COMPLEX ANALYSIS-II

Mathematics-I

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INDEX-II

(a) Contour Integrals.

(b) Cauchy's Integral Formulae and Related Theorems.

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(d) Classification of Isolated Singularities.

2.11.15

Contour Integrals

Example:

$$\oint \frac{dz}{z}$$

$C : |z| = 1.$

$z = x + iy$

on C, $x = r\cos\theta$

$y = r\sin\theta$

i.e. $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$dz = i e^{i\theta}\, d\theta$.

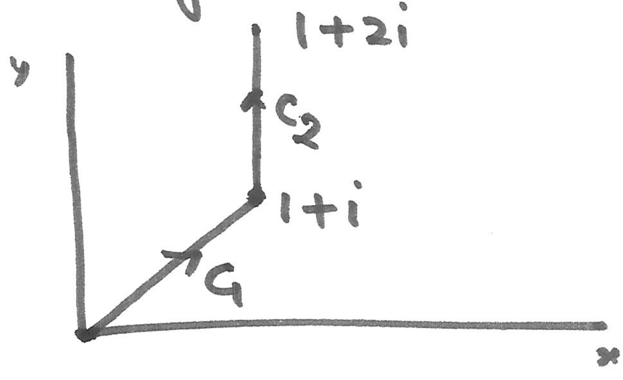
$$= \int_{\theta=0}^{2\pi} e^{-i\theta}\, i e^{i\theta}\, d\theta$$

$$= 2\pi i$$

Example:

$$\int_C (x^2 + iy^2)\, dz,$$

where C is the contour shown in the following figure.



$$= \int_{C_1} (x^2 + iy^2)\, dz + \int_{C_2} (x^2 + iy^2)\, dz$$

On C_1, $y = x$

$z = x + ix$

$dz = (1+i)\, dx$

$x \to 0$ to 1

$$\int_{C_1} (x^2 + iy^2)\, dz$$

$$= \int_0^1 (x^2 + ix^2)(1+i)\, dx$$

$$= 2i/3.$$

On C_2, $\underline{\underline{x = 1}}$, $\underline{1 \le y \le 2}$

$z = x + iy = 1 + iy$

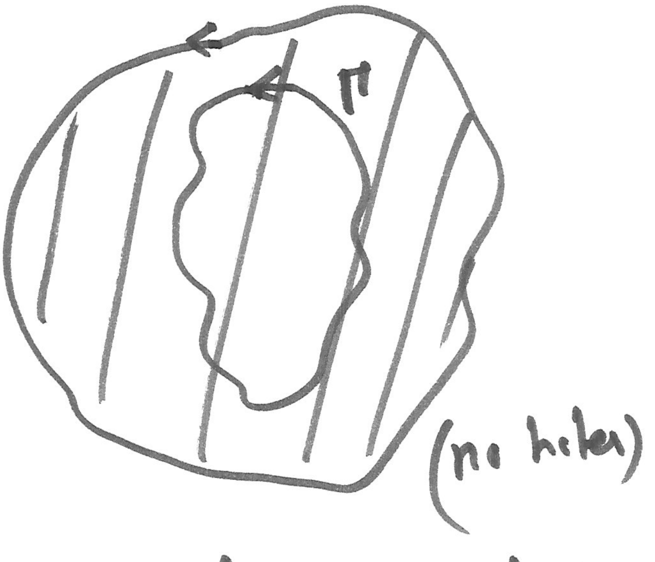
$dz = i\,dy$.

$$\int_{C_2} (x^2 + iy^2)\,dz = \int_1^2 (1 + iy^2)\, i\,dy$$

$$= i - 7/3.$$

$$\therefore \int_C (x^2 + iy^2)\,dz = \frac{2i}{3} + \left(i - \frac{7}{3}\right) = -\frac{7}{3} + \frac{5i}{3}$$

Simply & multiply connected domain

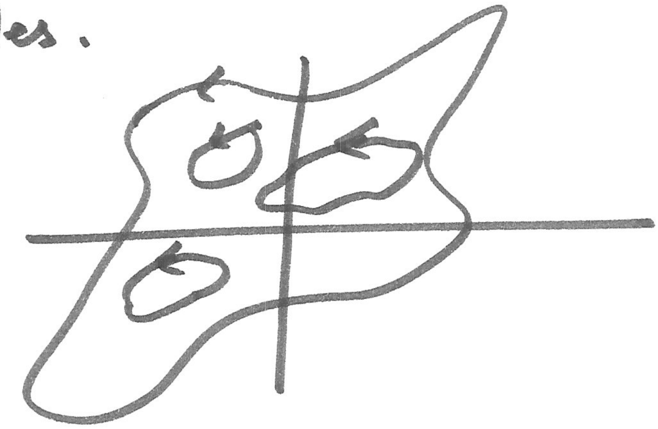


Simply connected

multiply connected.

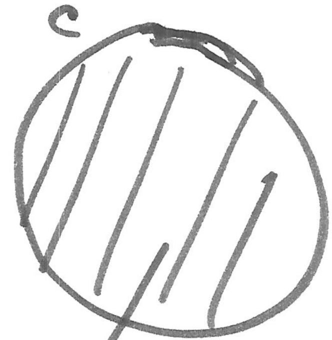
- curve Γ inside the region R
  → if shrunk to a pt. without leaving R.
  → simply connected region.

- Otherwise → multiply connected region.
  → holes.



Jordan curve Theorem.

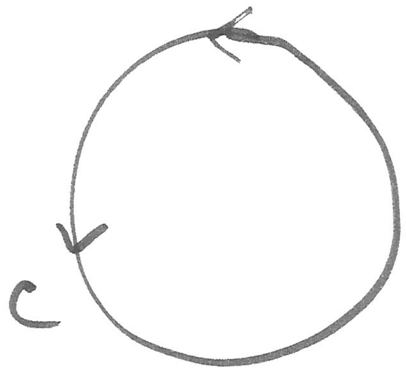
- C → <u>continuous</u> <u>closed</u> curve that <u>does not intersect itself</u>.
  (Jordan curve)



exterior
↓
unbdd region.

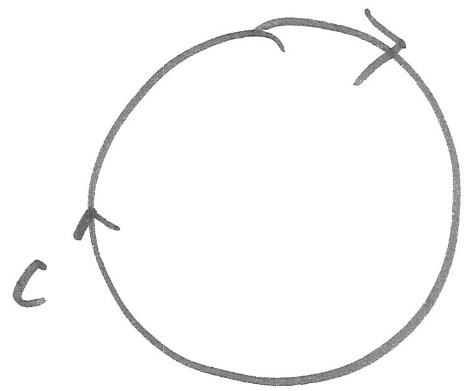
interior
↓
bdd. region
$|z| < M$.

· Convention regarding traversal of a closed path.



+ve sense

(counter clockwise)



-ve sense

(clockwise.

$\oint_C f(z)\,dz \rightarrow$ line integral of $f(z)$ around the curve C in the +ve sense.

when C is closed.

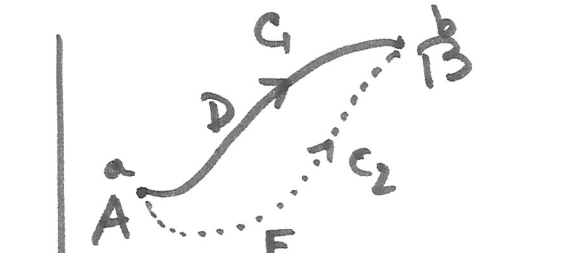
Cauchy-Goursat Theorem / Cauchy's Th. / Cauchy's Integral Theorem.

- $f(z)$ analytic in a domain D & on its boundary C.

domain: simply connected / multiply connected.

Then $\boxed{\oint_C f(z)\,dz = 0.}$

Example: $\int_a^b f(z)\,dz$. is independent of path joining a & b.

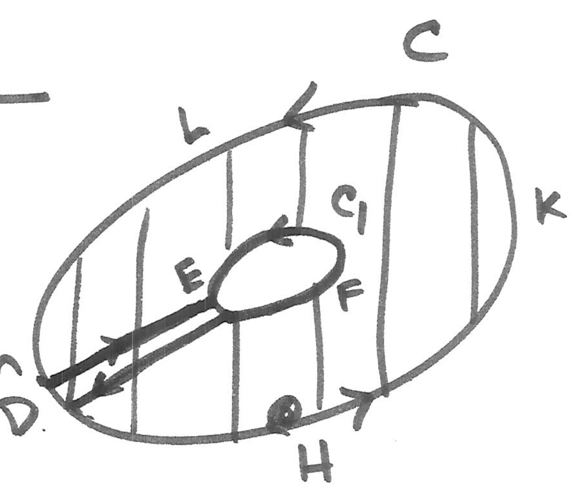


By Cauchy's Theorem,

$$\int_{ADBEA} f(z)\,dz = 0.$$

i.e. $0 = \int_{ADB} f(z)\,dz + \int_{BEA} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$

Deformation of contours



- $f(z)$ is analytic in the region between C & C_1

Then $$\boxed{\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz}$$

proof:

$$\oint_\Gamma f(z)\,dz = 0 \quad \text{by Cauchy's Theorem}$$

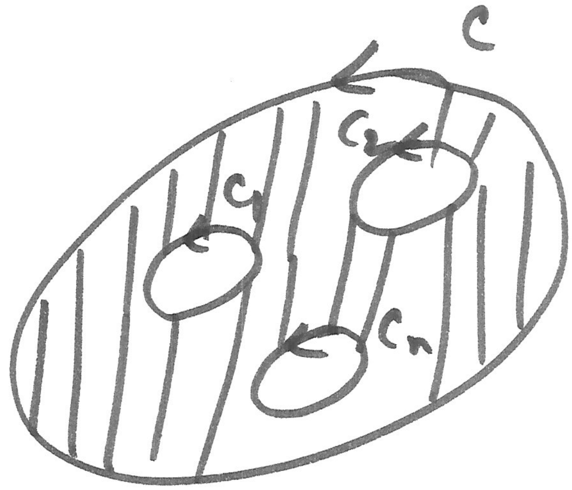
Γ: DEFEDHKLD.

$$\Rightarrow \cancel{\int_{DE}} - \oint_{C_1} + \cancel{\int_{ED}} + \oint_C f(z)\,dz = 0.$$

i.e. to integrate $f(z)$ along curve C, we can equivalently replace by any curve C_1 so long as $f(z)$ is analytic in the region between

- Value of integral over a complicated contour C $\xrightarrow{\text{same}}$ value of the integral over a more convenient contour C_1 lying inside C.

<u>Generalization</u>:



$C_1, C_2, \ldots, C_n$

→ non-overlapping

→ entirely within C.

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz,$$

provided $f(z)$ is analytic in the shadded region.

Morera's Theorem (Converse of Cauchy's Theorem).

- $f(z) \rightarrow$ continuous in a simply-connected domain D

& $\oint_C f(z)\,dz = 0$ around every simple closed curve in D.

$\Rightarrow$ $\boxed{f(z) \text{ is analytic}}$

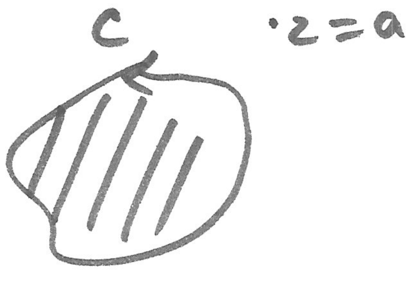
Applications of Cauchy's Theorem.

(in evaluating line integrals).

Example: Evaluate $\oint_C \frac{dz}{z-a}$ when

C is any simple closed curve & $z=a$ is (i) outside C

(ii) inside C.



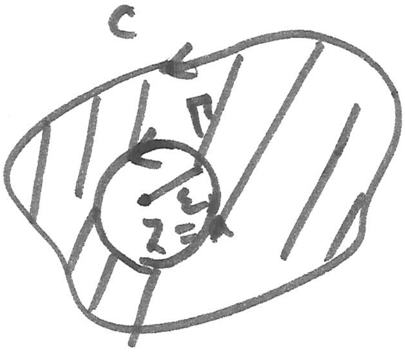
Sol^n.

(i) $z=a$ is outside C

$\Rightarrow \frac{1}{z-a}$ is analytic inside C

$\Rightarrow \oint \frac{dz}{z-a} = 0.$

(ii)



$\Downarrow$
by Cauchy's Th.

$f(z) = \frac{1}{z-a}$

$z = a$ is inside C

Γ: a circle of radius ε
with center at $z = a$
so that Γ is inside C.

$$\oint_C \frac{dz}{z-a} = \int_{\Gamma : |z-a| = \varepsilon} \frac{dz}{z-a}.$$

$|z-a| = \varepsilon$.
$\Rightarrow z = a + \varepsilon e^{i\theta}$
$dz = \varepsilon i e^{i\theta} d\theta$
$0 \le \theta \le 2\pi$.

$$= \int_0^{2\pi} \frac{\cancel{\varepsilon} i \cancel{e^{i\theta}}}{\cancel{\varepsilon} \cancel{e^{i\theta}}} d\theta$$

$= 2\pi i$

<u>Example:</u> Evaluate $\oint_C \dfrac{dz}{(z-a)^n}$, $n = 2, 3, 4, \ldots$

when $z = a$ is inside the simple closed curve C.

<u>Sol^n.</u>

$$\oint_C \frac{dz}{(z-a)^n} = \oint_{\Gamma: |z-a| = \varepsilon} \frac{dz}{(z-a)^n} \qquad \left|\ \begin{array}{l} z-a = \varepsilon e^{i\theta} \\ dz = \varepsilon i e^{i\theta} d\theta . \end{array}\right.$$

$$= \int_{\theta=0}^{2\pi} \frac{\varepsilon i e^{i\theta}\, d\theta}{\varepsilon^n e^{in\theta}}$$

$$= \varepsilon^{1-n}\, i \int_0^{2\pi} e^{(1-n)i\theta}\, d\theta .$$

$$= \frac{\varepsilon^{1-n} \cancel{i}}{(1-n)\cancel{i}} \Big[e^{(1-n)i\theta} \Big]_0^{2\pi}, \quad n \neq 1.$$

$$= \frac{\varepsilon^{1-n}}{1-n} \Big[\underset{1}{e^{2\pi i (1-n)}} - 1 \Big] = 0.$$

• $$\oint_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{if } n > 1 \\ & \text{+ve integer} \end{cases}$$

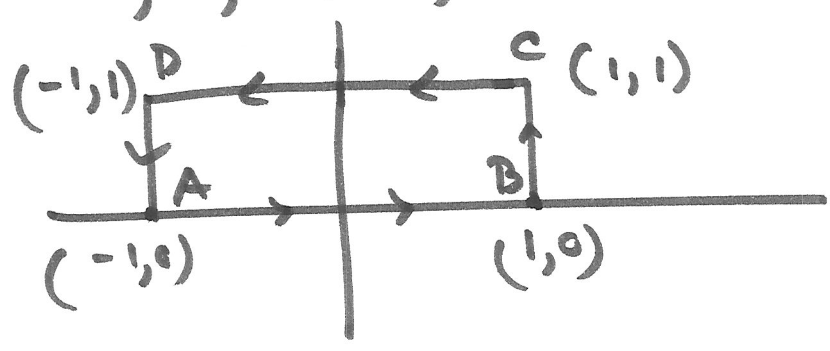
Example: $$F(\xi) = \int_C \frac{4z^2 + z + 5}{z - \xi}\, dz.$$

$$C : \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$$

$F(3.5) = 0.$

Example: Verify Cauchy's theorem for $\oint z^3 dz$ when C is the boundary of C the rectangle with vertices

$-1,\ 1,\ 1+i,\ -1+i$



Example: $$\int_{C} \frac{z+4}{z^2+2z+5}\,dz = 0.$$

$C: |z+1| = 1$

$$z^2+2z+5 = 0 \Rightarrow z = -1 \pm 2i$$

$$(-1,2),\ (-1,-2)$$

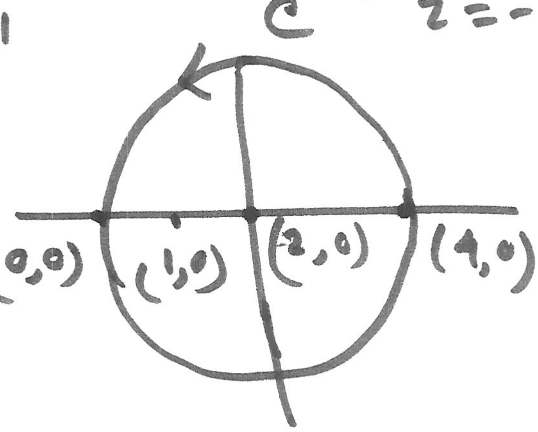
outside C.

Example: $$\oint_{C} \frac{5z+7}{z^2+2z-3}\,dz.$$

$C: |z-2| = 2$

$$= \oint \left[\frac{3}{z-1} + \frac{2}{z+3}\right] dz.$$

$C: |z-2| = 2$ $\quad z=1 \quad\quad z=-3.$



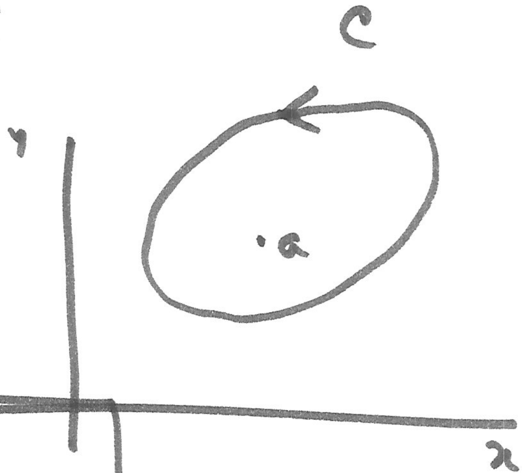
$$= 3 \cdot 2\pi i + 0$$

$$= 6\pi i$$

Cauchy's Integral formulae & Related Theorems.

- $f(z) \rightarrow$ analytic inside & on a simple closed curve C

- $a \rightarrow$ any pt. inside c.



- Then

$$\boxed{f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz} \quad \text{——} (1).$$

when c is traversed in the +ve sense.

- Also the n-th derivative of $f(z)$ at $z=a$ is given by

$$\boxed{f^{n}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz} \quad \text{—} (2).$$

- (1) is a special case of (2) when $n=0$.

Note 1. $f(z)$ known on a simple closed curve

$\Rightarrow$ the values of f & all its derivatives can be found at all pts. inside C.

Note 2: $f(z)$ is <u>analytic</u> in a simply connected domain D.

i.e. $f'(z)$ exists.

$\Rightarrow$ all the higher derivatives of $f(z)$ exists in D.

Not necessarily true for fn. of real variables

- $f(z)$ analytic in a domain D
  $\Rightarrow f'(z), f''(z), \ldots$ are analytic in D.

Applications

$$f^{n}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$\hookrightarrow$ $n = 0$ will be proved

Example:

$$I = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz.$$

$$= \oint_C \left(\sin \pi z^2 + \cos \pi z^2\right) \left[\frac{1}{z-2} - \frac{1}{z-1}\right] dz.$$

$$= \oint_C \frac{\overbrace{\left(\sin \pi z^2 + \cos \pi z^2\right)}^{f(z)}}{\underbrace{z-2}_{a=2}}\, dz - \oint_C \frac{\left(\sin \pi z^2 + \cos \pi z^2\right)}{\underbrace{z-1}_{a=1}}\, dz$$

$$= 2\pi i \cdot \left(\sin \pi 2^2 + \cos \pi 2^2\right)$$

$$- 2\pi i \left(\sin \pi 1^2 + \cos \pi 1^2\right)$$

$$= 0 \;?$$

$2\pi i + 2\pi i = 4\pi i$

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz.$$

Example: $I = \int\limits_{C : |z-1|=1} \frac{3z^2+z}{z^2-1}\, dz$

Sol^n.

$z = \pm 1 \rightarrow$ singularity.

$z = -1 \rightarrow$ outside C

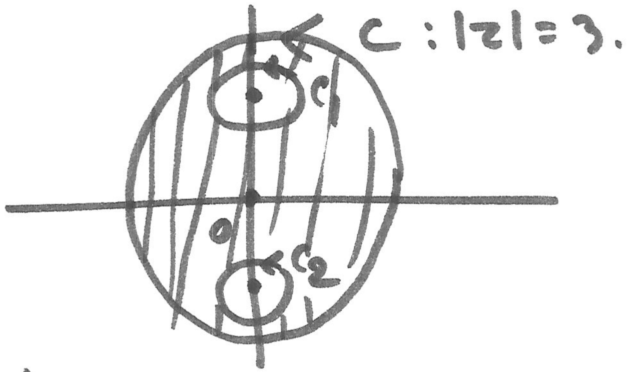
$z = 1 \rightarrow$ inside C.

$$I = \int\limits_{C : |z-1|=1} \frac{3z^2+z}{(z-1)(z+1)}\, dz \quad \rightarrow f(z)$$

$= 2\pi i \, f(1).$

<u>Example:</u> Evaluate $\oint_{C:|z|=3.} \frac{dz}{z^2+1}$

<u>Sol^n:</u> $\frac{1}{z^2+1} \rightarrow$ not analytic at $z = \pm i$, both lie inside $C: |z| = 3$



$$\oint_C \frac{dz}{z^2+1}$$

$$= \frac{1}{2i} \int_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

$$= \frac{1}{2i} \int_{C_1} \Big[\underbrace{\frac{1}{z-i}}_{2\pi i} - \underbrace{\frac{1}{z+i}}_{0}\Big] dz + \frac{1}{2i} \int_{C_2} \Big[\underbrace{\frac{1}{z-i}}_{0} - \underbrace{\frac{1}{z+i}}_{2\pi i}\Big] dz$$

$$= \frac{1}{2i}(2\pi i - 0) + \frac{1}{2i}(0 - 2\pi i)$$

$$= 0$$

Example: Prove that

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots (2n)} 2\pi,$$

where $n = 1, 2, 3, \ldots$

Sol^n: Let $z = e^{i\theta}$

$$dz = i e^{i\theta} d\theta$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

Let $C : |z| = 1$ (unit circle).

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \oint_{C:|z|=1} \left[\frac{z + \frac{1}{z}}{2}\right]^{2n} \frac{dz}{iz}$$

$$= \frac{1}{2^{2n} i} \oint_C \frac{1}{z} \left[z^{2n} + \binom{2n}{1} z^{2n-1} \cdot \frac{1}{z} + \cdots \right.$$
$$\left. \cdots + \binom{2n}{k} z^{2n-k} \cdot \frac{1}{z^k} + \cdots + \frac{1}{z^{2n}} \right] dz.$$

$$= \frac{1}{2^{2n} i} \oint_{C : |z|=1} \left[z^{2n-1} + \binom{2n}{1} z^{2n-3} + \cdots + \boxed{\binom{2n}{k} z^{2n-2k-1}} + \cdots + z^{-2n-1} \right] dz.$$

$$\oint_C \frac{1}{(z-a)^n} dz = \begin{cases} 0 & \text{if } n > 1 \text{ +ve int.} \\ 2\pi i & \text{if } n = 1. \end{cases}$$

$$= \frac{1}{2^{2n} \cancel{i}} \binom{2n}{n} 2\pi \cancel{i}$$

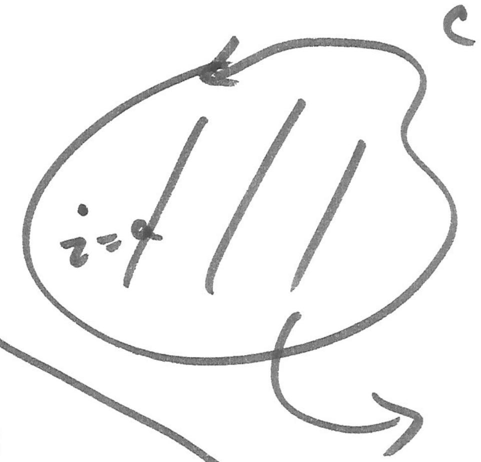
$k = 1, 2, \cdots, n-1 \rightarrow \oint_C = 0.$

$k = n, n+1, n+2, \cdots$

$\oint \frac{dz}{z} = 2\pi i$ $\qquad$ $\oint \frac{dz}{z^2} dz = 0$

. Cauchy's Integral formula

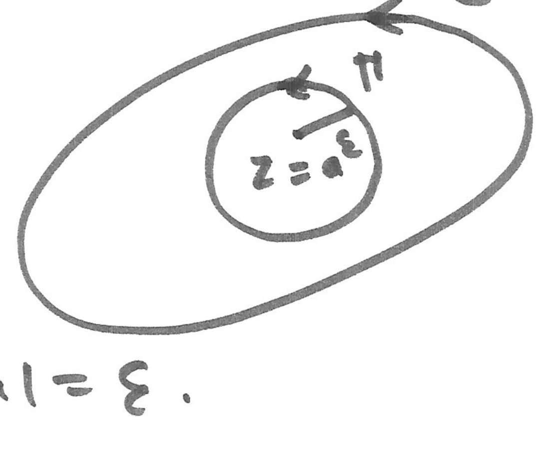
$$f(a) = \frac{1}{2\pi i} \int \frac{f(z)}{(z-a)} dz.$$



f(z) analytic

proof.

$\frac{f(z)}{z-a} \rightarrow$ analytic inside & on C except at $z=a$.



Let $\Gamma : |z-a| = \varepsilon$.

$z = a + \varepsilon e^{i\theta}$

$dz = \varepsilon i e^{i\theta} d\theta$.

$$\oint_C \frac{f(z)}{z-a} dz = \oint_\Gamma \frac{f(z)}{z-a} dz.$$

$$= \int_{\theta=0}^{2\pi} \frac{f(a+\varepsilon e^{i\theta})}{\cancel{\varepsilon} e^{i\cancel{\theta}}} \varepsilon i \cancel{e^{i\theta}} d\theta$$

letting $\varepsilon \to 0$, we get

$$\oint_C \frac{f(z)}{z-a} dz = \int_0^{2\pi} \lim_{\varepsilon \to 0} f(a+\varepsilon e^{i\theta}) i d\theta$$

(as f continuous).

$$= \int_0^{2\pi} f(a) i d\theta.$$

$$= 2\pi i f(a). \qquad \to f(a,z).$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz.$$

- $$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz.$$

differentiating w.r.to a using leibnitz's rule

4.10.15

Example:

$$I = \oint \frac{e^{-z}}{z+1}\, dz$$

$$c: |z| = \frac{1}{2}$$

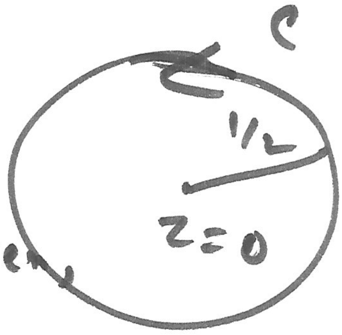
Soln.

$z = -1$ is the only singularity of

$$f(z) = \frac{e^{-z}}{z+1}$$

which lies outside the circle

$$c: |z| = \frac{1}{2}$$



Hence, by Cauchy's Theorem,

$$\oint_c f(z)\, dz = 0.$$

Example: $$I = \oint \frac{e^{2z}}{(z+1)^4}\, dz$$

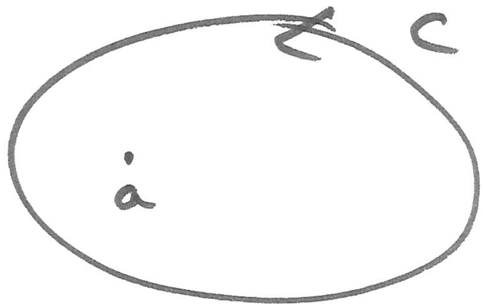
$C : |z| = 3.$

Sol^n.

Cauchy's Integral formula

$$f^{n}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z) = e^{2z}}{(z - \underset{-1}{\underset{\downarrow}{a}})^{n+1}}\, dz \quad , n = 0, 1, 2, \ldots$$

when $f(z)$ is analytic.



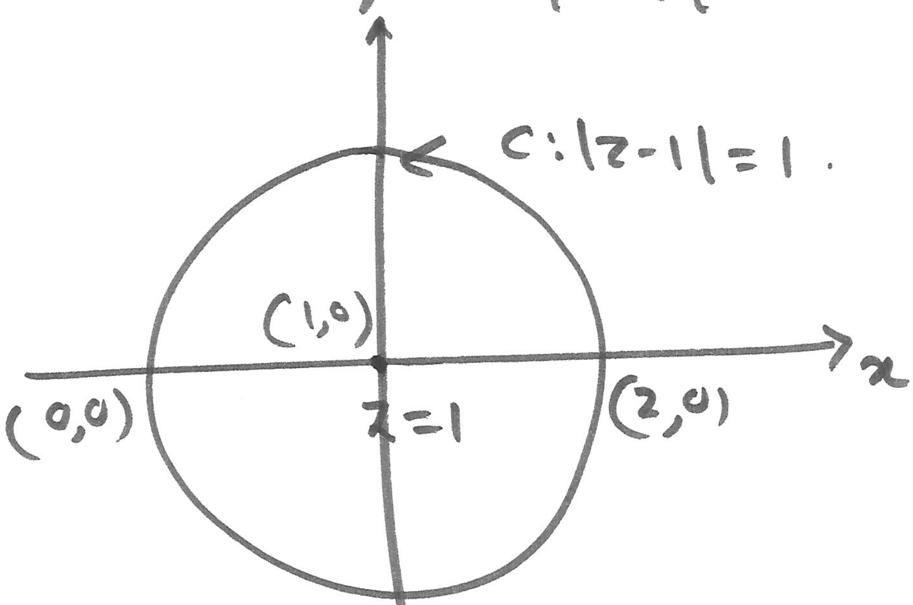
$$I = \frac{2\pi i}{3!} f^{(3)}(-1)$$

when $\boxed{f(z) = e^{2z}.}$

$$I = \frac{2\pi i}{3!}\, 8\, e^{-2}.$$

Example: $I = \int \frac{3z^2+z}{z^2-1}\, dz.$ $\quad c: |z-1| = 1$

$z^2 - 1 \rightarrow z = \pm 1 \downarrow$ only $z=1$ is inside



$$I = \oint_{c: |z-1|=1} \frac{\overbrace{\left(\frac{3z^2+z}{z+1}\right)}^{f(z)}}{(z-1)}\, dz.$$

$f(z)$ in analytic inside & on c.

$\therefore$ By Cauchy's Integral Theorem.

$$I = \frac{2\pi i}{0!} f^{0}(1) = 2\pi i\, f(1) = \underline{4\pi i}$$

- <u>M-L inequality (A bounding Theorem).</u>

- f <u>continuous</u> on a <u>smooth curve</u> C.

- $\cancel{|f(M)|}$ $|f(z)| \le M \;\; \forall z$ on C.

Then $$\left| \int_C f(z)\,dz \right| \le ML,$$

where L is the length of C.

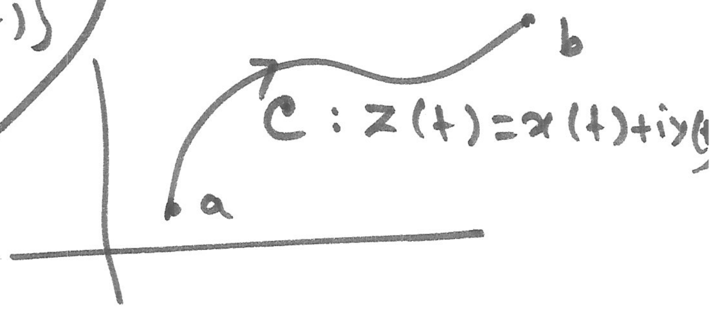
- <u>Length of a curve</u>

$$s = \int_a^b \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2}\, dt$$

$C: \; z = x + iy$

$\Downarrow$ parametric form

$x = x(t), \; y = y(t)$



$C: \; z(t) = x(t) + i\,y(t)$

$z'(t) = x'(t) + i\,y'(t) \implies |z'(t)| = \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2}$

$$\implies \boxed{\ell = \int_a^b |z'(t)|\,dt}\,.$$

Example: Find an upper bound for the absolute value of $\oint \frac{e^z}{z+1}\,dz$, when C is the circle $|z|=4$.

Sol^n: $\left| \oint_{C:\,|z|=4} \frac{e^z}{z+1}\,dz \right| \le ML$

M → ?, L → ?

C: $x = 4\cos t$, $y = 4\sin t$

. $L = \int_0^{2\pi} \sqrt{(-4\sin t)^2 + (4\cos t)^2}$

$= 8\pi$.

. $f(z) = \frac{e^z}{z+1}$

on $C: |z|=4$, $|z+1| \geqslant |z|-1 = 4-1 = 3$.

$|f(z)| \le \frac{|e^z|}{3} = \frac{|e^x(\cos y + i\sin y)|}{3}$

$= \frac{e^x}{3} \le \frac{e^4}{3}$ on C → M.

$\therefore \left| \oint_C \frac{e^z}{z+1} \right| \le ML = 8\pi \cdot \frac{e^4}{3}$

Infinite Series — Taylor's Series & Laurent's series

Taylor's Series
(Series ~~expansi~~ representation of an analytic fun.).

- $f(z) \rightarrow$ analytic inside & on a smooth closed curve C.
- $a, a+h \rightarrow$ two pts. inside C.

Then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

or

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + \cdots$$

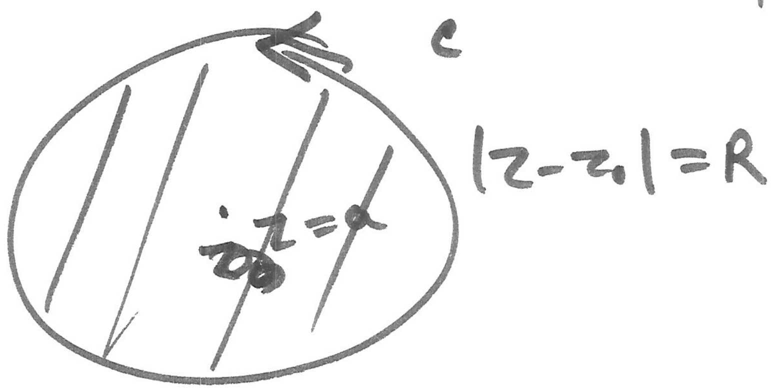
$$= \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n.$$

for $|z-a| < R$.

$\hookrightarrow$ region of convergence of the series

- $R \rightarrow$ radius of convergence of the series.

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where} \quad a_n = \frac{f^{n}(a)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz}$$



- $a=0 \rightarrow$ Maclaurin Series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} z^n$$

<u>Example:</u> $\quad f(z) = \dfrac{\sin z}{z^3}$.

$\rightarrow$ $f(z)$ is not analytic at $z=0$.

$\rightarrow$ hence cannot be expanded in Maclaurin series.

Maclaurin series expansion of $\sin z$

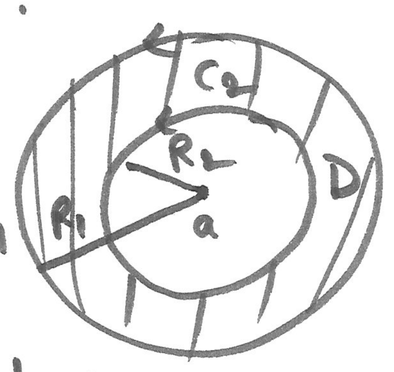
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$f(z) = \frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots$$

Converges $\forall z$, except $z=0$.
i.e. converges $\forall z$, $|z|>0$.

Laurent's Theorem

- $C_1, C_2 \rightarrow$ concentric circles of radii R_1 and R_2 resp'y., center at $z=a$.



- $f(z) \rightarrow$ single-valued, analytic on C_1, C_2 & in the region ~~between~~ (annular region) D between C_1, C_2.
- $a+h \rightarrow$ any pt. in D.

Then
$$\boxed{f(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \cdots}$$

where
$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_2} (z-a)^{n-1} f(z)\,dz.$$

- C_1, C_2 traversed in the +ve sense w.r. to their interiors.
- C_1, C_2 may be replaced by any concentric circle C between C_1 & C_2.

i.e.
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz,$$

$$n = 0, \pm 1, \pm 2, \ldots$$

in

Laurent's series expansion of $f(a+h)$

- $f(z) = \boxed{a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots}$

$$+ \boxed{\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots}$$

analytic part

when $a_n = \dfrac{1}{2\pi i} \oint_C \dfrac{f(\zeta)}{(\zeta-a)^{n+1}} d\zeta,$

$n = 0, \pm 1, \pm 2, \ldots$

- the principal part of the Laurent's series

- if the principal part is zero $\longrightarrow$ Taylor's series

- Laurent's series (about an isolated singularity).

- Taylor's series (for analytic fn.).

Classification of isolated Singularities.

$z = z_0$ (isolated singularity)	Laurent series
1. Removable singularity	$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$
2. Pole of order n	$\dfrac{a_{-n}}{(z-z_0)^n} + \dfrac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$
3. Simple pole	$\dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$
4. Essential singularity	$\cdots + \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$

$$\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0.$$