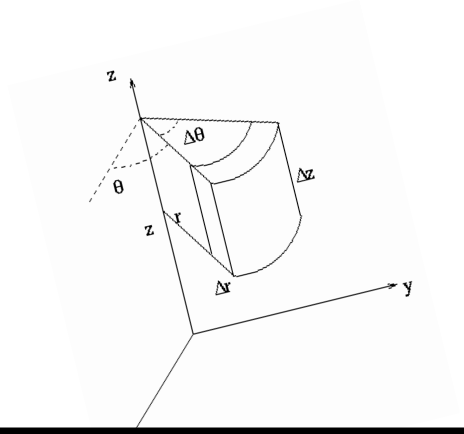
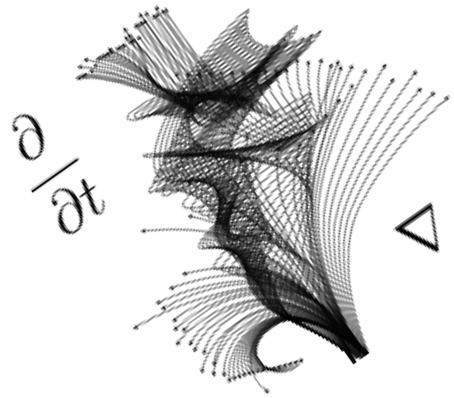
Mathematics-I

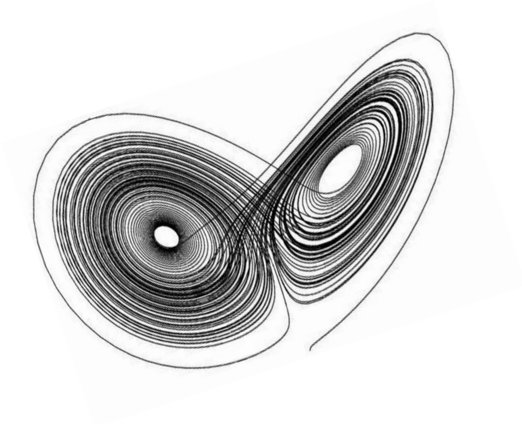
AUTUMN 2015

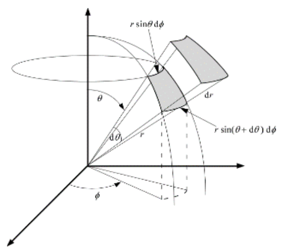
(MA10001)



Ordinary Differential Equation-II







Dr. Ratna Dutta

Department of Mathematics

Indian Institute of Technology

Kharagpur- 721302

INDEX-II

· linear DE with const. co-eff.

$$\boxed{f(D)y = X}, \quad D \equiv \frac{d}{dx}$$

$$f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n,$$

$P_1, \dots, P_n \to$ const.

$X \to$ a fun. of x.

$$y = \text{C.F} + \text{P.I}.$$

$$f(D)y = 0., \text{ say } Y$$

$u = $ P.I.

$$y = Y + u$$

$$f(D)y = f(D)Y + f(D)u.$$

$$X \qquad\qquad 0$$

$$f(D)u = X$$

$$\boxed{u = \frac{1}{f(D)} X}$$

Short methods to find P.I.

$X = $
- $\to e^{ax}$, a const.
- $\to x^{m}$, m +ve integer
- $\to \sin ax$ or $\cos ax$
- $\to e^{ax} V$, $V = V(x)$.

I) $\frac{1}{f(D)} e^{ax}$.

Case-I. $f(a) \neq 0$

$$P.I = \frac{1}{f(a)} e^{ax}, \quad f(a) \neq 0.$$

Case II $f(a) = 0$.

let $f(D) = (D-a)^{l} \phi(D)$

where $\phi(a) \neq 0$.

$$P.I = \boxed{\cancel{\frac{1}{\phi(a)} e^{ax} \cdot \frac{1}{(D-a)^{l}} \cdot 1}}$$

$$\boxed{P.I. = \frac{1}{\phi(a)} e^{ax} \cdot \frac{x^{l}}{l!}}$$

$$Dy = \frac{d}{dx} y.$$

$$\frac{1}{D} y = \int y \, dx.$$

Example: Solve

$$\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}.$$

Sol^n. **auxiliary eqn.**

$$y = e^{mx}.$$

$$m^3 + 1 = 0 \Rightarrow m = -1, \frac{1 \pm \sqrt{3}i}{2}$$

$$C.F = C_1 e^{-x} + e^{\frac{1}{2}x}\left[C_2 \cos \frac{\sqrt{3}}{2} x + C_3 \sin \frac{\sqrt{3}}{2} x\right].$$

$$P.I. = \frac{1}{D^3+1}\left(3e^{0.x} + e^{-x} + 5e^{2x}\right)$$ Let $D \equiv \frac{d}{dx}$

$$= \frac{1.3}{0^3+1} e^{0.x} + 5.\frac{1}{2^3+1} e^{2x}$$

$$+ \frac{1}{D^3+1} e^{-x}.$$

$$\frac{1}{D^3+1}\,e^{-x} = \frac{1}{(D+1)\underbrace{(D^2-D+1)}_{\phi(D)}}\,e^{-x}.$$

$$\cancel{= e^{-x} \cdot \frac{1}{(-1)^2-(-1)+1} \cdot \frac{1}{D+1} \cdot 1}$$

$$= \frac{e^{-x}}{3}\,x$$

$$\therefore \boxed{P.I = 3 + \frac{5}{9}e^{2x} + \frac{e^{-x}}{3}x.}$$

$\therefore$ The complete Sol^n: is

$$y = C.F + P.I.$$

$$= C_1 e^{-x} + e^{\frac{x}{2}}\left[C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right]$$

$$+ 3 + \frac{5}{9}e^{2x} + \frac{e^{-x}}{3}x.$$

II) $X = x^m$, m is +ve int. $m > 0$.

$$f(D) = D^n + P_1 D^{n-1} + \cdots + P_l D^l,$$

$l \rightarrow$ lowest degree of D.

$$P.I = \frac{1}{f(D)} x^m.$$

$$= \frac{1}{P_l D^l \left[1 + \frac{D^n + P_1 D^{n-1} + \cdots + P_{l-1} D^{l-1}}{P_l D^l}\right]} \times x^m.$$

$$= \frac{1}{P_l D^l} \left[1 + \left(\frac{P_{l-1}}{P_l} D + \cdots\right)\right]^{-1} x^m.$$

$\swarrow$ binomial expansion.

Example: $(D^3+3D^2+2D)\,y = x^2.$

Sol^n: C.F. $\quad y = e^{mx}.$

$$m^3+3m^2+2m=0$$

$$\Rightarrow m = 0, -1, -2$$

$$C.F = C_1 + C_2 e^{-x} + C_3 e^{-2x}, \quad C_1, C_2, C_3 \text{ const}^s.$$

$$P.I. = \frac{1}{D^3+3D^2+\underline{\underline{2D}}}\, x^2$$

$$= \frac{1}{2D}\left[1+\frac{D^3+3D^2}{2D}\right]^{-1} x^2$$

$$= \frac{1}{2D}\left[1 - \frac{D^2+3D}{2} + \left(\frac{D^2+3D}{2}\right)^2 \frac{(-1)(-2)}{2!} + \cdots\right] x^2$$

$$= \frac{1}{2D}\left[x^2 - \frac{1}{2}(3.2x+2) + \frac{\cancel{2}}{\cancel{2}}.\frac{1}{\cancel{4}_2}9.2\right]$$

$$= \frac{1}{2D}\left[x^2 - (3x+1) + \frac{9}{2}\right]$$

$$= \frac{1}{2D}\left[x^2 - 3x + \frac{7}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{7}{2}x\right].$$

(Note: $Dy \to$ differentiation once
$\frac{1}{D}y \to$ integration once)

III A) $\frac{1}{f(D)} \cos ax$, a is any const.

↳ replace D^2 by $(-a^2)$

• $P.I. = \frac{1}{f(-a^2)} \cos ax$ if $f(-a^2) \neq 0$.

• if $f(-a^2) = 0$

$$P.I. = Re\left[\frac{1}{f(D)} e^{iax}\right]$$

B) Similarly $\sin ax$

if $f(-a^2) \neq 0$, replace D^2 by $-a^2$

if $f(-a^2) = 0$, then $Im\left[\frac{1}{f(D)} e^{iax}\right]$

Example: find the P.I. of

$$(D^3+D^2-D-1)\,y = \cos 2x.$$

Solⁿ:

$$P.I. = \frac{1}{D^3+D^2-D-1}\cos 2x.$$

$$= \frac{1}{(-2^2)\cdot D + (-2^2) - D - 1}\cos 2x.$$

$$= \frac{1}{-4D-4-D-1}\cos 2x.$$

$$= \frac{1}{-5D-5}\cos 2x.$$

$$= -\frac{1}{5}\cdot\frac{1}{D+1}\cos 2x.$$

$$= -\frac{1}{5}\,\frac{D-1}{D^2-1}\cos 2x.$$

$$= -\frac{1}{5}\,\frac{(D-1)}{-(2)^2-1}\cos 2x.$$

$$= \frac{1}{25}(D-1)\cos 2x = \frac{1}{25}(-2\sin 2x - \cos 2x)$$

Example: $(D^2+a^2)y = \cos ax$.

Sol^n P.I. = ?

$$P.I. = \frac{1}{D^2+a^2}\cos ax.$$

$$= Re\left[\frac{1}{D^2+a^2}e^{iax}\right]$$

$$= Re\left[\frac{1}{(D+ia)(D-ia)}e^{iax}\right]$$

$$= Re\left[\frac{e^{iax}}{2ia}\cdot\frac{1}{(D-ia)}1\right].$$

$$= Re\left[\frac{e^{iax}}{2ia}x\right].$$

$$= Re\left[\frac{\cos ax + i\sin ax}{2ia}x\right]$$

$$= \frac{x\sin ax}{2a}$$

$\underline{IV} >$ $\dfrac{1}{f(D)} e^{ax} V$, V any fn. of x, V is not const.

$$P.I. = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V$$

Example: Solve $(D^2+1)y = xe^{2x}$.

Soln. C.F. $= A\cos x + B\sin x$ $\quad m = \pm i$

$$P.I. = \frac{1}{D^2+1} xe^{2x}.$$

$$= e^{2x} \cdot \frac{1}{(D+2)^2+1} x$$

$$= e^{2x} \cdot \frac{1}{D^2+4D+5} x$$

$$= \frac{e^{2x}}{5} \cdot \left[1 - \frac{D^2+4D}{5} + \cdots\right] x.$$

$$= \frac{e^{2x}}{5} \left[x - \frac{4}{5} \cdot 1\right] = \frac{e^{2x}}{25}(5x-4).$$

Example: $(D^2 - 2D + 1)y = x e^x \sin x$.

P.I. = ?

Ans. $-e^x (x \sin x + 2\cos x)$.

Example: $(D^2 - 1)y = x^2 \cos x$.

$$P.I = Re\left[\frac{1}{D^2 - 1} x^2 e^{ix}\right]$$

apply the rule for $X = e^{ax} V(x)$

replace D by $D+i$ to take e^{ix} out

Justification for the form of C.F

- $f(D)\,y = 0$ — (1)

↳ linear DE with const. co-efficient of order n.

$y_1, y_2 \to$ two sol^{n}s of (1).

$f(D)\,y_1 = 0 = f(D)\,y_2$

$$f(D)\left(c_1 y_1 + c_2 y_2\right) = c_1 \underbrace{f(D)\,y_1}_{=0} + c_2 \underbrace{f(D)\,y_2}_{=0}$$

c_1, c_2 consts.

$$= 0.$$

- auxiliary eqn. $y = e^{mx}$.

$f(m) = 0 \longrightarrow m_1, m_2, \ldots, m_n$

Case I all distinct.

↓ n consts. in C.F (done).

Case II $m_1 = m_2 = m$.

↓ contribution in the sol^n.

$(c_1 + c_2 x)\, e^{mx}$.

why ??

↓ ∵ 2 const in C.F part (done).

- $f(D) = (D-m)^2 \phi(D)$

$$(D-m)^2 y = 0.$$

$$(D-m) v = 0, \text{ when } \underline{\underline{v = (D-m) y.}}$$

$$\Downarrow$$

$$v = c_1 e^{mx}.$$

linear.

$$(D-m) y = c_1 e^{mx}$$

$$\text{I.F} = e^{-\int m\,dx} = e^{-mx}.$$

Sol^n.

$$d\left(y e^{-mx}\right) = c_1 e^{mx} \cdot e^{-mx} dx.$$

$$= c_1\,dx.$$

$$y = c_1 e^{mx} x + c_2 e^{mx}.$$

- $(D-m)^2 y = 0 \longrightarrow y = (c_2 + c_1 x) e^{mx}.$

Similarly,

- $(D-m)^3 y = 0. \longrightarrow y = (c_3 + c_2 x + c_1 x^2) e^{mx}$

- $X \begin{cases} \to e^{ax}, & \cancel{a\neq 0}\ .\ a \text{ is any const., may be } 0 \\ \to x^m, & m > 0, \text{ +ve integer} \\ \to \sin ax, \cos ax, & a \text{ is any const.} \\ \to e^{ax} V(x) \end{cases}$

$X = \sec ax$ or $\tan ax$ or $\cot ax$??

- General Method to find P.I.

a) $\dfrac{1}{f(D)}$ may be factored.

$$\text{P.I.} = \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)} X$$

$\dfrac{1}{D-m_n} X = y$. $\to$ linear in y.

$(D-m_n)y = X$., I.F $= e^{-m_n x}$.

$$d\left(y e^{-m_n x}\right) = X e^{-m_n x}\, dx$$

$$\boxed{y = e^{m_n x} \cdot \int X e^{-m_n x}\, dx.}$$

$\hat{X}$

$\hookrightarrow$ no const. as P.I.

$$P.I. = \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_{n-1})} \cdot \hat{X}$$

$$= \frac{1}{D-m_1} \cdot \frac{1}{D-m_2} \cdots \frac{1}{D-m_{n-2}} e^{m_{n-1}x} \int \hat{X} e^{-m_{n-1}x} dx$$

$= \ldots\ldots$

$$\boxed{P.I. = e^{m_1 x} \int e^{(m_2-m_1)x} \int e^{(m_3-m_2)x} \int \cdots \cdots \int X e^{-m_n x} (dx)^n.}$$

$\hookrightarrow$ check with

<u>Exercise.</u> $X = e^{ax}$.

$$P.I = \frac{1}{f(D)} X \to \begin{cases} f(a) \neq 0 \\ f(a) = 0 \end{cases}$$

(b) b) $\frac{1}{f(D)}$ may be decomposed into its partial fractions.

$$P.I. = \left[\frac{N_1}{D-m_1} + \frac{N_2}{D-m_2} + \dots + \frac{N_n}{D-m_n}\right] X.$$

$$\frac{N_n}{D-m_n} X = y.$$

$$(D-m_n) y = X N_n \quad \underline{\text{fu}^n \text{ of } x}$$

linear in y.

$$I.F = e^{-m_n x}$$

$$d\left(y e^{-m_n x}\right) = X N_n e^{-m_n x} dx$$

$$y = e^{m_n x} \int X N_n e^{-m_n x} \, dx.$$

Example: $(D^2+a^2)y = \sec ax$.

$$P.I. = \frac{1}{(D+ia)(D-ia)} \sec ax.$$

$$= -\frac{1}{2ia}\left[\frac{1}{D+ia} - \frac{1}{D-ia}\right] \sec ax.$$

$$\frac{1}{D+ia} \sec ax = y.$$

$(D+ia)\,y = \sec ax$ (linear in y)

$$I.F = e^{iax}.$$

$$y = e^{-iax} \int e^{iax} \sec ax \, dx.$$

$$= e^{-iax} \int \frac{\cos ax + i\sin ax}{\cos ax}\, dx.$$

$$= e^{-iax} \int (1 + i\tan ax)\, dx.$$

$$= e^{-iax}\left[x - \frac{i}{a}\log(\cos ax)\right].$$

$$\text{P.I.} = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log(\cos ax)$$

Exercise

i) $(D^2+a^2)y = \begin{cases} \tan ax \\ \cot ax. \end{cases}$

ii) $(D^2+1)y = \sec^2 x.$

Linear DE with variable co-efficients

$$\frac{d^i y}{dx^i} \to \text{coeff. } P_i,$$

$P_i =$ a f^n. of x

$$P_i = x^i \times (\text{const.})$$

Homogeneous linear eqn. (The Cauchy's linear eqn.)

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + p_{n-1} x \frac{dy}{dx} + p_n y = X$$

where $p_1, p_2, \dots, p_n$ are consts., & X is a f^n. of x.

• $X = 0 \rightarrow$ Cauchy - Euler's eqn
or
Euler's eqn.

Solution Method.

(Changing the independent variable)

put $x = e^z$ or $z = \log x$.

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\boxed{D \equiv \frac{d}{dz}}$$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y.$$

$$x^n\frac{d^ny}{dx^n} = D(D-1)\cdots(D-n+1)y$$

verify.

Example: Solve $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2\log x$ —— (1).

Sol^n. let $z = \log x$, $D \equiv \dfrac{d}{dz}$.

$$(1) \Rightarrow \left[D(D-1) - D + 1\right] y = 2z$$

$m = 1, 1$

$$C.F = (C_1 + C_2 z)\, e^{z}$$

$$P.I. = \frac{1}{(D-1)^2} \cdot 2z$$

$$= \cancel{2 \cdot \frac{1}{1-}}$$

$$= 2.\left[1 - (D^2 - 2D) + \cdots\right] z.$$

$$= 2\left[z + 2\right].$$

$\therefore$ The general sol^n is

$$y = x(C_1 + C_2 \log x)\, \cancel{e^{\log x}} + 2\left[\log x + 2\right]$$

C_1, C_2 consts.

Exercise $\quad x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4.$

Example: $\quad (5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 0.$

reducible to homogeneous eqn.

put $5+2x = z$.

$$2^2 z^2 \frac{d^2y}{dz^2} - 6 \cdot 2z \frac{dy}{dz} + 8y = 0.$$

rewrite as Euler's eqn.

$$z^2 \frac{d^2y}{dz^2} - 3z \frac{dy}{dz} + 2y = 0.$$

put $z = e^t$, $D \equiv \frac{d}{dt}$.

$$\left[D(D-1) - 3D + 2\right] y = 0.$$

$$y = c_1 (5+2x)^{2+\sqrt{2}} + c_2 (5+2x)^{2-\sqrt{2}},$$

c_1, c_2 consts.

Eqn^{ns} reducible to homogeneous linear form.

$$(a+bx)^n \frac{d^n y}{dx^n} + p_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots$$

$$\cdots + p_{n-1} (a+bx) \frac{dy}{dx} + p_n y$$

$$= F(x).$$

(legendre's linear eqn.).

7.10.15.

Example: find the value of λ for which all solutions of

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - \lambda y = 0 \quad \text{--- (1).}$$

tend to 0 as $x \to \infty$

Sol^n.

put $z = \log x$

$$D \equiv \frac{d}{dz}$$

(1) $\Rightarrow$

$$\left[D(D-1) + 3D - \lambda \right] y = 0$$

$$(D^2 + 2D - \lambda) y = 0. \quad \text{--- (2).}$$

put $y = e^{mz}$

$\therefore$ auxiliary eqn. $\rightarrow$ $m^2 + 2m - \lambda = 0.$

$$m = -1 \pm \sqrt{1+\lambda}.$$

Case I. $1+\lambda \geqslant 0$ i.e. $\lambda \geqslant -1$.

$$y = c_1 e^{-[1+\sqrt{1+\lambda}]z} + c_2 e^{-[1-\sqrt{1+\lambda}]z}$$

$= c_1 x^{-[1+\sqrt{1+\lambda}]} + c_2 x^{-[1-\sqrt{1+\lambda}]}$

$\longrightarrow 0$ as $x \to \infty$

provided $1 - \sqrt{1+\lambda} > 0$

i.e. if $\lambda < 0$.

Ans. $\underline{-1 \leq \lambda < 0}$ or $\underline{\lambda < -1}$

case II $1+\lambda < 0 \longrightarrow$

check.

~~Ans.~~

Example: Solve

$$x^3 \frac{d^3y}{dx^3} + 2x\frac{dy}{dx} - 2y = x^2 \log x + 3x \quad \text{--- (1)}$$

Sol^n. $z = \log x$, $D \equiv \frac{d}{dz}$, i.e. $x = e^z$

$$(1) \Rightarrow [D(D-1)(D-2) + 2D - 2]y$$
$$= e^{2z} z + 3e^z. \quad \text{--- (2)}.$$

C.F. put $y = e^{mz}$

$$m(m-1)(m-2) + 2m - 2 = 0.$$

$$m = 1, \quad 1 \pm i$$

$$\text{C.F} = C_1 e^{z} + e^{z}\left[C_2 \cos z + C_3 \sin z\right].$$

P.I.

$$\frac{1}{(D-1)(D^2-2D+2)}\left(ze^{2z} + 3e^{z}\right)$$

$$= e^{2z} \cdot \frac{1}{(D+2-1)\left((D+2)^2 - 2(D+2) + 2\right)} z$$

$$+ \frac{3}{1^2 - 2\cdot 1 + 2} e^{z} \cdot \frac{z}{1}$$

$$= \frac{x^2}{2}(\log x - 2) + 3x \log x$$

<u>Check.</u>

Example: Solve

$$(x^2D^2 + xD + 1)y = \log x . \sin \log x \quad \text{--- (1)}.$$

Sol^n: put $z = \log x$ i.e. $x = e^z$

& let $D' \equiv \frac{d}{dz}$.

(1) $\Rightarrow$

$$[D'(D'-1) + D' + 1]y = z \sin z.$$

put $y = e^{mz}$

auxiliary eqn $\longrightarrow$ $m(m-1) + m + 1 = 0$

$\Rightarrow$ $m = \pm i = 0 \pm i$

$$C.F = e^{0z}\left[A \cos z + B \sin z\right]$$

$= A \cos z + B \sin z$, A, B are consts.

$$P.I. = \frac{1}{(D_1^2+1)} z \sin z.$$

$$= \mathrm{Im}\left[\frac{1}{D_1^2+1}\, z e^{iz}\right]$$

$$= \mathrm{Im}.\left[e^{iz}\cdot\frac{1}{(D_1+i)^2+1}\, z\right]$$

$$= \mathrm{Im}.\left[e^{iz}\cdot\frac{1}{D_1^2+\underline{\underline{2iD_1}}}\, z\right]$$

$$= \mathrm{Im}.\left[e^{iz}\cdot\frac{1}{2iD_1}\left[1+\frac{D_1}{2i}\right]^{-1} z\right]$$

$$= \mathrm{Im}.\left[e^{iz}\cdot\frac{1}{2iD_1}\cdot\left(1-\frac{D_1}{2i}+\cdots\right)z\right]$$

$$= \mathrm{Im}\left[e^{iz}\cdot\frac{1}{2iD_1}\left(z-\frac{1}{2i}\right)\right]$$

$$= \mathrm{Im}\left[\frac{e^{iz}}{2i}\left(\frac{z^2}{2}-\frac{1}{2i}z\right)\right].$$

$$= \frac{1}{4}\log x \sin\log x - \frac{1}{4}(\log x)^2\cos\log x.$$

Variation of Parameters.

(linear equ^{ns}. with const. co-efficients).

① — $(D^n + P_1 D^{n-1} + \cdots + P_n)y = X$, $\quad$ || sec ax, tan ax

$P_1, \cdots, P_n \to$ consts.

$X \to$ a fn. of x., $\quad D \equiv \frac{d}{dx}$.

· $C.F = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$.

· To find particular integral

- Let P.I. be

② — $y = L_1(x) y_1(x) + L_2(x) y_2(x) + \cdots + L_n(x) y_n(x)$.

$\to$ replacing $C_1, C_2, \cdots, C_n$ in C.F by $L_1(x), L_2(x), \cdots, L_n(x)$ resp^{ly}.

- $\boxed{\text{Determine } L_1(x), L_2(x), \cdots, L_n(x) \text{ so that ② satisfies ①.}}$

<u>Example:</u> Solve $(D^2 - 2D)y = e^x \sin x$ —— (1)

using variation of parameters.

<u>Sol</u>

$m^2 - 2m = 0$

$m = 0, 2$

$C.F = C_1 e^{0x} + C_2 e^{2x}$, C_1, C_2 consts.

$= C_1 + C_2 e^{2x}$

Let P.I is $y = L_1(x) + L_2(x) e^{2x}$

$= L_1 + L_2 e^{2x}$.

$Dy = (L_1' + L_2' e^{2x}) + 2L_2 e^{2x}$.

Set $L_1' + L_2' e^{2x} = 0$ —— (2).

$$\left.\begin{aligned} \therefore Dy &= 2L_2 e^{2x} \\ \therefore D^2 y &= 2L_2' e^{2x} + 4L_2 e^{2x} \end{aligned}\right\} \text{—— (3)}$$

(3), (1) $\Rightarrow$

$$2L_2' e^{2x} + \cancel{4L_2 e^{2x}} - \cancel{2.2L_2 e^{2x}}$$

$$= e^x \sin x$$

$$④ \;—\; L_2' = \frac{e^{-x}\sin x}{2}$$ solving

$$\therefore L_2 = -\frac{1}{4}e^{-x}(\cos x + \sin x)$$

$$②, ④ \Rightarrow \quad L_1' = -L_2' e^{2x}$$

$$= -\frac{e^{x}\sin x}{2}$$ Solve

$$\therefore L_1 = -\frac{1}{4}e^{x}(-\cos x + \sin x)$$

$$\therefore \text{P.I.} = L_1 + L_2 e^{2x}$$

$$= -\frac{1}{2}e^{x}\sin x.$$

Method of Variation of Parameter

$$(D^2 + PD + Q)y = R \quad \text{—— (1).}$$

$\boxed{C.F = C_1 u + C_2 v}$, C_1, C_2 consts, u, v are two linearly independent sol^{n}. of $(D^2 + PD + Q)y = 0$.

Let P.I. be $y = L_1 u + L_2 v$, $L_1 = L_1(x)$, $L_2 = L_2(x)$

Then $Dy = \underbrace{L_1' u + L_2' v}_{= 0} + L_1 u' + L_2 v'$.

$= L_1 u' + L_2 v'$, Setting $\underline{L_1' u + L_2' v = 0}$ —— (2)

$$D^2 y = L_1 u'' + L_2 v'' + L_1' u' + L_2' v'.$$

Substitute Dy, $D^2 y$ in (1)

$$L_1(\underbrace{u''+Pu'+Qu}_{0}) + L_2(\underbrace{v''+Pv'+Qv}_{0}) + L_1'u' + L_2'v' = R.$$

$$\Rightarrow \quad L_1'u' + L_2'v' = R \qquad (3).$$

$$\left.\begin{aligned} L_1'u + L_2'v &= 0 \\ L_1'u' + L_2'v' - R &= 0 \end{aligned}\right\}$$

$$\frac{L_1'}{-Rv} = \frac{L_2'}{Ru} = \frac{1}{uv'-u'v}.$$

$$\boxed{\begin{aligned} L_1' &= -\frac{Rv}{uv'-u'v} \\ L_2' &= \frac{Ru}{uv'-u'v}. \end{aligned}}$$

- W = Wronskian of u, v.

$$= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

$$L_1 = -\int \frac{vR}{W} + A$$

$$L_2 = \int \frac{uR}{W} + B.$$

Gen. Soln.

$$y = l_1 u + l_2 v.$$

Third order DE.

$$(D^3 + PD^2 + QD + R)\, y = S. \quad \text{—— (1).}$$

$u, v, w \rightarrow$ 3 ind. Soln. of (1) with $S = 0$.

P.I.

$$y = L_1(x)\, u + L_2(x)\, v + L_3(x)\, w \quad \text{—— (2).}$$

Wronskian of u, v, w

$$W = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix}$$

$$L_1' = \frac{S}{W} \begin{vmatrix} v & w \\ v' & w' \end{vmatrix}, \quad L_2' = -\frac{S}{W} \begin{vmatrix} u & w \\ u' & w' \end{vmatrix}$$

$$L_3' = \frac{S}{W} \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

Solving $\longrightarrow$ $L_1, L_2, L_3 \longrightarrow$ P.I.

Exercise $(D^3 + D)y = \sec x$.

Solve using variation of parameter.