COMPLEX ANALYSIS-I

Mathematics-I

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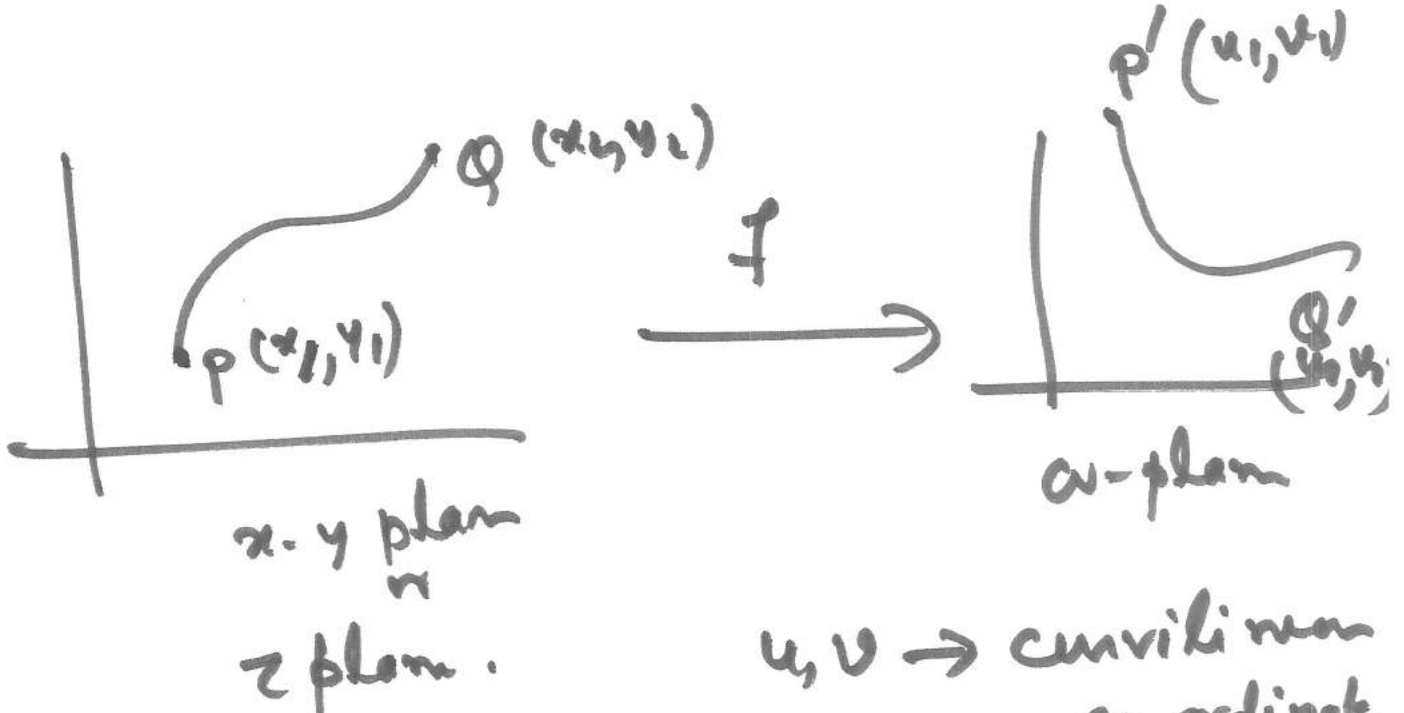
12.10.15

Complex Analysis

- $w = f(z)$, $z = x + iy$ → rectangular co-ordinate.

~~$= u + iv$~~

$= u(x,y) + i\,v(x,y)$



$u, v \rightarrow$ curvilinear co-ordinate.

- $w = f(z)$ $\to$ $u = u(x,y)$
  $= u + iv$ $\to$ $v = v(x,y)$

- $z = x + iy$

Example: $\quad w = f(z) = z^2$

$$z = x + iy \Rightarrow w = (x^2 - y^2) + (2ixy)$$
$$= u(x,y) + iv(x,y)$$

$$\left.\begin{aligned} \text{where} \quad u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy \end{aligned}\right\}$$

· $w = z^{1/2}$; $w = \log z$ → infinitely many branches

two branches

multiple valued fn.

$w = \sin^{-1} z$.

Limit & Continuity

w → mapping / transformation
$f(z)$

limit

· $w = f(z)$ defined in a domain D. (except perhaps at z_0 of D).

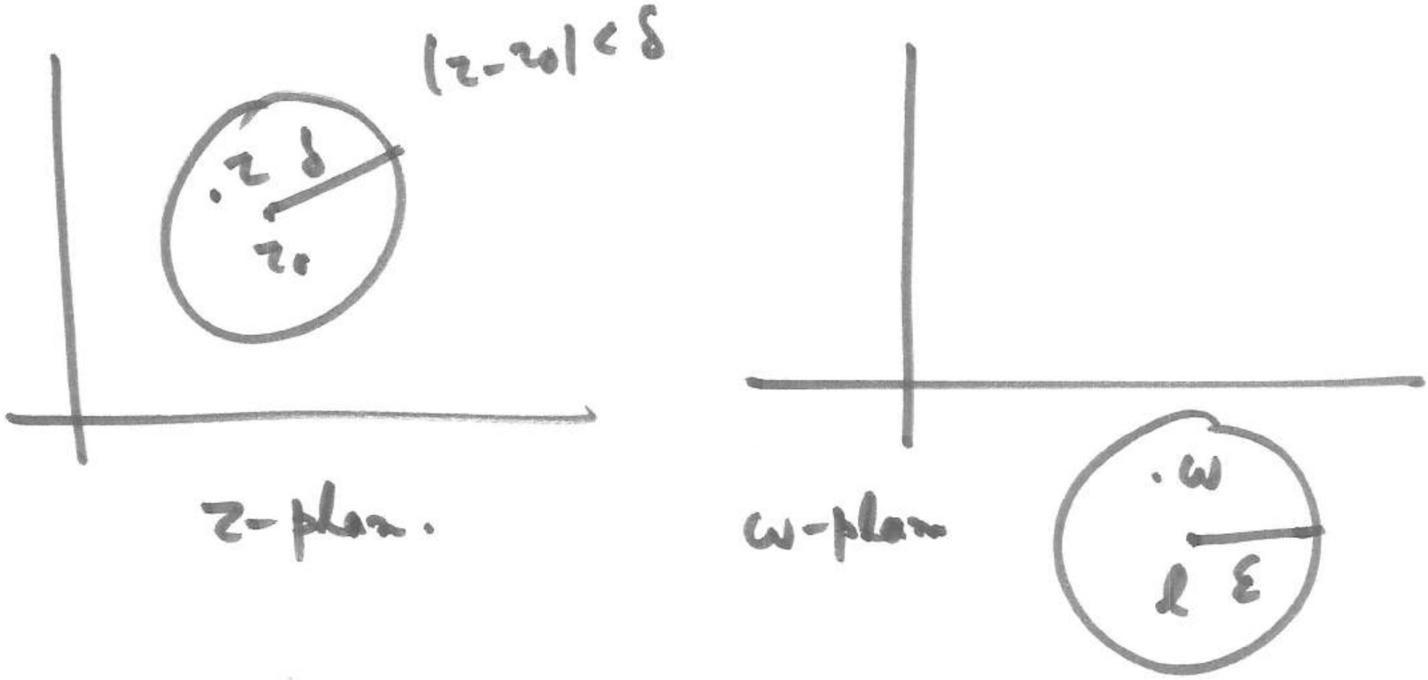
· l be a complex const.

- $\lim\limits_{z \to z_0} f(z) = l$ iff

to each +ve ε, a +ve δ can be found s.t. $\boxed{|f(z) - l| < \varepsilon}$

$\forall z \in \boxed{0 < |z - z_0| < \delta}$.

- l is the limit of $f(z)$ as $z \to z_0$ along any path whatsoever.



Example: Prove that $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z}$ does not exist.

$z = x+iy$

Solⁿ.

$z\to 0$ along x-axis

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} \frac{x}{x} = 1$$

$z \to 0$ along y-axis $(x=0)$

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} \frac{-iy}{iy} = -1$$

$\therefore$ The limit does not exist.

Example: Prove that $\lim\limits_{z\to z_0} f(z) = z_0^2$

if $f(z) = z^2$

Claim. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ (δ generally depends on ε & z_0)

s.t. $|z^2 - z_0^2| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Let $0 < |z - z_0| < \delta$.

$$|z^2 - z_0|^2 = |z - z_0|\,|z + z_0|$$

$$< \delta\,|z + z_0| = \delta\,|\underline{z - z_0} + 2z_0|$$

$$< \delta\,\left(|z - z_0| + 2|z_0|\right)$$

$$< \delta\,\left(1 + 2|z_0|\right) < \varepsilon$$

if $\delta \leq 1$.

Choose $\delta = \min\left\{1, \dfrac{\varepsilon}{1 + 2|z_0|}\right\}$

19.10.15.

- Complex fun^n.

$$\omega = f(z) \longrightarrow \text{a mapping from } z\text{-plane to } \omega\text{-plane.}$$

- $z = x + iy$

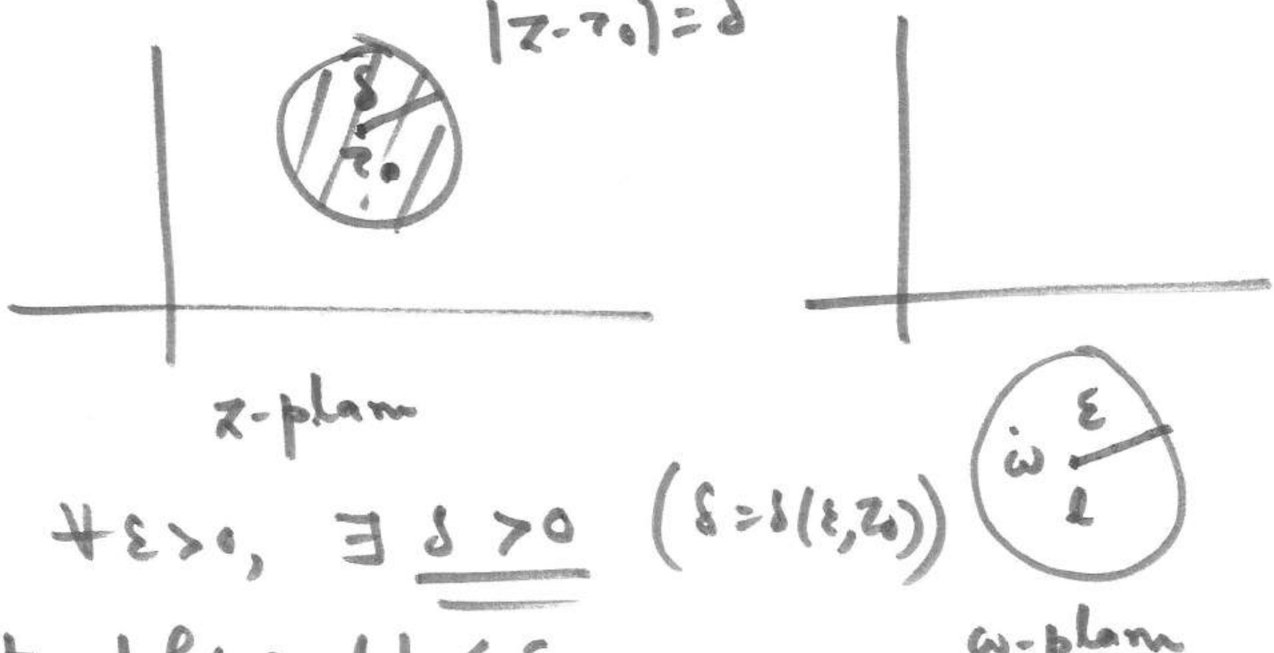
$$\omega = f(z) = u(x,y) + i v(x,y).$$

$(x,y) \to$ Rectangular co-ordinates

$(u,v) \to$ curvilinear co-ordinates.

- Limit / Continuity $\qquad w = f(z)$

$$\lim_{z \to z_0} f(z) = l.$$



$\forall \varepsilon > 0, \ \exists \ \delta > 0 \ \ (\delta = \delta(\varepsilon, z_0))$

s.t $|f(z) - l| < \varepsilon$

$\forall z \in \mathbb{C} \quad 0 < |z - z_0| < \delta.$

• $\lim_{z \to z_0} f(z) = l$, $\lim_{z \to z_0} g(z) = m$.

i) $\lim \left(f(z) \pm g(z) \right) = l \pm m$

ii) product

iii) division. provided denominator does not vanish.

<u>Example:</u> If $\boxed{\lim_{z \to z_0} g(z) = m}$ $(\neq 0)$,

then prove that $\exists \, \delta > 0$ s.t.

$$|g(z)| > \frac{1}{2}|m| \quad \text{for } 0 < |z - z_0| < \delta.$$

<u>Solⁿ.</u> $\forall \, \underline{\varepsilon > 0}$, $\exists \, \delta > 0$ s.t.

$$|g(z) - m| < \underline{\underline{\varepsilon}} \quad \text{for } 0 < |z - z_0| < \delta$$

Take $\varepsilon = \frac{1}{2}|m|$.

$$|m| = |m - g(z) + g(z)| \leq |m - g(z)| + |g(z)|$$
$$< \frac{1}{2}|m| + |g(z)|$$

$$\Rightarrow |g(z)| > \frac{1}{2}|m|.$$

Example:
$$\lim_{z\to 2e^{\pi i/3}} \frac{(z^3+8)\,(z^2-4)}{(z^4+4z^2+16)\,(z^2-}$$

$$= \lim_{z\to 2e^{\pi i/3}} \frac{(z^3+8)(z^2-4)}{z^6-64}$$

$$= \lim_{z\to 2e^{\pi i/3}} \frac{z^2-4}{z^3-8}$$

$$= \frac{1}{8}\left(3-i\sqrt{3}\right) \quad \text{(check)}.$$

- Behavior of $f(z)$ at $z=\infty$.

$\longrightarrow$ examine the behavior of $f\left(\frac{1}{w}\right)$ at $w=0$.

- pt. $\underline{z=0}$ $\xrightarrow{\;w=1/z\;}$ $w=\infty$.

  pt $z=\infty$ $\xrightarrow{\;w=1/z\;}$ $w=0$.

Continuity $\quad w = f(z)$ continuous at $z_0 \in D$.

[$D \to$ domain of defn. of w].

if $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

- Continuity in a domain D.
- $w = f(z)$, $\boxed{z_0 = x_0 + iy_0}$.

defined over a domain D.

let $w = f(z)$ is equivalent to real fun. $u(x,y)$ & $v(x,y)$.

i.e. $\boxed{w = u(x,y) + iv(x,y)}$

Then $\lim\limits_{z \to z_0} f(z) = l = \underline{\underline{a+ib}}$ (say)

iff

$\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = a$ & $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = b$.

proof. $\lim\limits_{z\to z_0} f(z) = l \Rightarrow \lim\limits_{z\to z_0} \overline{f(z)} = \bar{l}$

$$\Rightarrow \lim_{z\to z_0}\left[\operatorname{Re} f(z)\right] = \operatorname{Re} l$$

as

$$\lim_{z\to z_0}\left[f(z)+\overline{f(z)}\right] = l+\bar{l}$$

Illy, $\lim\limits_{z\to z_0} \operatorname{Im} f(z) = \operatorname{Im} l$.

Continuity. i) $f(z)$, $g(z)$ continuous at $z=z_0$.

$\Rightarrow f(z)\pm g(z),\ f(z)\cdot g(z),\ \dfrac{f(z)}{g(z)}$ all continuous at $z=z_0$.

ii) $f(z)$ continuous in a closed region $\Rightarrow$ $f(z)$ is bdd in that region.

i.e. $|f(z)| < M$

$\exists$ const. M s.t $|\;\;|$

$\forall z$ in that region.

iii) $f(z) = u(x,y) + i v(x,y)$

f continuous in a region $\Leftrightarrow$ u, v continuous in that region

iv) Continuous fn. of a continuous fn. is continuous.

<u>Example</u>: a) all polys.

b) e^z

c) $\sin z$, $\cos z$.

<u>Example</u>: Is the fn.

$$f(z) = \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$$

continuous at $z = i$?

<u>Sol.</u> Not continuous.

→ Removable discontinuity at $z = i$

→ Can be removed by redefining $f(z)$ at $z = i$ as

$$\boxed{f(z) = 4 + 4i}$$

· <u>Complex Differentiation : Analytic fn</u>

· $f(z) \rightarrow$ a single valued fn. defined on a domain D of the complex plane.

· $f(z)$ is differentiable at $z_0 \in D$ if

$$\boxed{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

e.g. $\log z$ → multiple valued fun

exists, is finite & is independent of the manner in which $z \to z_0$ in D, provided of course z always remains a pt. of D.

or

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists.}$$

or

$$\boxed{\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)}$$

- <u>Analytic fu^n:</u> (Holomorphic fu^n or regular fu^n.)

A fu^n. f is said to be analytic at a pt. z_0, if $\exists$ some δ-nbd of z_0 at all pts. of which $f'(z)$ exists.

$|z-z_0|<\delta$

<u>Example</u> $f(z) \rightarrow$ Single valued, diff. at every pt. of domain D. (except possibly for a finite no. of pts, called singular pts.)

$\rightarrow$ analytic fu^{ns}.

· Differentiability $\Longrightarrow$ Continuity

$$\not\Longleftarrow$$

<u>Example:</u> (i) $f(z) = \bar{z}$

(ii) $f(z) = |z^2|$.

Continuous $\forall z \in \mathbb{C}$, but is differen-tiable nowhere except at the origin.

$\cancel{f(z-z_0)}$

$$f(z_0+h) - f(z_0)$$

$$= \frac{f(z_0+h) - f(z_0)}{h} \cdot h.$$

When $z \neq z_0$, $z_0 \neq 0$, we have

$$\frac{|z^2| - |z_0^2|}{z - z_0} = \frac{z \cdot \bar{z} - z_0 \cdot \bar{z}_0}{z - z_0}$$

$$= \frac{\bar{z}(z - z_0) + z_0(\bar{z} - \bar{z}_0)}{z - z_0}$$

$$= \bar{z} + \frac{z_0(\bar{z} - \bar{z}_0)}{z - z_0}.$$

Let $z - z_0 = re^{i\theta}$.

$$\therefore \frac{|z^2| - |z_0^2|}{z - z_0} = \bar{z} + \frac{z_0 r e^{-i\theta}}{re^{i\theta}}.$$

$$= \bar{z} + z_0 e^{-2i\theta}$$

$$= \bar{z} + z_0(\cos 2\theta - i\sin 2\theta)$$

$\not\Rightarrow$ a unique limit as $z \to z_0$ in any manner

But when $z_0 = 0$, we get a finite limit, namely $\bar{z}_0$

- <u>Common examples of Complex Differentiability</u>

i) $f(z) = z^n \rightarrow$ analytic over the entire complex plane.

ii) $f(z) = \operatorname{Re} z$ and $f(z) = \operatorname{Im} z$

$\rightarrow$ not differentiable.

iii) $f(z) = \frac{1}{z}$ $\rightarrow$ differentiable $\forall z \in \mathbb{C}$ except at $(0,0)$.

$\rightarrow$ iv) $f(z) = \bar{z}$ $\rightarrow$ not differentiable anywhere

$\rightarrow$ however continuous at z_0

v) $f(z) = |z^2|$ $\rightarrow$ differentiable only at $z=0$.

$\rightarrow$ however continuous at every pt. z.

$$\frac{d}{dz}(\bar{z}) = \lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \quad \left\{ \begin{array}{l} z = x+iy \\ h = \Delta x + i\Delta y. \end{array} \right.$$

if $\Delta y = 0$ $\longrightarrow$ $= 1$

if $\Delta x = 0$ $= -1$.

26.10.15

- Analytic fn.

$|z-z_0| \le \delta$

- Necessary condition for a fn. to be analytic (Cauchy-Riemann eqn.).

$f'(z)$ exists $\forall z \in N_\delta(z_0)$

- $f(z) = u(x,y) + i\vartheta(x,y)$, $z = x+iy$.

- CR eqn.

$$\left.\boxed{\begin{array}{l} u_x = \vartheta_y \\ u_y = -\vartheta_x \end{array}}\right\}.$$

f analytic at z $\Rightarrow$ CR eqn. are satisfied.

proof.

$f'(z)$ exists ~~at a pt.~~

i.e. $\dfrac{f(z+h) - f(z)}{h} \to$ a unique lim $f'(z)$ as $h \to 0$.

in any manner.

· if we take h to be real, then

$$\frac{u(x+h,y)+iv(x+h,y) - u(x,y) - iv(x,y)}{h}$$

$z+h = \underline{x+h}+iy$

$f(z) = u(x,y) + iv(x,y)$

$z = x+iy$

$\longrightarrow f'(z)$ as $h \to 0$.

i.e. $\boxed{f'(z) = u_x + i v_x.}$ — (1)

· if we take h to be imaginary, then

$\underline{h = ik}$

$z+h = x+i(y$

$$\frac{u(x,y+k)-u(x,y)}{ik} + \cancel{i}\,\frac{v(x,y+k)-v(x,y)}{\cancel{i}k}$$

$\longrightarrow f'(z)$ as $k \to 0$.

i.e. $\boxed{f'(z) = -i u_y + v_y}$ — (2).

(1), (2) $\Rightarrow$ $\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$ - CR eqn.

· $f(z)$ analytic $\Rightarrow$ u_x, u_y, v_x, v_y all must
$\parallel$ exist & CR eqn^n.
$u(x,y) + i v(x,y)$ are satisfied.
with $z = x+iy$

$$\not\Leftarrow$$

Observation Conditions of the above result are not sufficient

Example: $f(z) = \sqrt{|xy|}$

Show that At the origin, CR eqn^{ns} are satisfied, but the fn^n. is not analytic (regular) there.

Sol^n $f(z) = \underbrace{u(x,y)}_{\sqrt{|xy|}} + i \underbrace{v(x,y)}_{0}$

$$u_x(0,0) = \lim_{h \to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h}$$

$$= 0.$$

$$u_y(0,0) = 0$$

$$v_x(0,0) = 0 = v_y(0,0).$$

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \text{ at } (0,0) \Rightarrow \text{CR equ}^n\text{s are satisfied}$$

But
$$\frac{f(h)-f(0)}{h} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}-0}{x+iy}$$

let $h = x+iy$.

along $y = mx$

$$= \lim \frac{\sqrt{|m^2|}}{1+im}$$

which depends on m.

$\Rightarrow f'(0)$ does not exist.

Exercise

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Prove that CR equ^{n}s are satisfied, but the fu^n is not analytic at $z=0$.

· Sufficient conditions for a fu^n. $f(z)$ to be regular $\longrightarrow$ a fu^n. which is analytic with no singularity.

u_x, u_y, v_x, v_y all exist, continuous and ~~at~~ CR-eqn^{n}! are satisfied at z_0

$\Rightarrow$ $f(z)$ is analytic at z_0.

~~if $f(z)$~~

Observations:

- $\omega = f(z) = u(x,y) + iv(x,y)$,
- $z = x+iy$, $\bar{z} = x - iy$.
- $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$

· if $f(z)$ is an analytic fu^n of z, then x & y can occur in $f(z)$ only in the combination of $x+iy$.

$u(x,y), v(x,y) \rightarrow fu^{n}$! of two independent variables $z, \bar{z}$.

· if u_x, u_y, v_x, v_y exist & are continuous, then condition that ω shall be independent of $\bar{z}$ is

$$\frac{\partial \omega}{\partial \bar{z}} = 0. \qquad x, y \rightarrow z, \bar{z}$$

$$x = \frac{z+\bar{z}}{2}, \qquad y = \frac{z-\bar{z}}{2i}.$$

i.e. $$\frac{\partial}{\partial \bar{z}}\left(u + i v\right) = 0.$$

$$\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) = 0.$$

$$\left\{u_x \cdot \frac{1}{2} + u_y\left(-\frac{1}{2i}\right)\right\} + i\left\{v_x \cdot \frac{1}{2} + v_y\left(-\frac{1}{2i}\right)\right\} = 0.$$

or $$(u_x - v_y) + i\,(v_x + u_y) = 0 = 0 + i0$$

$$\Rightarrow \left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\}$$ i.e. the CR eqn^s are satisfied.

· $f(z)$ analytic $\Rightarrow$ $f(z)$ is a fun^n of z, not $\bar{z}$

· $u(x,y)$, $v(x,y) \rightarrow$ conjugate fun^s.

Harmonic fun^{n}!

- $\phi = \phi(x,y)$ is harmonic if

$$\boxed{\begin{aligned} &\nabla^2 \phi = 0. \\ \text{i.e. } &\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \end{aligned}}$$ Laplace's eqn.

- Real & Imaginary parts of an analytic fun^{n} satisfy Laplace eqn.

$f(z) = u(x,y) + iv(x,y)$, $z = x + iy$
is analytic $\Rightarrow$ u, v are harmonic.

proof

$$\boxed{u_x = v_y, \quad u_y = -v_x}$$ as $f(z)$ is analytic

$$u_{xx} = v_{xy} = v_{yx} = -u_{yy}$$

$$\Rightarrow \quad u_{xx} + u_{yy} = 0.$$

Similarly, v is harmonic.

- $f(z)$ analytic $\Rightarrow$ u, v both are harmonic.
$\downarrow$
conjugate harmonic fun^{n}

Theorem If the harmonic fun^{n}! u & v satisfy CR eqn^{ns}., then $u+iv$ is an analytic fun^{n}.

Example: If $u = e^{x}(x\cos y - y\sin y)$, find the analytic fun^{n}. $u+iv$.

Sol^{n}.

$u_x = v_y$
$u_y = -v_x$

$$dv = v_x\,dx + v_y\,dy.$$

$$= -u_y\,dx + u_x\,dy.$$

exact differential

$$= -e^{x}(-x\sin y - \sin y - y\cos y)\,dx$$
$$+ e^{x}(x\cos y - y\sin y + \cos y)\,dy.$$

$$v = \int_{y\,\text{const.}} e^{x}(x\sin y + \sin y + y\cos y)\,dx$$
$$+ \int (\text{those terms which do not contain } x)\,dy.$$

$$= \sin y\,(xe^{x} - e^{x}) + e^{x}\sin y + e^{x}y\cos y$$
$$+ 0 + c,\ c\ \text{const}$$

$f(z) = u + iv$ [express it in terms of only z.

$= z e^{z} + ci$

Alternatively,

$f(z) = u(x,y) + i v(x,y)$.

$= u(z,0) + i v(z,0)$. (To be proved)

$= e^{z}(z-0) + i\{0+0+c\}$

$= z e^{z} + ci$

proof

- $\omega = f(z) = u(x,y) + i v(x,y)$, $z = x+iy$, $\bar{z} = x - iy$.
- $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$

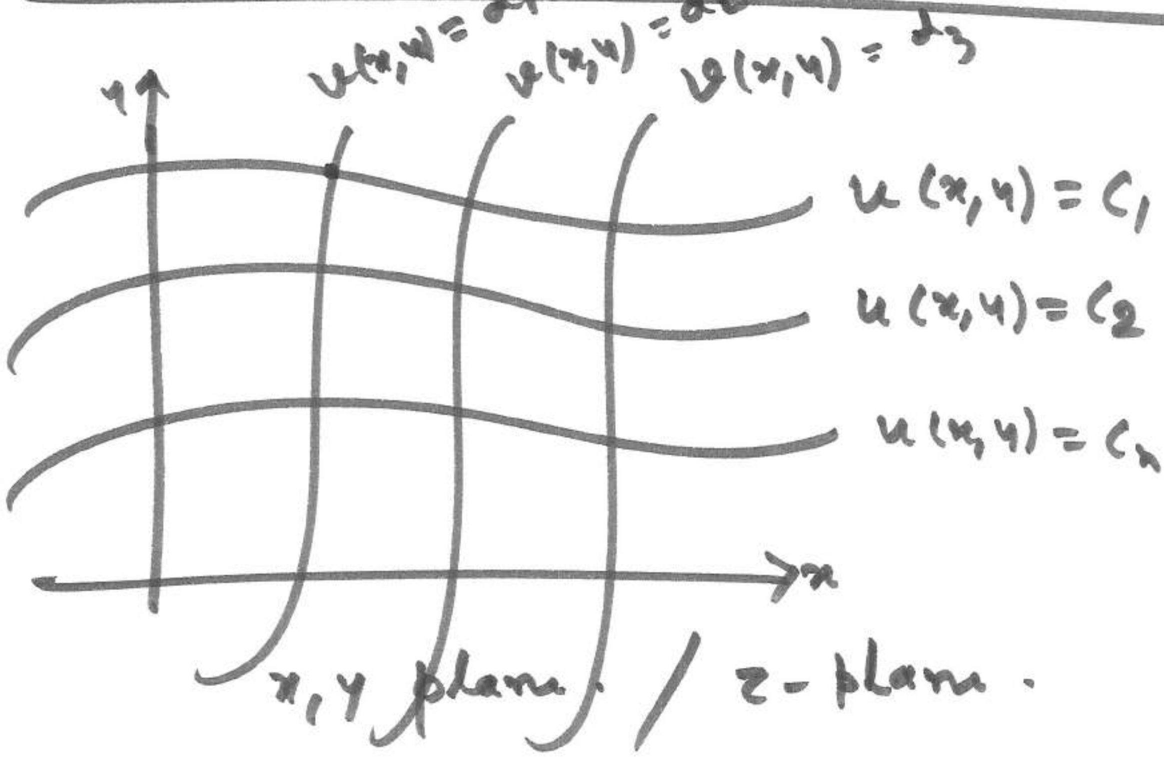
$$\Rightarrow f(z) = u\left[\frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z})\right] + i v\left[\frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z})\right].$$

$\rightarrow$ identity in two ind. variables z & $\bar{z}$

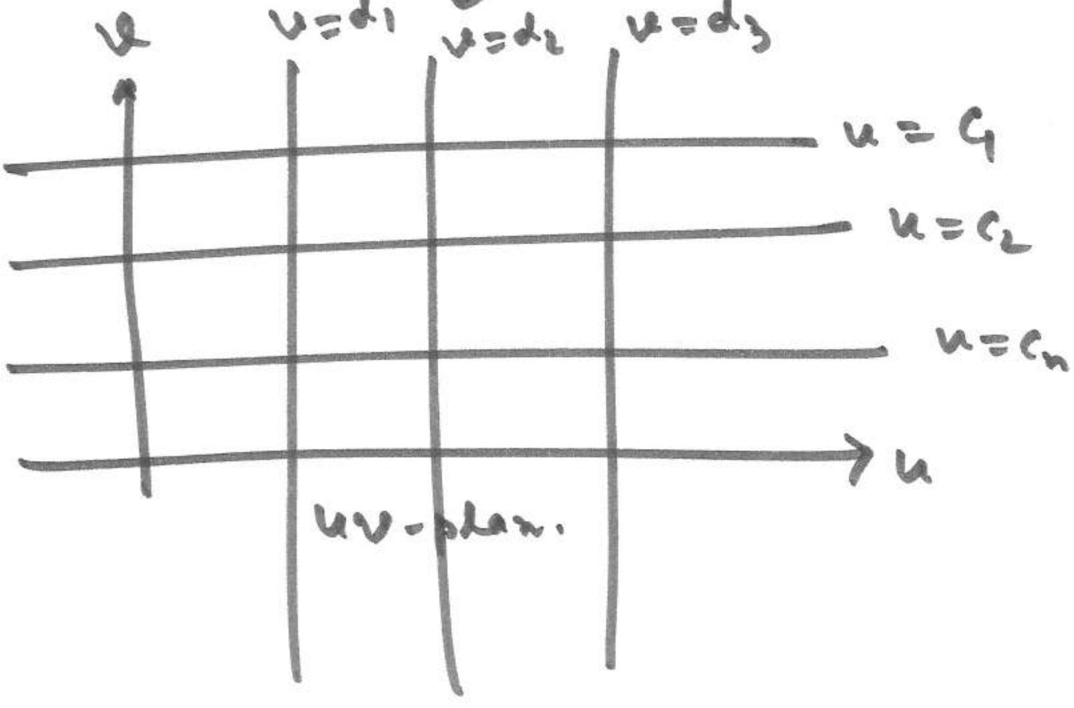
. put $z = \bar{z}$ (i.e. putting $y = 0$).

$$\Rightarrow f(z) = u(z, 0) + i\,v(z, 0).$$

Orthogonal system of curves



$\downarrow f$



- Two families of curves

$$u(x,y) = c_1, \quad v(x,y) = c_2$$

are said to be an <u>orthogonal system</u> if they intersect at right angles at each pt. of their intersection.

- Two such families intersect orthogonally if

$$\boxed{u_x \cdot v_x + u_y v_y = 0.}$$

<u>proof.</u>

$$u(x,y) = c_1 \Rightarrow u_x + u_y \frac{dy}{dx} = 0.$$

$$m_1 = \frac{dy}{dx} = -\frac{u_x}{u_y}$$

$$v(x,y) = c_2 \Rightarrow m_2 = \frac{dy}{dx} = -\frac{v_x}{v_y}.$$

$$m_1 m_2 = -1 \Rightarrow \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right) = -1$$

$$\Rightarrow u_x v_x + u_y v_y = 0.$$

- if $w = f(z) = u + iv$ is analytic fu^n of $z = x + iy$, then the curves $\boxed{u = \text{const}, \ v = \text{const}}$. on the z-plane intersect at right angles. ($f'(z) \neq 0$).

proof. f analytic

$$\Rightarrow \quad \begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

Now $u_x v_x + u_y v_y = u_x v_x + (-v_x) u_x$
$= 0.$

- conformal mapping

$f(z)$ analytic, $f'(z) \neq 0$

$$C_1, C_2 \xrightarrow{f} C_1', C_2'$$

angle between C_1, C_2 in z-plane
$=$ angle between C_1', C_2' in w-plane.

Method of Constructing Regular Fun^n.

The Milne-Thompson Method.

- $\omega = f(z) = u(x,y) + i v(x,y)$,

  $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2}$

- $\boxed{f(z) = u(z,0) + i v(z,0).}$

- $f'(z) = u_x + i v_x$.

  $= u_x - i u_y$

CR eqn^n:

$u_x = v_y$

$u_y = -v_x$

Let $u_x = \phi_1(x,y)$, $\cancel{v_x}$ $u_y = \phi_2(x,y)$

Then $f'(z) = \phi_1(x,y) - i\,\phi_2(x,y)$

$= \phi_1(z,0) - i\,\phi_2(z,0)$.

Integrating,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c,$$

$c \rightarrow$ an arb. const.

Similarly, if $v(x,y)$ is given, then we have

$$f'(z) = v_y + i v_x = \psi_1(x,y) + i\psi_2(x,y)$$

$$= \psi_1(z,0) + i\,\psi_2(z,0)$$

$$\Rightarrow f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c',$$

$c' \rightarrow$ arb. const.

When $v_y = \psi_1(x,y)$, $v_x = \psi_2(x,y)$.

<u>Exercise</u> Find the analytic fn: of which the real part is

$$e^{-x}\left\{ (x^2-y^2)\cos y + 2xy\sin y \right\}.$$

<u>using Milne-Thomson method.</u>

<u>Ans:</u> $f(z) = C + z^2 e^{-z}$, C is an arb. const.

Example:

a) Prove that $u(x,y) = e^{-x}(x\sin y - y\cos y)$ is harmonic

b) find $v(x,y)$ s.t. $f(z) = u + iv$ is analytic

c) Find $f(z)$ in terms of z.

Polar form of CR eqn.

$x = r\cos\theta,\ y = r\sin\theta$

$$\boxed{x^2 + y^2 = r^2, \quad \theta = \tan^{-1}\frac{y}{x}.} \longrightarrow$$

$$\left.\begin{array}{l} u_x = u_y \\ u_y = -v_x \end{array}\right\} \Longrightarrow \left.\begin{array}{l} u_r = \frac{1}{r} v_\theta \\ \cancel{v_\theta =} \\ v_r = -\frac{1}{r} u_\theta . \end{array}\right\}$$

$$u_x = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$u_y = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

$v_x =$

$v_y =$

A simple method of constructing an analytic f^n. (without involving the use of integration).

If the real part of an analytic f^n. $f(z)$ is a given harmonic f^n. $u(x,y)$, then prove that

$$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0)}$$

(A purely imaginary const. can be added)

Sol^n. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

Then $\overline{f(z)} = \overline{f(x+iy)} = u(x,y) - iv(x,y)$

(add) $f(x+iy) + \overline{f(x+iy)} = 2u(x,y)$ — (1)

Note: The conjugate f^n $\overline{f(z)}$ has the partial derivative zero w.r.to z.

So we may consider $\overline{f(z)}$ as a f^n. of $\bar{z}$

Let us denote $\overline{f(z)}$ by $\bar{f}(\bar{z})$.

With this notation,

$$\overline{f(x+iy)} = \bar{f}(x-iy).$$

① $\Rightarrow$ ~~$f(x+iy)$~~

$$u(x,y) = \frac{1}{2}\left[f(x+iy) + \bar{f}(x-iy) \right] \quad \text{—— ②.}$$

$\rightarrow$ This identity holds even when x, y are complex.

Hence substituting $x = \frac{z}{2}$, $y = \frac{z}{2i}$ in ②, we get

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}\left[f\left(\frac{z}{2} + \frac{iz}{2i}\right) + \bar{f}\left(\frac{z}{2} - \frac{iz}{2i}\right) \right]$$

$$= \frac{1}{2}\left[f(z) + \bar{f}(0) \right].$$

or $f(z) = 2\, u\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0)$.

$u(0,0) = iv(0,0)$

$$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0)} + C$$

purely imaginary

proved

Exercise: $f(z) = u(x,y) + iv(x,y)$.

$$\Rightarrow \boxed{f(z) = 2iv\left(\frac{z}{2}, \frac{z}{2i}\right) + c,}$$

c is a real const.

prove

Example: Construct the analytic fun.

$f(z) = u + iv$, when $u = y^3 - 3x^2y$.

using the above method. (without involving any integration).

Example: find the values of consts. a, b, c, d s.t. the fn.

$$f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$$

is analytic.

Solⁿ. CR eqns. are satisfied.

$$\Downarrow$$

$$d = 2, \ a = 2, \ c = -1, \ b = -1.$$

Example: If $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$,

find the analytic fn. $\underline{\underline{f(z) = u + iv.}}$

Solⁿ.

$$\underline{\underline{(1+i) f(z)}} = (1+i)(u+iv)$$

$$= (u - v) + i(u + v).$$

$$= U + iV$$

$$\swarrow$$

· L' Hospital's Rules holds

i) $\lim_{z\to 0} (\cos z)^{1/z^2} \longrightarrow$

ii) $\lim_{z\to 0} \dfrac{1-\cos z}{\sin z^2}$

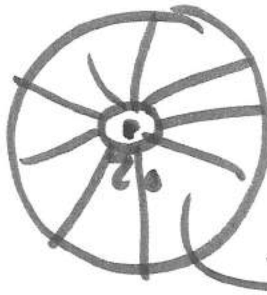
Singular pts.

↳ A pt. at which $f(z)$ fails to be analytic

· Various types of singularity exist

---

1. Isolated singularity.



$0 < |z - z_0| < \delta$.

→ No other singularity

· if no such δ can be found, then z_0 is a non-isolated singularity.

2. Poles

→ if we can find a +ve integer n s.t.

$$\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0, \text{ then}$$

$z = z_0$ is a pole of $f(z)$ of order n.

e.g. a) $f(z) = \dfrac{1}{(z-2)^3}$ → pole of order 3 at $z=2$

b) $$f(z) = \frac{3z-2}{(z-1)^2 (z+1)(z-4)}.$$

$z = 1$ → a pole of order 2.

$z = -1, 4$ → simple poles.

3. Removable Singularity

↓

an isolated singularity pt. z_0 is called a removable singularity of $f(z)$ if $\lim_{z \to z_0} f(z)$ exists.

• By defining $f(z_0) = \lim_{z \to z_0} f(z)$,

it can be shown that $f(z)$ is not only continuous at z_0, but it is also analytic at z_0.

e.g. $f(z) = \dfrac{\sin z}{z}$.

$\downarrow$

has removable $\boxed{\text{singularity}}$ at $z = 0$.

4. Essential singularity

An isolated singularity which is not a pole, or removable singularity is called an essential singularity.

e.g. $f(z) = e^{\frac{1}{z-3}}$ has an essential singularity at $z = 3$

5. Singularities at infinity

The type of singularity of $f(z)$ at $z = \infty$ is the same as that of $f\left(\frac{1}{w}\right)$ at $w = 0$.

e.g. $f(z) = z^3$. has a pole of order 3 at $z = \infty$

as $f\left(\frac{1}{w}\right) = \frac{1}{w^3}$ has a pole of order 3 at $w = 0$

6. Branch pts. e.g. $w = z^{1/2}$.

$\boxed{\begin{array}{l}\nexists \text{ no +ve int. } n \text{ s.t.} \\ \lim\limits_{z \to z_0} (z - z_0)^n f(z) = A \neq 0.\end{array}}$

$\hookrightarrow$ essential singularity of $f(z)$ at $z = z_0$.

28.10.15

- $f(z) = u + iv$ analytic
- C-R eqn. $\rightarrow$ $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$
- $u, v \rightarrow$ harmonic.
- $u = c_1 \perp v = c_2$ $\rightarrow$ orthogonal system of curves.
- given u $\rightarrow$ compute v

  given v $\rightarrow$ compute u.

  u $\downarrow$ $v = ?$

  $dv =$

  Method I $\rightarrow$ C-R

  Method II $\rightarrow$ exact DE.

  Method III $\rightarrow$ MT method.

  Method IV $\rightarrow$ identity.
- Polar co-ordinate $\rightarrow$ CR $\rightarrow$ $\boxed{\begin{array}{l} u_r = \frac{1}{r} v_\theta \\ v_r = -\frac{1}{r} u_\theta \end{array}}$
- L'Hospital's rules
- Singular pts. $\longleftrightarrow$ ordinary pts.

  $\rightarrow$ Isolated singularities

  $\rightarrow$ removable singularities

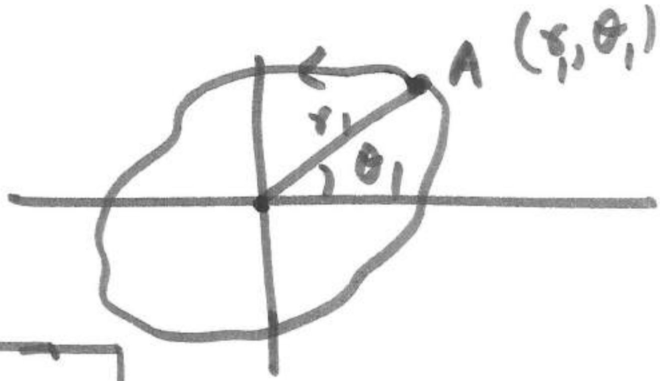
  $\rightarrow$ poles

  $\rightarrow$ essential singularity.

↳ Branch point.

6. Branch point

- $\omega = z^{1/2}$



- Let $z = r_1 e^{i\theta_1}$

$$\Rightarrow \boxed{\omega = r_1^{1/2} e^{i\theta_1/2}}$$

z-plane.

at A $(r = r_1, \theta = \theta_1)$.

- After 1 complete circuit, $\theta = \theta_1 + 2\pi$

So $\omega = r_1^{1/2} e^{i(\theta_1 + 2\pi)/2}$

$= -r_1^{1/2} e^{i\theta_1/2}$ at A

- We have not achieved the same value of ω with which we started.

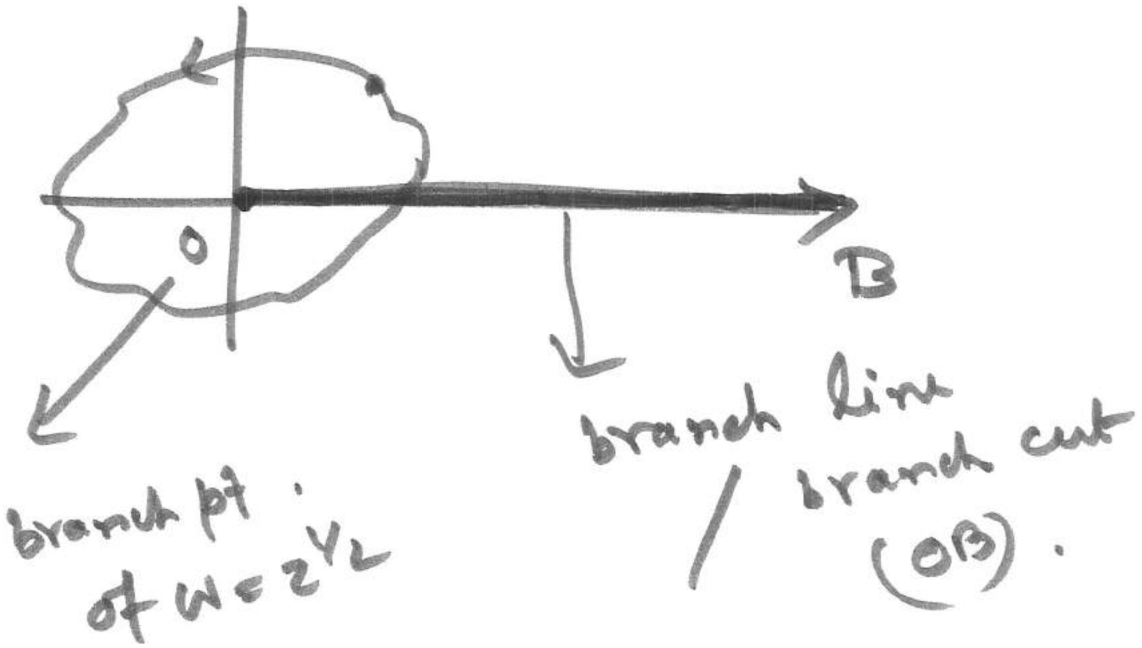
- After completing a second circuit, $\theta = \theta_1 + 4\pi$ & so $\omega = r_1^{1/2} e^{i\theta_1/2}$ at A.

same value of ω with which ~~we~~ we started.

- if $0 \leq \theta < 2\pi$, we are on one branch of the multiple valued fn. $w = z^{1/2}$

- if $2\pi \leq \theta < 4\pi$, we are on the other branch of the fn. $w = z^{1/2}$.

- each branch of this fn. is then single-valued.



- In order to keep the fn. single-valued, we set up an <u>artificial barrier</u> such as OB, which we agree not to cross.

branch cut

e.g. (i) $f(z) = (z-2)^{1/2}$ has a branch p[oint]
at $z=2$

(ii) $f(z) = \ln(z^2+z-2) = \ln(z+2)(z-$
has branch pts. at $z=-2, 1$.

<u>Example</u>:

(i) $$f(z) = \frac{z}{(z^2+9)^2} = \frac{z}{(z+3i)^2(z-3i)}$$

→ <u>poles</u> of order 2 at $z = \pm 3i$

→ isolated singularities (why?).

<u>$z = 3i$</u>

$|z-3i| = \delta$.

<u>Choose $\delta = 1$</u>

$z = -3i$ → isolated singularities

Example: $f(z) = \sec\left(\frac{1}{z}\right)$

$$= \frac{1}{\cos\left(\frac{1}{z}\right)}.$$

- singularity occurs when $\cos\frac{1}{z} = 0$.

$$\Rightarrow z = \frac{2}{(2n+1)\pi}, \quad n = 0, \pm 1, \ldots$$

$$\lim_{z \to \frac{2}{(2n+1)\pi}} \left(z - \frac{2}{(2n+1)\pi}\right) f(z) = A \neq 0$$

simple poles (isolated).

- $z = 0 \rightarrow$ essential singularity
as $\exists$ no +ve integer n s.t.

$$\lim_{z \to 0} (z-0)^n f(z) = A \neq 0.$$

$\hookrightarrow$ non-isolated singularity.
as every circle of radius δ with center at $z = 0$ contains singularities

other than $z=0$.

~~$f(z)$~~

<u>Example:</u> $f(z) = (z-3)^{1/2} \rightarrow z=3$ is a branch pt

<u>Example:</u> $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}} \Rightarrow z=0$ is not a branch pt.

Let $z = \underline{re^{i\theta}} = \underline{re^{i(\theta+2\pi)}}$, $0 \le \theta \le 2\pi$

$$f(re^{i\theta}) = \frac{\sin\sqrt{re^{i\theta}}}{\sqrt{re^{i\theta}}} = \frac{\sin(r^{1/2}e^{i\theta/2})}{r^{1/2}e^{i\theta/2}}$$

$$f(re^{i(\theta+2\pi)}) = \frac{\sin(-r^{1/2}e^{i\theta/2})}{-r^{1/2}e^{i\theta/2}}$$

$$= \frac{\sin(r^{1/2}e^{i\theta/2})}{r^{1/2}e^{i\theta/2}}$$

$\Rightarrow f(z)$ has only one branch.

& $z=0$ is not a branch pt.

Since $\lim\limits_{z\to 0} \dfrac{\sin\sqrt{z}}{\sqrt{z}} = 1$, it follows that

$z=0$ in a removable singularity

Exercise: find analytic fn: $f(z) = u(r,\theta) + i v(r,\theta)$

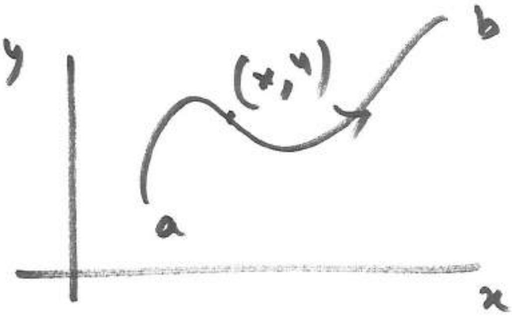
where $v(r,\theta) = r^2 \cos 2\theta - r\cos\theta + 2$.

Ans: $f(z) = i r^2 e^{2i\theta} - i r e^{i\theta} + c + 2i$

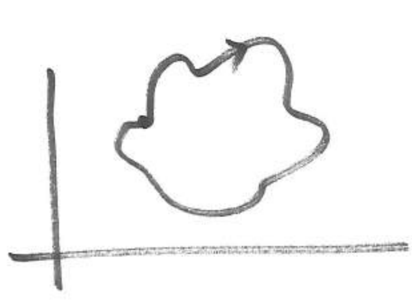
Complex Integration & Cauchy's Theorem.

- **Curve**

$$z = x + iy$$
$$= \phi(t) + i\psi(t)$$
$$= z(t), \quad t_1 \le t \le t_2 .$$



- $x = \phi(t)$, $y = \psi(t)$ $\rightarrow$ parametric eqn. of the curve

- 

$z(t_1) = z(t_2)$.

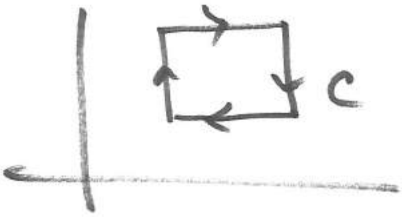
- Simple closed curve $\rightarrow$ a closed curve that does not intersect itself anywhere.

- $\phi'(t), \psi'(t)$ continuous $\rightarrow$ smooth curve.

- piecewise smooth curve / <u>contours</u>

  $\rightarrow$ composed of finite no. of smooth curves.

$\rightarrow$



- $\int_C f(z)\,dz$, $\oint_C f(z)\,dz$ $\rightarrow$ line integral or contour integral

- $f(z) = u(x,y) + iv(x,y) = u + iv$.

  $z = x + iy$
  $dz = dx + idy$.

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Example: $\int_C \bar{z}\, dz$ from $z=0$ to $z=4+2i$

i) along the curve $C: \ z = t^2 + it$

ii) along the ~~curve~~ line from $z=0$ to $z=2i$

& then the line from $z=2i$ to $z=4+2i$

Solⁿ. (i) $\displaystyle\int_{C:\, z=t^2+it} \bar{z}\, dz = \int_0^2 (t^2 - it)(2t+i)\, dt$

from $z=0$ to $z=4+2i$

$z = t^2 + it$

$dz = (2t + i)\, dt$

$z = 0 \Rightarrow t = 0$

$z = 4+2i \Rightarrow t = 2$

$= 10 - \dfrac{8i}{3}$.

Alternatively

parametric eqⁿ. of

$C: \ z = t^2 + it$

in

$\cancel{x(t) = t^2}$

$\boxed{x = t^2, \ y = t}$

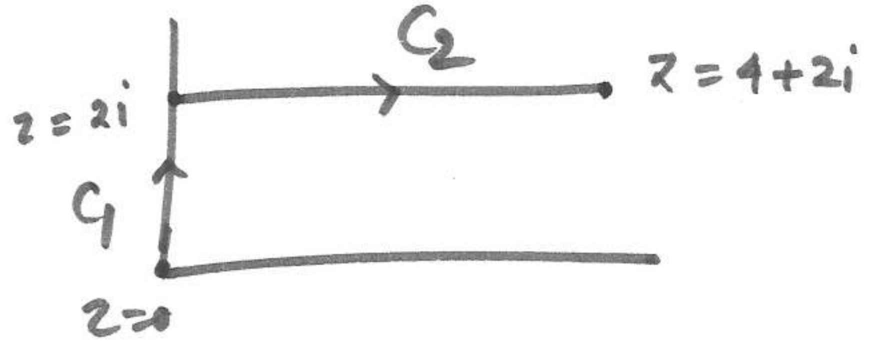
$$\int_C \bar{z}\,dz = \int_C (x-iy)(dx+idy)$$

$$= \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

$$= \int_0^2 t^2\,d(t^2) + t\,d(t) + i\int_0^2 t^2 d(t) - t\,d(t$$

$$= 10 - \frac{8i}{3}.$$

(ii)



$$\int_C \bar{z}\,dz = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

$C = C_1 \cup C_2$

Along C_1, $x=0$, $dx=0$, y from 0 to 2

$$\longrightarrow \int_0^2 0(0) + y\,dy + i\int_0^2 0\,dy - y0$$

$$= \frac{y^2}{2}\Big]_0^2 = 2$$

Along C_2, $y = 2$, $dy = 0$, x from 0 to 4

$$\rightarrow \int_0^4 x\,dx + 2(0) + i\int x(0) - 2\,dx$$

$$= 8 - 8i$$

$$\therefore \int_C \bar{z}\,dz = 2 + 8 - 8i = \boxed{10 - 8i}$$