ $R = \frac{1}{L}$

- Circle of convergence $\longrightarrow$

$|z-z_0|=$



$\to$ Power series converges $\to |z-z_0|<R$

diverges $\to |z-z_0|>R$.

undecided $\to |z-z_0|=R$.

Example: $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(z-1-i)^k}{k!}$

$R = ?$ $\qquad a_n = \dfrac{(-1)^{n+1}}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$\Rightarrow R = \infty$

Example: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{6n+1}{2n+5}\right)^n (z-2i)^n.$

$R = \frac{1}{3}.$

Circle Region of convergence!

$|z-2i| < \frac{1}{3}.$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\lim_{n\to\infty} \sqrt[n]{a_n}.$

$|z-2i| \cancel{\frac{1}{3}}$

Example:

Expand $f(z) = \dfrac{1}{1-z}$ in Taylor's series with center $z_0 = 2i$

$\displaystyle\sum_{n=1}^{\infty} u_n \rightarrow$ conv if $L<1$, div if $L>1$, $L=1 \rightarrow$ fails.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = L.$$

$$f(z) = f(2i) + (z-2i) f'(2i) + \cdots$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{\cancel{n!}}{(1-2i)^{n+1}} \frac{(z-2i)^n}{\cancel{n!}}$$

$|z - \underline{\underline{2i}}| = \sqrt{5}$.

<u>circle of convergence</u>

$$\frac{1}{1-z}.$$

$(0,2)$, $(1,0)$.

about which singularity

$\frac{1}{1-z}$ in expansion of $\frac{1}{1-z}$.

$$\left.\begin{array}{l} \dfrac{1}{(1-z)^2} \\[2ex] \log\left(\dfrac{1}{1-z}\right). \end{array}\right\} \text{about } z_0 = 2i$$

Sol^n. Hint: $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$

$|z| < 1$

a).

Integrating

$\log(1+z) =$ $\rightarrow$ region of convergence

$\rightarrow |z| < 1.$

(b).

(c). $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$

$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots$

$\log\left(\frac{1+z}{1-z}\right) = 2z + \frac{2z^3}{3} + \frac{2z^5}{5} + \cdots$

Exercise. Prove that $1 + z + z^2 + \cdots = \frac{1}{1-z}$ if $|z| < 1$.

proof.

$$S_n = 1 + \cancel{z} + \cancel{z^2} + \cdots + \cancel{z^n}$$
$$\underset{-}{z} S_n = \underset{-}{\cancel{z}} \underset{-}{+} \cancel{z^2} + \cdots \underset{-}{+} \cancel{z^n} \underset{-}{+} z^{n+1}$$

$$(1-z) S_n = 1 - z^{n+1}$$

$$S_n = \frac{1 - z^{n+1}}{1-z}.$$

Claim $\boxed{\lim_{n\to\infty} z^n = 0}$ if $|z| < 1$.

$\forall \varepsilon > 0$, $\exists$ a +ve integer $N > 0$

s.t. $\boxed{|z^n - 0| < \varepsilon \text{ whenever } n > N.}$

use this to establish your claim.

$\{u_n = z^n\}$.

Example: Expand $f(z) = \dfrac{1}{z(z-1)}$

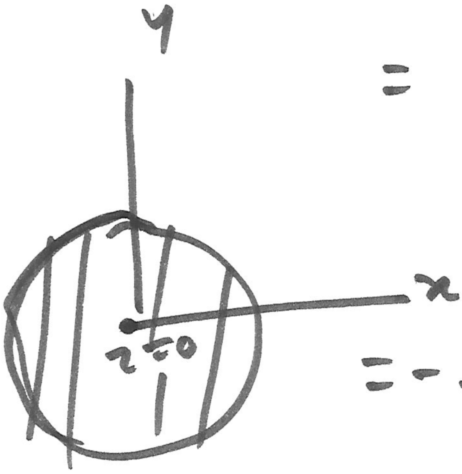
in Laurent series valid for

a) $0 < |z| < 1$, b) $1 < |z|$

c) $0 < |z-1| < 1$, d) $1 < |z-1|$

Sol. a) $0 < |z| < 1.$

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\cdot\frac{1}{1-z}$$



$$= -\frac{1}{z}\Big[\underbrace{1 + z + z^2 + \cdots}_{\text{converges for } |z|<1}\Big]$$

$$= -\frac{1}{z} - 1 - z - z^2 - \cdots\cdots$$

converges for $\underline{0 < |z| < 1.}$

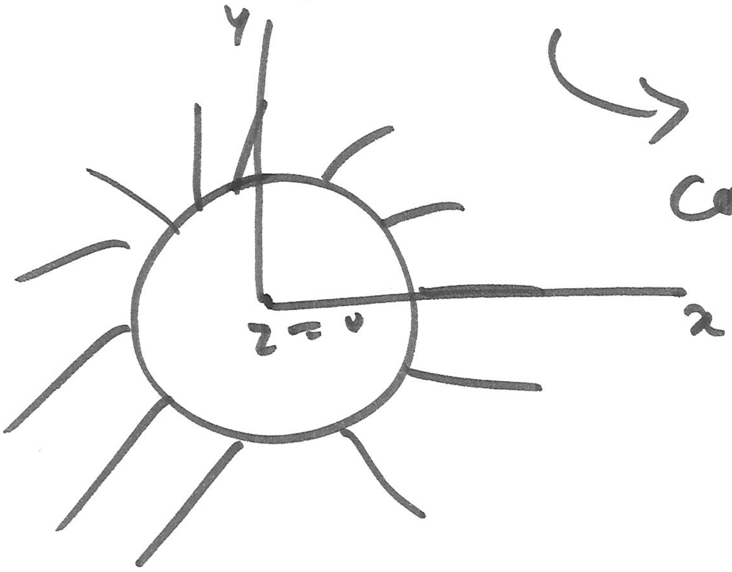
b) $1 < |z|.$

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{1 - \frac{1}{z}}.$$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots$$

$$|z| < 1.$$

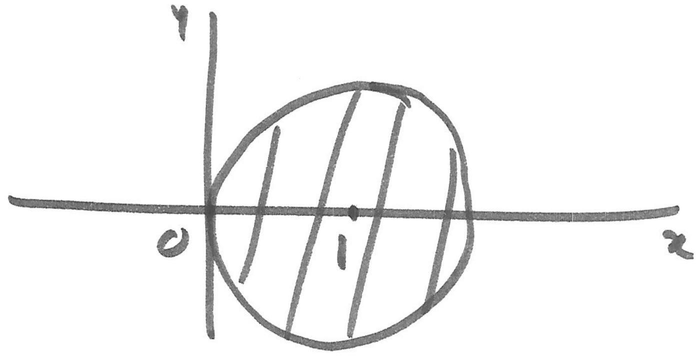
$$= \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$



converges for $\left|\frac{1}{z}\right| < 1.$

i.e. $|z| > 1.$

c) $0 < |z-1| < 1$.



$$f(z) = \frac{1}{z(z-1)}$$

$$= \frac{1}{(z-1)\left(1+\overline{z-1}\right)}.$$

$$= \frac{1}{(z-1)}\Big[1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots\Big]$$

converges for $|z-1| < 1$.

$$= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots$$

converges for $0 < |z-1| < 1$.

d) Exercise.

Exercise. $f(z) = \dfrac{1}{(z-1)^3 (z-3)}$ find Laurent's series expansion

a) $0 < |z-1| < 2$

b) $0 < |z-3| < 2$.

Residue & Residue Theorem.

- $f(z) \rightarrow$ isolated singularity at z_0.
- $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

(Laurent's series expansion).

which converges $\forall z$ near z_0.

- $$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

- $n = -1$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\,dz$$

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}$$

$n = 0, \pm 1, \pm 2, \ldots$

Residue of $f(z)$ at $z = z_0$ $\Rightarrow$

$$\oint_C f(z)\,dz = 2\pi i\, a_{-1}.$$

find it by Laurent series expansion.

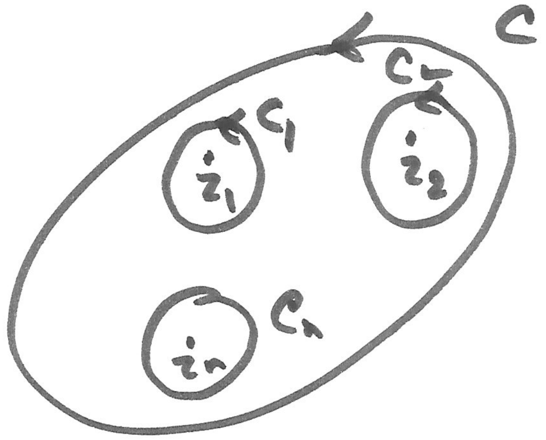
$\rightarrow$ isolated singularities.

Cauchy's Residue Theorem.

$D \rightarrow$ simply connected domain.

$C \rightarrow$ closed contour lying entirely within D.

$f(z) \rightarrow$ analytic within & on C, except a finite no. of pts. $z_1, z_2, \ldots, z_n$ within C.



Then

$$\oint_C f(z)\,dz = \sum_{K=1}^{n} \oint_{C_K} f(z)\,dz$$

$$= \sum_{K=1}^{n} 2\pi i \ \mathrm{Res}\,(f(z), z_K).$$

$$\boxed{a_{-1} = \mathrm{Res}\,(f(z), z_K)}$$

Residue at a simple pole

- f has a <u>simple pole</u> at $z = z_0$.

Then $$\boxed{\operatorname{Res}\left(f(z), z_0\right) = \lim_{z \to z_0} (z - z_0) f(z)}$$

proof:

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

$$(z - z_0) f(z) = a_{-1} + a_0 (z - z_0) + a_1 (z - z_0)^2 + \cdots$$

$$\longrightarrow a_{-1} \quad \text{as } z \to z_0.$$

$$a_{-1} \to \operatorname{Res}\left(f(z), z_0\right)$$

Residue at a pole of order n.

- f has a pole of order n at $z = z_0$.

Then $$\boxed{\operatorname{Res}\left(f(z), z_0\right) \cancel{= \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]}$$

__proof:__

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}$$
$$+ a_0 + a_1(z-z_0) + \cdots$$

$$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1}$$
$$+ a_0 (z-z_0)^n + a_1 (z-z_0)^{n+1} + \cdots$$

$$\frac{d^{n-1}}{dz^{n-1}} \cancel{f(z)} \left[(z-z_0)^n f(z) \right]$$
$$= a_{-1}(n-1)! + a_0\, n!\,(z-z_0)$$
$$+ a_1 (n-1)!\, (z-z_0)^2 + \cdots$$

$$\longrightarrow a_{-1}(n-1)! \quad \text{as} \quad z \to z_0.$$

__Example:__ $$f(z) = \frac{1}{(z-1)^2 (z-3)}$$

$$\mathrm{Res}\,(f(z), 3) = \lim_{z\to 3} f(z)(z-3) = 1/4.$$

$$\mathrm{Res}\,(f(z), 1) = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right].$$

pole of order 2

$$= -1/4.$$

Line integral evaluation by Residue Theorem

Example: Evaluate $\oint_C \frac{dz}{(z-1)^2(z-3)}$ by the Residue Theorem

a) C : rectangle defined by
$x=0,\ x=4,\ y=-1,\ y=1.$

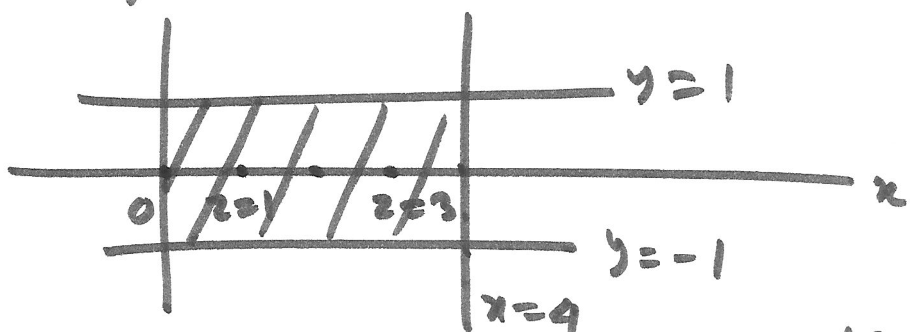
b) $C: |z|=2.$

Sol^n b). $\oint_C \frac{dz}{(z-1)^2(z-3)} = 2\pi i \, Res(f(z), 1)$

$= 2\pi i \left(-\frac{1}{4}\right)$

$= -\frac{\pi i}{2}.$

a)



$$\oint_C \frac{dz}{(z-1)^2(z-3)} = 2\pi i \left[Res(f(z),1) + Res(f(z),3) \right]$$
$$= 2\pi i \left[-\frac{1}{4} + \frac{1}{4} \right] = 0.$$

Example: $\oint_{C:|z|=1} e^{3/z}\,dz. = 2\pi i\, a_{-1}$
$= 2\pi i . 3$
$= 6\pi i$

$z=0 \rightarrow$ essential singularity.

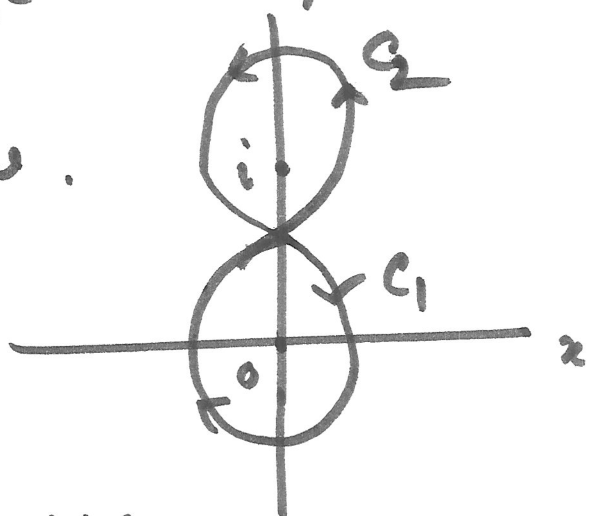
$$e^{3/z} = 1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \cdots$$

Example: $\oint_{C:|z|=2} \frac{e^z}{z^4+5z^3}\,dz = \frac{17\pi i}{125}.$

Example: (Cauchy's Integral Theorem for derivatives)

Evaluate $\oint_C \frac{z^3+3}{z(z-i)^2}\,dz$ when

C is the contour below.



Sol^n: $C \rightarrow$ not simple closed contour.

~~Think~~ Think of C as $C = C_1 \cup C_2$.

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz.$$

$$= -\oint_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz.$$

$z=0$ inside C_1 $\qquad$ $z=i$ inside C_2.

$$= -\oint_{C_1} \frac{\dfrac{z^3+3}{(z-i)^2}}{z}\,dz + \oint_{C_2} \frac{(z^3+3)/z}{(z-i)^2}\,dz$$

$$= -2\pi i\left[\frac{0^3+3}{(0-i)^2}\right] + \frac{2\pi i}{1!}\left[\frac{d}{dz}\left(\frac{z^3+3}{z}\right)\right]_{z=i}$$

$$= 4\pi(-1+3i)$$ <u>check</u>