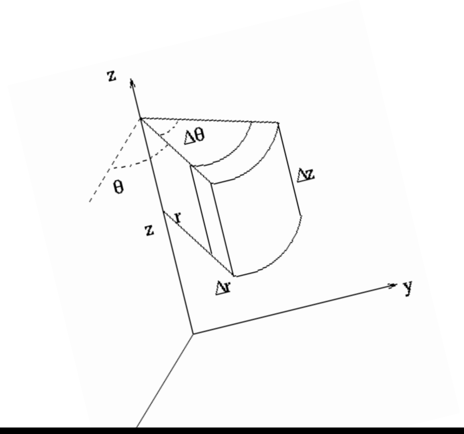
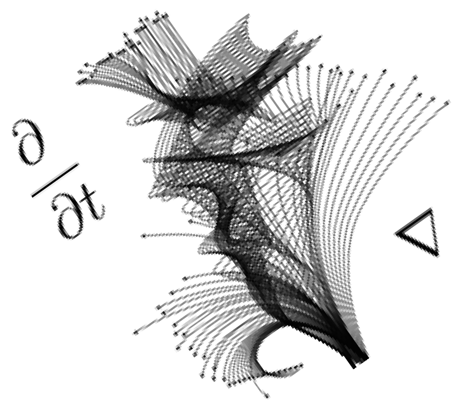
Mathematics-I

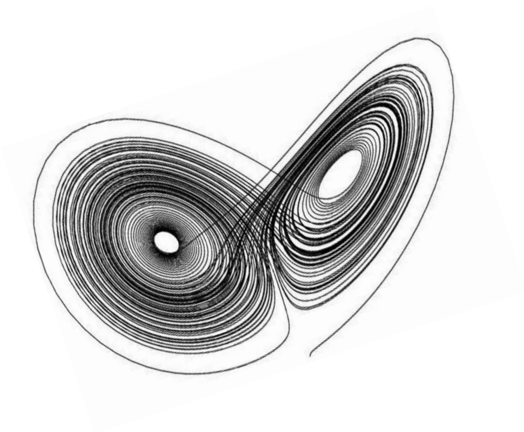
AUTUMN 2015

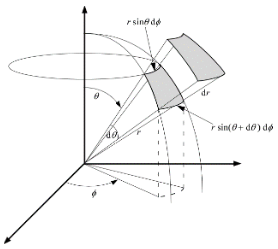
(MA10001)



Ordinary Differential Equation-III







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INDEX-III

(a) System of Simultaneous Linear Equations.

(b) System of Homogeneous Linear Differential Equation with Constant Co-efficient.

(c) Exact Differential Equation(1^{st} order and 1^{st} degree).

(d) Integrating Factor.

12.10.15

Systems of simultaneous linear eqns.

Example:

$$\frac{d^2x}{dt^2} - 3x - 4y = 0$$
$$\frac{d^2y}{dt^2} + x + y = 0.$$

$(D^2-3)x - 4y = 0$

$x + (D^2+1)y = 0$

$D \equiv \frac{d}{dt}.$

$$\cancel{(D^2-3)}x - 4y = 0$$
$$\cancel{(D^2-3)}x + (D^2+1)(D^2-3)y = 0$$
$$\underline{-} \qquad \underline{-}$$

$$(D^4 - 2D^2 + 1)\,y = 0.$$

Approach $\rightarrow$ Reducing to a single eqn. with const. co-efficients.

$$\rightarrow (m^2-1)^2 = 0$$

$$m = \pm 1, \pm 1$$

$$y = (c_1 + c_2 t)e^{t} + (c_3 + c_4 t)e^{-t}.$$

$$x = -(D^2+1)y = \; ?$$

Example:

$$\left.\begin{aligned} \frac{dy}{dx} &= y + z + x \\ \frac{dz}{dx} &= -4y - 3z + 2x \end{aligned}\right\} \quad \begin{aligned} (y)_{x=0} &= 1 \\ (z)_{x=0} &= 0. \end{aligned}$$

i.e.

$$\left.\begin{aligned} (D-1)y - z &= x \\ 4y + (D+3)z &= 2x \end{aligned}\right\}, \quad D \equiv \frac{d}{dx}$$

$$(D+3)(D-1)y - \cancel{(D+3)z} = \underline{(D+3)x}.$$

$$4y + \cancel{(D+3)z} = 2x$$

$$(D^2 + 2D + 1)y = 1 + 3x + 2x = 1 + 5x$$

$$(m+1)^2 = 0 \Rightarrow m = -1, -1.$$

$$y = (\underline{c_1} + \underline{c_2} x)\, e^{-x} + \text{P.I.}$$

$$\text{P.I} = \frac{1}{D^2 + 2D + 1}(1 + 5x).$$

$$= \left[1 - (D^2 + \underline{2D})\right](1 + 5x).$$

$$= 1 + 5x - 2\cdot 5 = 5x - 9$$

Sol^n. is
$$y = (10+6x)e^{-x} + 5x - 9$$
$$z = (-14 - 12x)\,e^{-x} - 6x + 14.$$

Example:

$$\frac{dx}{x^2-y^2-z^2} = \underbrace{\frac{dy}{2xy} = \frac{dz}{2xz}}$$
$$= \frac{x\,dx + y\,dy + z\,dz}{2x(x^2+y^2+z^2)}.$$

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log a$$
$$\Rightarrow y = az \quad \text{—} \ (1).$$

$$\frac{dz}{2xz} = \frac{d(x^2+y^2+z^2)}{(x^2+y^2+z^2)}.$$

$$\log z = 2\log(x^2+y^2+z^2) + \log b.$$

$$\Rightarrow \begin{cases} x^2+y^2+z^2 = cz \quad \text{—} \ (2). \\ y = az \quad \text{—} \ (1). \end{cases}$$

Sol^n., a, c are arb. consts.

Exercise:

(i) $$\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{dz}{(a-b)xy}$$

(ii) $$\frac{dx}{mz-ny} = \frac{dy}{nz-lz} = \frac{dz}{ly-mx}.$$

Example: Integrate the system

$$\left.\begin{aligned} \frac{dx}{dt} &= y+z \\ \frac{dy}{dt} &= x+z \\ \frac{dz}{dt} &= x+y. \end{aligned}\right\} \longrightarrow \frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{dz}{dt}$$

$$= 2x + z + y$$

$$= 2x + \frac{dx}{dt}.$$

$m = -1, 2.$

$$\boxed{x = c_1 e^{-t} + c_2 e^{2t},}$$ c_1, c_2 consts.

find z, y

$$\frac{dx}{dt} = y+z = -c_1 e^{-t} + 2c_2 e^{2t}$$

$$y = -c_1 e^{-t} + 2c_2 e^{2t} - z.$$

$$\frac{dz}{dt} = x + y = 3c_2 e^{2t} - z.$$

$\hookrightarrow$ linear in z.

$$\frac{dz}{dt} + z = 3c_2 e^{2t}.$$

$$I.F = e^{t}$$

$$\boxed{\begin{aligned} z &= c_3 e^{-t} + c_2 e^{2t}. \\ y &= -(c_1+c_3) e^{-t} + c_2 e^{2t}. \end{aligned}}$$ check

Approach II

Systems of homogeneous linear DE with const. co-efficients.

$$(1)\quad \begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \dfrac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ \dfrac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{cases}$$

↓ solve without reducing to an eqn. of n-th order.

. Seek for a particular soln. of ① in the following form:

$$② \text{---}\ x_1 = \alpha_1 e^{kt}, \quad x_2 = \alpha_2 e^{kt}, \ldots, x_n = \alpha_n e^{kt}$$

when $\alpha_1, \alpha_2, \ldots, \alpha_n, k$ are unknown, to be determined s.t. fun. in ② satisfy ①.

$$\cdot \begin{cases} k\alpha_1 e^{kt} = (a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n)e^{kt} \\ k\alpha_2 e^{kt} = (a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n)e^{kt} \\ \vdots \\ k\alpha_n e^{kt} = (a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n)e^{kt} \end{cases}$$

or

$$\left.\begin{aligned}
(a_{11}-k)\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n &= 0 \\
a_{21}\alpha_1 + (a_{22}-k)\alpha_2 + \cdots + a_{2n}\alpha_n &= 0 \\
\cdots \\
a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + (a_{nn}-k)\alpha_n &= 0
\end{aligned}\right\} - (3)$$

$$\Delta(k) = \begin{vmatrix} a_{11}-k & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-k & \cdots & a_{2n} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-k \end{vmatrix}$$

- $\Delta(k) \neq 0 \Rightarrow$ (3) has only trivial sol^n

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

$$\Downarrow$$

$$x_1(t) = x_2(t) = \cdots = x_n(t) = 0.$$

only trivial sol^n.

- (3) has non-trivial sol^n only for k s.t. $\Delta(k) = 0$.

$$\begin{vmatrix} a_{11}-k & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-k & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-k \end{vmatrix} = 0. \quad \text{--- (4)}$$

auxiliary equ[illegible] of the system (1).

k

real, distinct ✓

complex, distinct

equal (real/complex) ✗

Case I

(The roots of the auxiliary equ^n. are real & distinct).

- $k_1, k_2, \ldots, k_n$

$k_i \longrightarrow \alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_n^{(i)}$

$\longrightarrow x_1^{(i)} = \alpha_1^{(i)} e^{k_i t}, \; x_2^{(i)} = \alpha_2^{(i)} e^{k_i t}, \ldots, x_n^{(i)} = \alpha_n^{(i)} e^{\cdots}$

General sol^n. of (1)

$$x_1 = \sum_{i=1}^{n} c_i \, \alpha_1^{(i)} e^{k_i t}$$

$$\left.\begin{aligned} x_2 &= \sum_{i=1}^{n} c_i \alpha_2^{(i)} e^{k_i t} \\ &\vdots \\ x_n &= \sum c_i \alpha_n^{(i)} e^{k_i t}. \end{aligned}\right\}$$

$c_1, c_2, \dots, c_n$ are arb. consts.

Example:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + 2x_2 \\ \frac{dx_2}{dt} &= x_1 + 3x_2 \end{aligned}\right\} \text{—} (1).$$

Sol^n:

put
$$\boxed{\begin{aligned} x_1 &= \alpha_1 e^{kt} \\ x_2 &= \alpha_2 e^{kt} \end{aligned}}$$

$$\begin{vmatrix} 2-k & 2 \\ 1 & 3-k \end{vmatrix} = 0$$

$$k = 1, 4. \quad (2) \text{—} \left\{\begin{aligned} (2-k)\alpha_1 + 2\alpha_2 &= 0 \\ \alpha_1 + (3-k)\alpha_2 &= 0 \end{aligned}\right.$$

$k_1 = 4$

$$(2) \Rightarrow \left.\begin{aligned} (2-4)\alpha_1^{(1)} + 2\alpha_2^{(1)} &= 0. \\ \alpha_1^{(1)} + (3-4)\alpha_2^{(1)} &= 0. \end{aligned}\right\} \Rightarrow \alpha_2^{(1)} = \alpha_1^{(1)}$$

Take $\alpha_1^{(1)} = 1$, Then $\alpha_2^{(1)} = 1$.

$$\boxed{\begin{aligned} x_1^{(1)} = \alpha_1^{(1)} e^{k_1 t} = e^{4t}, \quad x_2^{(1)} &= \alpha_2^{(1)} e^{k_1 t} \\ &= e^{4t} \end{aligned}}$$

<u>$k_2 = 1$</u>

$$\left.\begin{aligned} (2-1)\,\alpha_1^{(2)} + 2\,\alpha_2^{(2)} &= 0 \\ \alpha_1^{(2)} + (3-1)\,\alpha_2^{(2)} &= 0 \end{aligned}\right\}$$

$$\Downarrow$$

$$\alpha_2^{(1)} = -\frac{1}{2}\,\alpha_1^{(2)}.$$

Let $\alpha_1^{(2)} = 1$

Thus $\alpha_2^{(1)} = -\frac{1}{2}$

$$\therefore \boxed{\begin{aligned} x_1^{(2)} = \alpha_1^{(2)} e^{k_2 t} = e^{t}, \quad x_2^{(2)} &= \alpha_2^{(2)} e^{k_2 t} \\ &= -\frac{1}{2} e^{t}. \end{aligned}}$$

$\therefore$ General soln of ① is

$$\left.\begin{aligned} x_1 &= C_1 e^{4t} + C_2 e^{t} \\ x_2 &= C_1 e^{4t} - \frac{1}{2} C_2 e^{t} \end{aligned}\right\} \quad C_1, C_2 \text{ arb. consts.}$$

Case II (The roots of the auxiliary eqn. are complext, distinct).

- $K_1 = \alpha + i\beta, \quad K_2 = \alpha - i\beta.$

$\downarrow$ cor. sol^n.

$$x_j^{(1)} = \alpha_j^{(1)} e^{(\alpha+i\beta)t}$$
$$x_j^{(2)} = \alpha_j^{(2)} e^{(\alpha-i\beta)t}.$$

$\alpha_j^{(1)}, \alpha_j^{(2)}$ are determined from system (3).

$\Downarrow$ gen. sol^n.

$$x_j^{(1)} = e^{\alpha t}\left(\lambda_j^{(1)} \cos \beta t + \lambda_j^{(2)} \sin \beta t\right).$$

$$x_j^{(2)} e^{\alpha t}\left(\bar{\lambda}_j^{(1)} \cos \beta t + \bar{\lambda}_j^{(2)} \sin \beta t\right).$$

<u>Exercise</u>: $\frac{dx_1}{dt} = -7x_1 + x_2$

$\frac{dx_2}{dt} = -2x_1 - 5x_2$.

aux. eqn.

$$\begin{vmatrix} -7-k & 1 \\ -2 & -5-k \end{vmatrix} = 0. \Rightarrow \underline{k = -6 \pm i}$$

. System of <u>linear</u> DE of higher order with <u>const. co-efficients.</u>

<u>Example</u>:
$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= a_{11}x + a_{12}y \\ \frac{d^2y}{dt^2} &= a_{21}x + a_{22}y. \end{aligned}\right\}$$

. Seek soln. of the form
$x = \alpha e^{kt}$, $y = \beta e^{kt}$.

aux. eqn. $\longrightarrow$ $\begin{vmatrix} a_{11}-k^2 & a_{12} \\ a_{21} & a_{22}-k^2 \end{vmatrix} = 0.$

<u>check.</u>

Exact DE. (1st. order 1st degree).

- $M\,dx + N\,dy = 0$, $M = M(x,y)$, $N = N(x,y)$
- $M, N \rightarrow$ continuous differentiable fns. s.t.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

- $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x} \rightarrow$ continuous in some region

Soln. $\int M\,dx + \int \left(\text{part of } N \text{ not } \sout{\text{indep of}} \text{ involving } x \right) dy = c$

why?

- $M\,dx + N\,dy$ exact $\Rightarrow \exists$ some $u = u(x,y)$

s.t. $M\,dx + N\,dy = du = u_x\,dx + u_y\,dy$, $u_x = M$, $u_y = N$

$\Downarrow$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$\frac{\partial M}{\partial y} = u_{yx}$, $\frac{\partial N}{\partial x} = u_{xy}$.

- $u = ?$ $M\,dx + N\,dy = du = 0$

$\Rightarrow$ $\underline{u = \text{const}}$

$$\cdot\quad M = \frac{\partial u}{\partial x} \Rightarrow u(x,y) = \int M\,dx + \phi(y).$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int M\,dx + \phi'(y)$$

$$N = \int \underbrace{\frac{\partial M}{\partial y}}_{=\frac{\partial N}{\partial x}}\,dx + \phi'(y).$$

- using Leibnitz's rule of differentiation under the integral sign.

$$\Rightarrow N = \int \frac{\partial N}{\partial x}\,dx + \phi'(y).$$

$$\text{or } \phi'(y) = N - \int \frac{\partial N}{\partial x}\,dx.$$

$$\text{or } \phi(y) = \int \left[N - \int \frac{\partial N}{\partial x}\,dx\right] dy.$$

$$\therefore u(x,y) = \int M\,dx + \phi(y).$$

Const.

$$= \int M\,dx + \int \underbrace{\left[N - \int \frac{\partial N}{\partial x}\,dx\right]}_{\text{part of } N \text{ involving only } y} dy.$$

Integrating factor (Justification).

- $M\,dx + N\,dy = 0$, not exact.

Imp!.

(i) if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, then

$$\text{I.F.} = e^{\int f(x)\,dx}.$$

(ii) if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$, then

$$\text{I.F.} = e^{\int f(y)\,dy}.$$

- let $\mu \to$ an integrating factor

$\mu = \mu(x, y)$.

$\Rightarrow M\mu\,dx + N\mu\,dy = 0$. is exact.

i.e. $$\frac{\partial}{\partial y}(M\mu) = \frac{\partial}{\partial x}(N\mu).$$

or $$M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}.$$

or $$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

or $$\underbrace{M\frac{\partial}{\partial y}(\log \mu)}_{=0} - N\frac{\partial}{\partial x}(\log \mu) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

(i) $\mu = \mu(x)$.

$$- N\frac{\partial}{\partial x}(\log \mu) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

or $$\mu = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx}.$$

$\mu(x)$

a fn. of x alone.

(ii) $\Rightarrow$ $\underline{\mu = \mu(y)}$