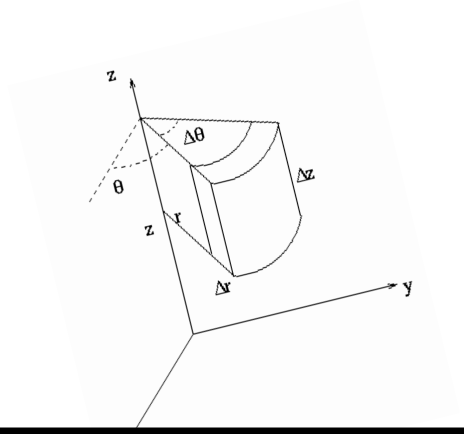
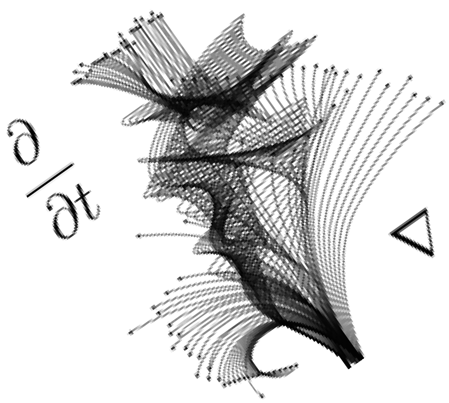
Mathematics-I

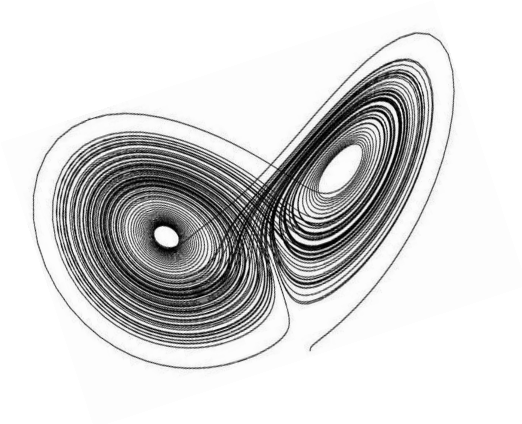
AUTUMN 2015

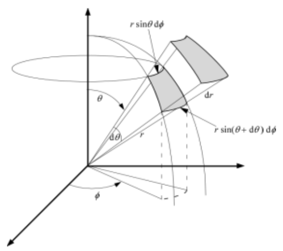
(MA10001)



Ordinary Differential Equation-I







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23.9.15

Ordinary Differential Equations.

(only one independent variable)

Example:

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = r$$

order $\rightarrow$ 2

degree $\rightarrow$ 2

make it free from radicals & fractions

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = r^2 \left(\frac{d^2y}{dx^2}\right)^2$$

order $\rightarrow$ order of the highest derivative appearing in the ODE.

degree $\rightarrow$ degree of the highest derivative, when the DE is free from radicals & fractions.

Example: $\frac{d^2y}{dx^2} = 0$

order $\rightarrow$ 2

degree $\rightarrow$ 1

DE → ODE. (Single ind. variable)

DE → PDE (partial DE). (more than one ind. variable)

Example:

$$y^2 \frac{\partial^2 z}{\partial x^2} + xy\frac{\partial z}{\partial y} = nxz.$$

Example: From $x^2+y^2+2ax+2by+c=0$, derive a DE not containing a, b or c.

A.

Sol^n.

w.r.to x

$$x + y\frac{dy}{dx} + a + b\frac{dy}{dx} = 0.$$

w.r.to x

$$1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} + b\frac{d^2y}{dx^2} = 0 \quad \text{--- (1)}$$

w.r.to x

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3} + b\frac{d^3y}{dx^3} = 0 \quad \text{--- (2)}$$

$$(1) \times \frac{d^3y}{dx^3} - (2) \times \frac{d^2y}{dx^2}$$

$$\frac{d^3y}{dx^3}\left[1+\left(\frac{dy}{dx}\right)^2\right] - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0 \quad \text{--- (3)}$$

Complete integral / General Sol^n.

(A) is called the complete integral of the DE (3).

↓ 3 consts.

↓ 3rd order DE.

- # of consts. in a Complete integral of a DE
  = order of the DE.

Particular Sol^n.

→ giving particular values to the consts. in the Complete integral of a DE.

Exercise Form the DE of which

$$e^{2y} + 2cxe^{y} + c^{2} = 0$$

is the complete integral.

DE of 1st. order, 1st. degree

1. Equns. of the form

$$f_1(x)\,dx + f_2(y)\,dy = 0.$$

(Variables are separable)

Example: $(1-x)\,dy - (1+y)\,dx = 0.$

trivial $\qquad \int f_1(x)\,dx + \int f_2(y)\,dy = C$

(c arb. const.)

2. Equns. homogeneous in x, y.

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)},$$ f_1, f_2 hom. in x, y of same degree.

put $y = vx$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

<u>Example:</u>

$$(x^2+y^2)\,dx - 2xy\,dy = 0.$$

<u>Sol^n.</u> $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy}.$

$y = vx$

$$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}.$$

Complete integral $\rightarrow$

$$x^2 - y^2 = cx, \quad c \text{ const}.$$

3. <u>Non-homogeneous eqn^n. of 1st. degree in x, y</u>

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}.$$

<u>Case I:</u> $\dfrac{a}{a_1} = \dfrac{b}{b_1} = m$

$$\frac{dy}{dx} = \frac{m(a_1x+b_1y)+c}{(a_1x+b_1y)+c_1} \quad \text{—} \ (1)$$

put $v = a_1x + b_1y.$

This will make ① separable.
(check it)

<u>Case II</u> : $\frac{a}{a_1} \neq \frac{b}{b_1}$

put $x = x_1 + h$
$y = y_1 + k.$

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1 x + b_1 y + c_1}$$

$$\Rightarrow \frac{dy_1}{dx_1} = \frac{ax_1 + by_1 + c + ah + bk}{a_1 x_1 + b_1 y_1 + c_1 + a_1 h + b_1 k}$$

Choose h, k s.t. $\left.\begin{array}{l} ah + bk + c = 0 \\ a_1 h + b_1 k + c_1 = 0 \end{array}\right\}$

becomes homogeneous

put $y_1 = v x_1$

<u>Example:</u> Solve
$(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0.$

Sol^n: $$\frac{dy}{dx} = \frac{-3y + 7x - 7}{7y - 3x + 3}$$

put $x = x_1 + h$, $y = y_1 + k$.

$$-3k + 7h - 7 = 0.$$
$$7k - 3h + 3 = 0.$$

$h = 1$
$k = 0$

$$\left\{\begin{array}{l} x = x_1 + 1 \\ y = y_1 \end{array}\right.$$

$$\frac{dy_1}{dx_1} = \frac{-3y_1 + 7x_1}{7y_1 - 3x_1}$$

put $y_1 = v x_1$

$\rightarrow$ find complete integral.

$\downarrow$ put $x_1 = x - 1$
$y_1 = y$

get the sol^n of the original DE.

4. Exact DE.

$$M\,dx + N\,dy = 0,$$ M, N fun^{ns} of x, y.

$\downarrow$ exact iff

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

- $M\,dx + N\,dy$ exact $\Rightarrow du = M\,dx + N\,dy$

Since $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$$\Rightarrow M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}.$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.}$$

- $M\,dx + N\,dy = 0$. $\rightarrow$ exact (1)

Complete integral of (1) is.

$$\int M\,dx + \int \left(\begin{array}{c}\text{terms of } N \text{ free} \\ \text{from } x\end{array}\right) dy = C, \quad c \text{ const.}$$

Example: $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$ — (1)

Sol^n:

$M = x^2 - 4xy - 2y^2$, $N = y^2 - 4xy - 2x^2$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -4x - 4y.$$

$\Rightarrow$ (1) is exact.

$\therefore$ general sol^n. of (1) is

$$\int (x^2 - 4xy - 2y^2)\,dx + \int y^2\,dy = c,$$

(c const.)

$$\frac{x^3}{3} - \frac{4x^2y}{2} - 2xy^2 + \frac{y^3}{3} = c.$$

Integrating factor

Example: $y\,dx - x\,dy = 0.$

$\longrightarrow$ not exact

- but when multiplied by $\frac{1}{y^2}$

$$\frac{y\,dx - x\,dy}{y^2} = 0.$$

$$d\left(\frac{x}{y}\right) = 0. \Rightarrow \frac{x}{y} = c, \text{ const.}$$

- when multiplied by $\frac{1}{xy}$.

$$\frac{dx}{x} - \frac{dy}{y} = 0. \rightarrow \text{exact.}$$

- when multiplied by $\frac{1}{x^2}$,

$$\frac{y\,dx - x\,dy}{x^2} = 0 \rightarrow \text{exact.}$$

$$-d\left(\frac{y}{x}\right) = 0.$$

- Any factor μ that changes a DE into an exact DE $\longrightarrow$ is called an integrating factor (IF).
- # of integrating factors $\rightarrow$ infinite.

28.9.15

Example: $\boxed{M\,dx + N\,dy = 0, \quad M, N \to fn^{ns}. \text{ of } x.}$

IF by inspection

$$y\,dx - x\,dy + \log x\,dx = 0.$$

$$\frac{y\,dx - x\,dy}{x^2} + \frac{\log x}{x^2}\,dx = 0.$$

$$-d\left(\frac{y}{x}\right) + \frac{\log x}{x^2}\,dx = 0.$$

gen. soln.

$$cx + y + \log x + 1 = 0 \quad , \; c \text{ const.}$$

Example: $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0.$

$$\frac{y\,dx + x\,dy}{x^2y^2} + \frac{xy^2\,dx - x^2y\,dy}{x^2y^2} = 0.$$

$$\frac{d(xy)}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0.$$

gen soln

$$x = cy\,e^{\frac{1}{xy}}$$

Rules for finding Integrating Factors (IF)

$$\boxed{M\,dx + N\,dy = 0}$$

Rule I. if $Mx + Ny \neq 0$, equ^n is homogeneous.

$$\boxed{\text{Then } \frac{1}{Mx+Ny} \text{ is an IF.}}$$

Example: $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0.$

Sol^n. $M = x^2y - 2xy^2, \; N = -(x^3 - 3x^2y).$

$$Mx + Ny = x^2y^2 \neq 0.$$

$$I.F = \frac{1}{x^2y^2}.$$

$$\frac{x^2y - 2xy^2}{x^2y^2}dx - \frac{x^3 - 3x^2y}{x^2y^2}dy = 0.$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0.$$

which is exact DE

$\therefore$ general sol^n is

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c, \quad c \text{ const.}$$

$$\Rightarrow \frac{x}{y} + \log \frac{x^3}{y^2} = c.$$

Rule II

if $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x),$$

then $$IF = e^{\int f(x)\,dx}.$$

Rule III

if $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y),$$

then $$IF = e^{\int f(y)\,dy}.$$

Example:

$$(x^2+y^2)\,dx - 2xy\,dy = 0.$$

$$M = x^2+y^2, \quad N = -2xy.$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y+2y}{-2xy} = -\frac{2}{x}.$$

$$I.F = e^{-\int \frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}.$$

$$\frac{x^2+y^2}{x^2}\,dx - \frac{2xy}{x^2}\,dy = 0.$$

$$\left(1+\frac{y^2}{x^2}\right)dx - 2\frac{y}{x}\,dy = 0.$$

general sol^n is

$$\int\left(1+\frac{y^2}{x^2}\right)dx + \int 0\,dy = c$$

Example: $\left(3x^2y^4 + 2xy\right)dx + \left(2x^3y^3 - x^2\right)dy = 0$

Try it.

Rule IV

$$\underbrace{f_1(xy)\,y}_{M}\,dx + \underbrace{f_2(xy)\,x}_{N}\,dy = 0.$$

if $Mx - Ny \neq 0$, then $\boxed{\frac{1}{Mx-Ny}}$ → an IF

Example: $\underbrace{y\left(xy + 2x^2y^2\right)}_{M}dx + \underbrace{x\left(xy - x^2y^2\right)}_{N}dy = 0.$

Sol^n:

$$Mx - Ny = 3x^3y^3.$$

$$IF = \frac{1}{3x^3y^3}.$$

find the gen. sol^n

Rule V

$$\underbrace{x^{\alpha_1} y^{\beta_1} (m_1 y\,dx + n_1 x\,dy)} + \underbrace{x^{\alpha_2} y^{\beta_2} (m_2 y\,dx + n_2 x\,dy)} = 0.$$

$$IF_1 = x^{k_1 m_1 - \alpha_1 - 1}\, y^{k_1 n_1 - \beta_1 - 1}.$$

$$IF_2 = x^{k_2 m_2 - \alpha_2 - 1}\, y^{k_2 n_2 - \beta_2 - 1}.$$

Commom IF. $\rightarrow$ find k_1, k_2 s.t.

$$k_1 m_1 - \alpha_1 - 1 = k_2 m_2 - \alpha_2 - 1.$$
$$k_1 n_1 - \beta_1 - 1 = k_2 n_2 - \beta_2 - 1.$$

Exampl: $(y^2 + 2x^2 y)\,dx + (2x^3 - xy)\,dy = 0.$

Sol^n: $\underbrace{y(y\,dx - x\,dy)} + x^2\underbrace{(2y\,dx + 2x\,dy)} = 0.$

$$IF_1 = x^{k_1 \cdot 1 - 0 - 1}\, y^{k_1(-1) - 1 - 1}$$

$$IF_2 = x^{k_2 \cdot 2 - 2 - 1} \quad y^{k_2 \cdot 2 - 0 - 1}$$

letting,

$$\left.\begin{aligned} k_1 - 1 &= 2k_2 - 3 \\ -k_1 - 2 &= 2k_2 - 1. \end{aligned}\right\} \Rightarrow \begin{aligned} k_2 &= 1/4 \\ k_1 &= -3/2. \end{aligned}$$

$$\therefore \text{Common } IF = x^{-\frac{5}{2}} y^{-\frac{1}{2}}.$$

$\therefore$ The general sol^n. of the given DE is

$$\int x^{-5/2} y^{-\frac{1}{2}} y^2 \, dx + \int 0 \, dy$$

$$+ \int 2x^2 y \, x^{-5/2} y^{-\frac{1}{2}} \, dx + \int 0 \, dy = c,$$

c const.

Linear DE. (1st. order)

$$\frac{dy}{dx} + Py = Q, \quad P, Q \to \text{fn}^{ns} \text{ of } x.$$

(dependent variable & its derivatives appear only in 1st. order)

$$\boxed{IF = e^{\int P dx}}$$

How?

$$\underbrace{(Py - Q)}_{M} dx + \underbrace{1}_{N=1} dy = 0.$$

fn^ns. of x.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = P = \text{fn}^n \text{ of } x.$$

$$IF = e^{\int P dx}.$$

Ex.

Example: Solve

$$\cos^2 x \frac{dy}{dx} + y = \tan x.$$

Sol^n: $\frac{dy}{dx} + \underbrace{\sec^2 x}_{P}\, y = \underbrace{\tan x \sec^2 x}_{Q}.$

$$\text{I.F} = e^{\int \sec^2 x\, dx} = e^{\tan x}.$$

$$d\left(e^{\tan x}.\, y\right) = \tan x \sec^2 x.\, e^{\tan x} dx$$

$$y e^{\tan x} = \int \tan x \sec^2 x\, e^{\tan x} dx + c.$$

$$= (\tan x - 1)\, e^{\tan x} + c.$$

$$y = (\tan x - 1) + c e^{-\tan x}, \quad c \text{ const.}$$

$$\frac{dy}{dx} + Py = Q.$$

$$I.F = e^{\int P dx}.$$

$$\underbrace{d\left(y e^{\int P dx}\right)}_{\text{why?}} = Q e^{\int P dx} dx$$

Equations reducible to the linear form

(Bernoulli's eqn.)

$$\frac{dy}{dx} + Py = Q \underline{\underline{y^n}}, \quad P, Q \to \text{fns. of } x.$$

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q. \quad \text{---} (1)$$

put $v = \dfrac{1}{y^{n-1}}$

$$\frac{dv}{dx} = \frac{-n+1}{y^n}\frac{dy}{dx}$$

① $\Rightarrow$

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q.$$

$$\frac{dv}{dx} + P(1-n)v = Q(1-n). \rightarrow \text{linear in } v.$$

Example: Solve $\dfrac{dy}{dx} + \dfrac{xy}{1-x^2} = xy^{1/2}.$

$$\frac{1}{y^{1/2}}\frac{dy}{dx} + \frac{x}{1-x^2}\cdot y^{\frac{1}{2}} = x \quad ①.$$

put $v = y^{\frac{1}{2}}$

$$\frac{dv}{dx} = \frac{1}{2y^{1/2}}\frac{dy}{dx}.$$

$$\Rightarrow \quad \frac{1}{y^{1/2}}\frac{dy}{dx} = 2\frac{dv}{dx}$$

$$\therefore ① \Rightarrow \quad 2\frac{dv}{dx} + \frac{x}{1-x^2}v = x.$$

$$② \longrightarrow \frac{dv}{dx} + \frac{x}{2(1-x^2)}v = \frac{x}{2}$$

$$\text{If} = e^{\int \frac{x}{2(1-x^2)}dx} \quad \text{linear in } v.$$

$$= (1-x^2)^{-1/4}.$$

general sol^n.

$$y^{1/2} = c\,(1-x^2)^{1/4} - \frac{1}{3}(1-x^2).$$

Example: Solve

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy.$$

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}.$$

not linear

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}\,x = \frac{\tan^{-1}y}{1+y^2}.$$

$P = P(y)$ $\qquad$ $Q = Q(y)$.

gen. sol^n.

$$x = (\tan^{-1}y - 1) + c\,e^{-\tan^{-1}y}, \quad c \text{ const}$$

(check it.)

DE of 1st. order, higher degree

- denote $\frac{dy}{dx}$ as p.

$$\frac{dy}{dx} + Px = Q.$$

$$p + Px = Q,$$

$P, Q \Rightarrow$ fn^{s} of x

- $f(x, y, p) = 0$. where

$$f(x,y,p) = p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_{n-1} p + P_n.$$

where $P_1, P_2, \dots, P_n$ are fn^{s}. of x.

I) . Solvable for p.

$$f(x,y,p) = (p - R_1)(p - R_2) \cdots (p - R_n) = 0.$$

$$p = R_1, R_2, \dots, R_n$$

$\rightarrow$ Sol^n. easy to get

Example: $4y^2p^2 + 2pxy(3x+1) + 3x^3 = 0$

$$\underbrace{(2yp + 3x^2)}_{0}(2yp + x) = 0.$$

$$2y\frac{dy}{dx} + 3x^2 = 0. \Rightarrow x^3 + y^2 = c$$

II > Equations solvable for y.

$f(x,y,p)=0 \Rightarrow y = F(x,p)$. —— ①

↓

diff. w.r. to x.

$p = \phi\left(x, p, \frac{dp}{dx}\right)$.

↓

try to deduce a relation of the form

$\psi(x,p,c)=0$.

—— (i).

Example: Solve $4y = x^2+p^2$ —— ①.

Solⁿ. w.r.to x differenting ①,

$$4p = 2x + 2p\frac{dp}{dx}.$$

or $\frac{dp}{dx} = \frac{2p-x}{p}$, hom. in x, p.

put $p = vx$.

$$v + x\frac{dv}{dx} = \frac{2v-1}{v}.$$

$$x\frac{dv}{dx} = \frac{2v-1}{v} - v = \frac{2v-1-v^2}{v}$$

$$= -\frac{(v-1)^2}{v}.$$

$$\frac{v-1+1}{(v-1)^2}\,dv = -\frac{dx}{x}.$$

$$\frac{dv}{v-1} + \frac{dv}{(v-1)^2} = -\frac{dx}{x}.$$

$$\log(v-1) - \frac{1}{v-1} = -\log x + c.$$

$$\Rightarrow \log(p-x) - \frac{x}{p-1} = c. \quad \text{——— (2)}.$$

equn. (2) together with equn. (1).
gives the general soln.

III > equations solvable for x.

$f(x, y, p) = 0 \Rightarrow x = F(y, p)$ — ①

↓ diff. w.r.to y.

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right).$$

↓ try to get a relation of the form

$$\psi(y, p, c) = 0. \quad — ②.$$

eliminating p from ①, ②

⟶ gives the gen. soln.

Example: Solve $x = y + p^2$ — ①.

Soln Diff. ① w.r.to y,

$$\frac{1}{p} = 1 + 2p\frac{dp}{dy}.$$

$$\frac{p^2}{1-p}\, dp = \frac{dy}{2}$$

$$\hookrightarrow y = c - \left[p^2 + 2p + 2\log(p-1) \right] \quad \text{—— (2)}.$$

Ans. (1) along with (2).

IV Equns that do not contain x (or do not contain y).

$f(y,p)=0 \to$ Solvable for p.

$\to \dfrac{dy}{dx} = \phi(y)$

$f(y,p)=0 \to$ Solvable for y.

$\to y = F(p)$.

$\downarrow$ Diff. w.r.t x

$\vdots$

Case II.

$f(x,p)=0 \to$ Case II

$f(x,p)=0 \to$ Case III.

Exercise: $x(1+p^2)=1$.

$\hookrightarrow p^2 = \dfrac{1-x}{x}$.

$$\frac{dy}{dx} = p = \sqrt{\frac{1-x}{x}}$$

$$x = \frac{1}{1+p^2}$$

Diff. w.r.to y

$$\frac{1}{p} = -\frac{1}{1+p^2} \, 2p \frac{dp}{dy}$$

$$p \cdot \frac{2p}{(1+p^2)^2} dp = -dy.$$

$$-\frac{p}{1+p^2} + \tan^{-1} p = -y - c.$$

<u>g^n Sol.</u>

$$y + c = \sqrt{\frac{1-x}{x}} \cdot x - \tan^{-1}\sqrt{\frac{1-x}{x}}$$

<u>V. Equ^n! hom. in x, y</u>

$F\left(\frac{dy}{dx}, \frac{y}{x}\right) = 0$ → Solvable for $\frac{dy}{dx}$ ✓

→ Solvable for $\frac{y}{x}$.

i.e. $\frac{y}{x} = f(p)$

i.e. $y = x f(p)$

$\downarrow$

Case II.

<u>Exercise:</u> $y^2 + xyp - x^2p^2 = 0$

<u>Exercise:</u> $y = yp^2 + 2px$

VI <u>Equations of the 1st. degree in x, y.</u>

· eqn. $\rightarrow$ Solvable for x $\rightarrow$ Case III

$\quad\;\;\rightarrow$ Solvable for y $\rightarrow$ Case II

· <u>Clairaut's eqn.</u>

$$y = px + f(p). \quad (1).$$

$\downarrow$ <u>gen. soln.</u>

$$y = cx + f(c), \quad c \text{ arb. const.}$$

How?

w.r.to x

$$\cancel{p} = \cancel{p} + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}.$$

$$\frac{dp}{dx}\left[x + f'(p)\right] = 0.$$

$$\frac{dp}{dx} = 0 \Rightarrow p = c. \quad \text{——} \quad (2).$$

$$\therefore \quad \boxed{y = x\,\underline{\underline{f_1(p)}} + f_2(p).}$$

p means Clairaut's form

diff. w.r.to x

$$p = f_1(p) + x f_1'(p)\frac{dp}{dx} + f_2'(p)\frac{dp}{dx}.$$

$$\frac{dp}{dx}\left[x f_1'(p) + f_2'(p)\right] = p - f_1(p)$$

or $$\frac{dx}{dp} = \frac{x f_1'(p) + f_2'(p)}{p - f_1(p)}.$$

$$\frac{dx}{dp} - \frac{f_1'(p)}{p - f_1(p)}\,x = \frac{f_2'(p)}{p - f_1(p)}.$$

linear in x & is solvable.

Example: Solve

$$x^2(y-px) = yp^2 \quad \text{—— (1).}$$

(This eqn. is reducible to clairaut's form).

Soln. put $x^2 = u$, $y^2 = v$.

$\therefore\ 2x\,dx = du$, $2y\,dy = dv$.

$$\Rightarrow \frac{y}{x}p = \frac{dv}{du}$$

$$\text{or } p = \frac{x}{y}\frac{dv}{du}$$

$$(1) \Rightarrow x^2\left(y - \frac{x}{y}\frac{dv}{du}\right) = y \cdot \frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2$$

$$x^2\left(y^2 - x^2\frac{dv}{du}\right) = x^2\left(\frac{dv}{du}\right)^2.$$

$$v - u\frac{dv}{du} = \left(\frac{dv}{du}\right)^2$$

$$\text{or } v = \frac{dv}{du}u + \left(\frac{dv}{du}\right)^2.$$

$\hookrightarrow$ ~~claur~~ clairaut's form

gen. sol $v = cu + c^2$, c const.

i.e. $\boxed{y^2 = cx^2 + c^2}$

30.9.15

- $y = px + f(p)$ — ①, $p = \frac{dy}{dx}$. $\rightarrow$ clairaut's form.

$\downarrow$ w.r.to x

$$\cancel{p} = \cancel{p} + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}.$$

$$\frac{dp}{dx}\left[x + f'(p)\right] = 0. \rightarrow \text{singular sol}^n.$$

$$\frac{dp}{dx} = 0 \Rightarrow p = c. \text{ — ②}$$

eliminating p from ① & ② $\Rightarrow$

$$\boxed{y = cx + f(c)} \rightarrow \text{gen. sol}^n \text{ for ①.}$$

can not be obtained from the gen. sol^n. (by putting a particular value to c).

- $f(x, y, p) = 0.$

$\frac{\partial f}{\partial p} = 0.$

$\downarrow$

eliminate p.

$\downarrow$

p-disc.

Example: Clairaut's form

$$f(x,y,p) = y - px - f(p) = 0$$

$$\frac{\partial f}{\partial p} = 0 \Rightarrow -x - f'(p) = 0$$

- $\phi(x,y,c) = 0$ $\longrightarrow$ soln. of given DE $f(x,y,p)=0$.

$\frac{\partial \phi}{\partial c} = 0$ $\longrightarrow$ eliminate c

$\downarrow$

c-disc.

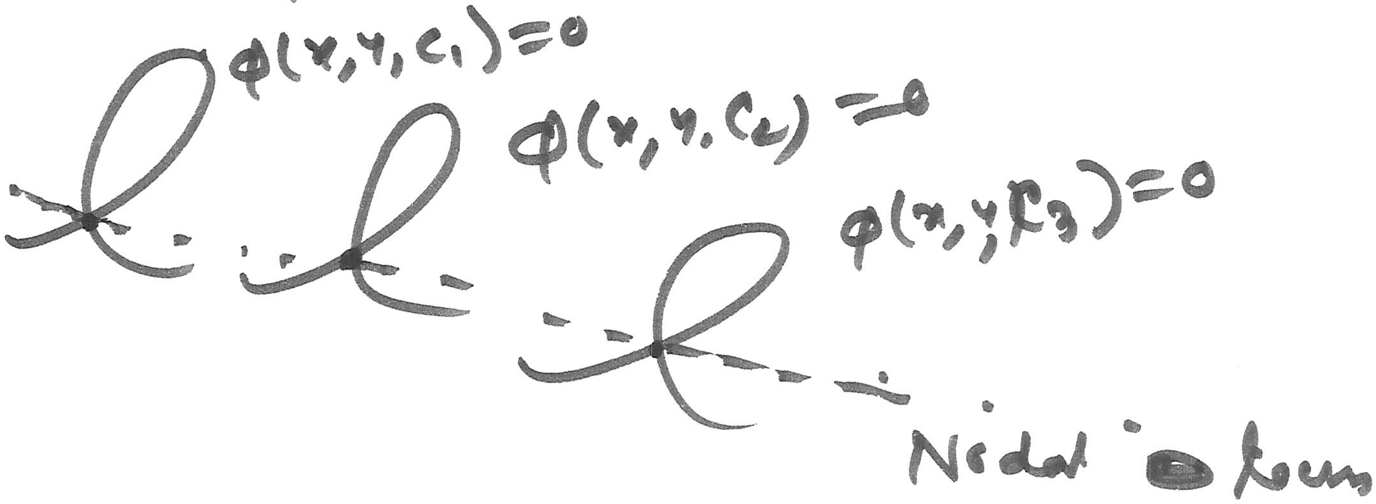
- p. disc. $\longrightarrow$ ET^2C

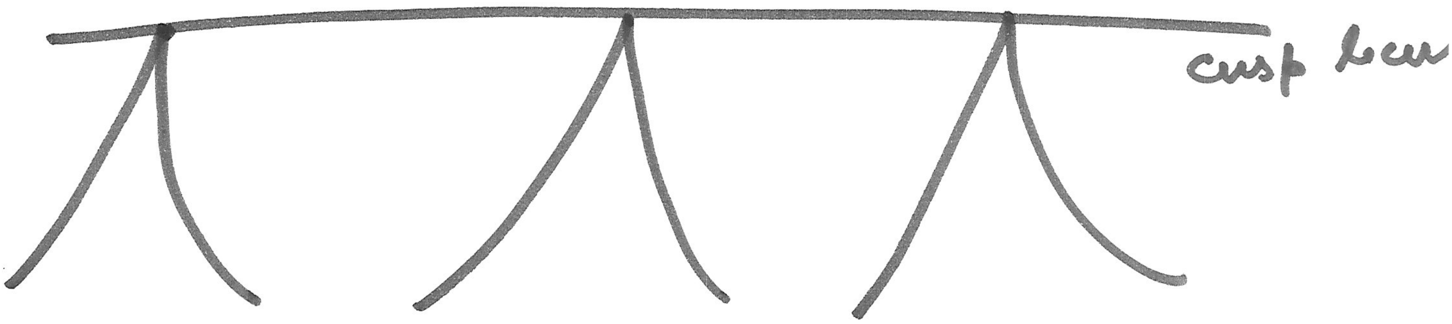
c-disc. $\longrightarrow$ EN^2C^3.

E : Envelop, N : Nodal,

C : cuspidal, T : tac loci

· $\phi(x, y, c) = 0$.





· E $\rightarrow$ envelop.

$\downarrow$ in p-disc

in c-disc.

satisfies the given DE.

$\downarrow$

singular solⁿ.

Example:

$$xp^2 - (x-a)^2 = 0 \quad \text{—①.}$$

find Envelop, tac locus, Nodal locus, cusp locus if any.

Solⁿ.

p-disc

$$\left.\begin{aligned} f(x,y,p) &= xp^2-(x-a)^2 = 0 \\ \frac{\partial f}{\partial p} &= 0 \Rightarrow xp = 0. \end{aligned}\right\}$$

eliminate p

$$x(x-a)^2 = 0. \rightarrow ET^2C$$

$x=0 \rightarrow$ Envelop as it satisfies ①.

$$\left[\ \frac{1}{p} = 0. \quad ① \Rightarrow \underset{0}{\underset{\shortparallel}{x}} - \underset{0}{\underset{\shortparallel}{\frac{(x-a)^2}{p^2}}} = 0\ \right]$$

c-disc

Solve the given DE ①.

$$p = \frac{x-a}{\sqrt{x}}.$$

gen solⁿ.

check

$$\frac{9}{4}(y+c)^2 = x(x-3a)^2, \quad c \text{ const.}$$

|

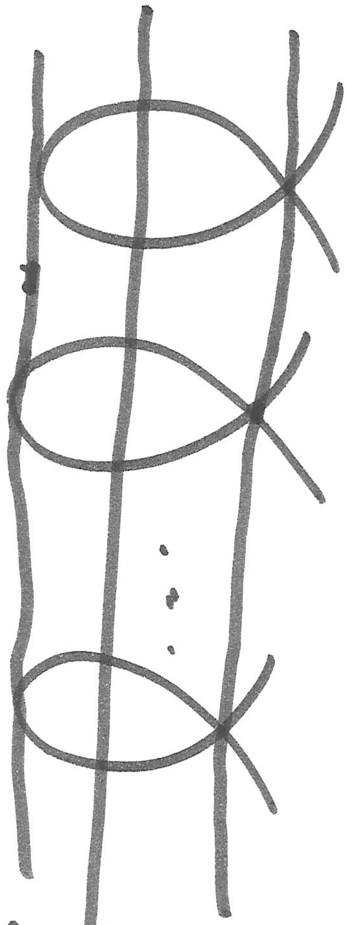
$$\left\{ \begin{array}{l} \Phi(x, y, c) = \frac{9}{4}(y+c)^2 - x(x-3a)^2 = 0 \\ \frac{\partial \Phi}{\partial c} = 0 \Rightarrow y = -c. \end{array} \right.$$

$\hookrightarrow$ Eliminate c

$x(x-3a)^2 = 0 \longrightarrow EN^2C^3.$

$x = 0 \rightarrow$ Envelop.

$x = 3a \rightarrow$ Nodal locus.



$x=0$ (E) $x = 3a$ (N)

$x = a$ (T)

General linear DE. with const. co-efficients.

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X.$$

or

$$\left(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n\right) y = X, \quad \left(D \equiv \frac{d}{dx}\right)$$

- $P_1, P_2, \ldots, P_n, X \rightarrow$ either <u>consts.</u> or fn^ns. of x.

$\rightarrow$ general linear DE.

- <u>Linear DE with const. co-eff.</u> $\rightarrow P_1, P_2, \ldots, P_n$ are consts.

- <u>$n=1$</u> $\quad \frac{dy}{dx} + Py = X, \quad P, X \Rightarrow$ fn^ns. of x.

(already considered)

$\cdot \quad f(D)\,y = X$ —— ①.

when $f(D) = D^n + P_1 D^{n-1} + \cdots + P_n$,

$\longrightarrow$ linear DE with const. co-efficients.

gen. sol^n. of ① $\to$ Complementary fn. (CF).

$\to$ Particular integral (P.I).

CF $\to$ sol^n of $f(D)\,y = 0$

P.I $\to$ $P.I = \dfrac{1}{f(D)}\,X$.

gen. sol^n. of ① $\boxed{y = C.F. + P.I.}$

To find CF.

$f(D)\,y = 0$ —— ②

put $y = e^{mx}$

auxiliary eqn. $\boxed{f(m) = 0.}$

$$f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n.$$

$$\begin{cases} y = e^{mx} \\ Dy = m e^{mx} \\ D^2 y = m^2 e^{mx}. \\ \vdots \end{cases}$$

- $f(m) = 0 \longrightarrow m = m_1, m_2, \cdots, m_n.$ (real).

i) $C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}.$
if all distinct

(all m_i are distinct)

ii) $C.F = (C_1 + C_2 x) e^{mx} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$
if two equal

($m_1 = m_2 = m$, all other m_i are distinct).

iii) $C.F = (C_1 + C_2 x + C_3 x^2) e^{mx} + C_4 e^{m_4 x} + \cdots + C_n e^{m_n x}$
if 3 equal

($m_1 = m_2 = m_3 = m$, other distinct)

Case. When auxiliary eqn. has imaginary ~~parts~~ roots.

$\rightarrow$ $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

$$C.F = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$$

$$C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$= C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

Example: $(9D^2 + 18D - 16)y = 0$, $\left(D \equiv \frac{d}{dx}\right)$

Soln. put $y = e^{mx}$.

auxiliary eqn.

$$9m^2 + 18m - 16 = 0.$$

$$m = -8/3,\ 2/3.$$

$\therefore$ general sol^n. of the given DE is

$$y = c_1 e^{-\frac{8}{3}x} + c_2 e^{2/3 x},$$

$c_1, c_2 \to$ arb. consts.

<u>Example:</u> $$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0.$$

<u>Sol^n.</u> put $y = e^{mx}$.

$$m^4 - m^3 - 9m^2 - 11m - 4 = 0.$$

$$\rightarrow \boxed{m = -1, -1, -1, 4}$$

<u>gen. sol^n.</u> $$y = (c_1 + c_2 x + c_3 x^2) e^{-x} + c_4 e^{4x},$$

$c_1, c_2, c_3, c_4 \to$ arb. consts.

<u>Example:</u> $$(D^4 - 4D^3 + 8D^2 - 8D + 4) y = 0$$ $$\left(D \equiv \frac{d}{dx}\right).$$

<u>Sol^n.</u> $$(m^2 - 2m + 2)^2 = 0.$$

$$\Rightarrow \quad m = 1 \pm i, \; 1 \pm i$$

<u>gen. sol^n</u>

$$y = e^{x}\left[(c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x\right]$$

$c_1, c_2, c_3, c_4 \rightarrow$ arb. consts.