Mathematics-I

FUNCTIONS OF SEVERAL VARIABLES-IV

(a) Extreme values for functions of two more variables.

(b) Saddle Point.

(c) Constrained Extremes.

(d) Lagrange's Multiplier Method.

(e) Lagrange's Method of Undetermined Multipliers.

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Extreme values for fun^{ns}. of two & more variables

- $f(x,y) \rightarrow$ real valued fun^{n}. of x,y defined in a certain nbd. of (a,b) of its domain.

- $f(x,y) < f(a,b) \quad \forall\ (x,y) \in N$
  $\longrightarrow$ f has max^{m}.
- $f(x,y) > f(a,b) \quad \forall\ (x,y) \in N$
  $\longrightarrow$ f has min^{m}.

$\boxed{\text{Necessary condition}}$ $\quad \boxed{f(x,y)}$

$$\underbrace{f_x = 0, \quad f_y = 0.}_{(a,b)} \text{ at } (a,b) \longrightarrow \text{stationary pts or Critical pts}$$

extreme pt. $\qquad$ saddle pt.

$f(x,y)$ has extrema at (a,b)

$\Rightarrow \quad f_x(a,b)=0, \; f_y(a,b)=0,$ if they exist.

$\not\Leftarrow$

Example: $f(x,y) = \begin{cases} 0 & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{elsewhere.} \end{cases}$

$f_x(0,0) = 0 = f_y(0,0)$.

$N \rightarrow$ a nbd. of $(0,0)$.

$f(x,y) - f(0,0)$ does not maintain a const. sign.

$\Rightarrow$ f has no extrema at $(0,0)$ although $f_x = f_y = 0$ at $(0,0)$.

$\Rightarrow$ $(0,0)$ is a saddle pt.

• <u>Saddle pt.</u> → Stationary / critical pts. which are not extreme pts

$f_x = 0 = f_y$

• $f(x,y)$ may have extrema at (a,b) even if f_x, f_y do not exist at (a,b)

<u>Example:</u> $f(x,y) = |x| + |y|$.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \cancel{\lim_{h \to 0}}$$

f_x, f_y do not exist at $(0,0)$, but f has <u>min</u> at $(0,0)$.

<u>Sufficient condition</u>

- $f(x,y)$
- $f_x, f_y, f_{xx}, f_{yy}, f_{xy} \rightarrow$ all continuous in some nbd. of (a,b).

- $df = f_x\,dx + f_y\,dy$.

  $df = 0 \Rightarrow f_x(a,b) = 0 = f_y(a,b)$.

  $\longrightarrow$ (a,b) is a stationary / critical pt

- $d^2f = f_{xx}(dx)^2 + 2f_{xy}\,dx\,dy + f_{yy}(dy)^2$

$A = f_{xx}(a,b)$, $B = f_{xy}(a,b)$,

$C = f_{yy}(a,b)$.

$$H = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$$

i) $H < 0 \rightarrow$ no extrema at (a,b)
(saddle pt.)

ii) $H > 0 \rightarrow f$ has extrema at (a,b).

$\Downarrow$

$AC - B^2 > 0$

maxima if $A, C < 0$
(i.e. $A < 0$).

minima if $A, C > 0$
(i.e. $A > 0$).

(iii) $H = 0 \rightarrow$ undecided.

$\Rightarrow$ principal minors $\rightarrow A, \begin{vmatrix} A & B \\ B & C \end{vmatrix}$ & H.

$\longrightarrow$ all +ve $\rightarrow$ min
$\longrightarrow$ ~~all +ve~~ alternatively +ve $\rightarrow$ max.

Generalization.

- $f(x,y,z)$
- $df = f_x\,dx + f_y\,dy + f_z\,dz$.
- $df = 0 \Rightarrow \boxed{f_x = 0 = f_y = f_z} \rightarrow$
- $d^2 f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2$

$$+ 2f_{xy}\,dx\,dy + 2f_{xz}\,dx\,dz$$
$$+ 2f_{yz}\,dy\,dz.$$

Sufficient condition

- $H = \begin{pmatrix} \boxed{f_{xx}} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}$

$\downarrow$

all the principal minors

all +ve $\Downarrow$ min

alternatively +ve $\Downarrow$ max.

$\Downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\Downarrow$

$d^2 f > 0$ $\qquad\qquad\qquad\qquad$ $d^2 f < 0.$

Example: Find the extreme values, if any, of the eqn.

$$f(x,y) = 2(x-y)^2 - x^4 - y^4.$$

Soln:

$$f_x = 4(x-y) - 4x^3 = 0$$

$$f_y = -4(x-y) - 4y^3 = 0.$$

$$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$$

$x+y=0$ $\quad$ or $\quad$ $x^2 - xy + y^2 = 0.$

pts. $\boxed{(0,0), \ (\sqrt{2}, -\sqrt{2}), \ (-\sqrt{2}, \sqrt{2})}$

Stationary / Critical pts.

$x - y - x^3 = 0 \Rightarrow$

i) $(0,0)$

$$H = \begin{vmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{vmatrix}$$

$= 0 \rightarrow$ further investigation needed.

ii) $(\sqrt{2}, -\sqrt{2}) \rightarrow \boxed{\text{max}^m.}$

iii) $(-\sqrt{2}, \sqrt{2}) \rightarrow$ ~~min~~^{m}. $\boxed{\text{max}^m.}$

Case $(0,0)$.

$$f(x,y) = 2(x-y)^2 - x^4 - y^4.$$

along x-axis

$$f(x,y) = 2x^2 - x^4$$
$$= x^2(2-x^2).$$
> 0 in the nbd of $(0,0)$

along the line $x=y$

$$f(x,y) = -2x^4. < 0$$
in the nbd. of $(0,0)$.

$\Rightarrow \exists$ nbd. of $(0,0)$ where

$f(x,y) - f(0,0)$ does not maintain a const. sign.

$\Rightarrow$ f has no extrema at $(0,0)$.

<u>Ans.</u> $f(\sqrt{2}, -\sqrt{2}) = 8 \rightarrow$ max^m. value.

$f(-\sqrt{2}, \sqrt{2}) = 8 \rightarrow$ max^m. value

<u>Exercise:</u> $f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz - a^3$.

has <u>max^m.</u> at $(-1,-1,-1)$ & <u>min</u> at $(1,1,1)$.

$f_x =$

$f_y =$

$f_z =$

f_{xx}, f_{yy}, f_{zz}

f_{xy}, f_{xz}, f_{yz}

Example: Find the maxima & minima of the f^n.

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20.$$

Sol^n. $f_x = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$f_y = 0 \Rightarrow 3y^2 - 12 = 0 \Rightarrow y = \pm 2$

$\therefore f$ has four stationary pts.:

$(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$.

$f_{xx} = 6x$, $f_{yy} = 6y$

At $(1,2)$

$f_{xy} = 0$.

$$\left. f_{xx} f_{yy} - f_{xy}^2 \right]_{(1,2)} = 36 \times 1 \times 2 > 0.$$

& $f_{xx}(1,2) > 0$.

$\longrightarrow$ min., $f_{min} = ?$

$(1,-2)$, $(-1,2)$, $(-1,-2)$

$(1,-2) \rightarrow$ neither max nor min.

$(-1,2) \rightarrow$ neither max nor min.

$(-1,-2) \rightarrow$ maxima.

Example: Show that $\boxed{\phi(x,y) = 2x^4 - 3x^2y + y^2}$ has <u>neither a maximum nor a minimum at $(0,0)$</u>, but

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 0.$$

Sol^n.

$$\phi_x = 8x^3 - 6xy = 0 \text{ at } (0,0)$$

$$\phi_y = -3x^2 + 2y = 0 \text{ at } (0,0).$$

$$\phi_{xx} = 24x^2 - 6y = 0 \text{ at } (0,0)$$

$$\phi_{yy} = 2 \text{ at } (0,0).$$

$$\phi_{xy} = -6x = 0 \text{ at } (0,0)$$

Claim.

$$\phi(x,y) - \phi(0,0).$$

$$= (2x^2 - y)(x^2 - y)$$

$$> 0 \text{ for } y < 0 \text{ or } x^2 > y > 0$$

Exercise

$$< 0 \text{ for } 0 < x^2 < y < 2x^2$$

Constrained Extremum.

Example: Find the shortest distance from the pt. (a,b,c) to the plane $px+qy+rz-s=0$.

Sol^n.

min
$$(x-a)^2+(y-b)^2+(z-c)^2$$

s.t. $px+qy+rz-s=0$.

let $\phi(x,y) = (x-a)^2+(y-b)^2+(z-c)^2$.

where $\underline{px+qy+rz-s=0}$.

$$p\,dx+q\,dy+r\,dz=0.$$

$$\Rightarrow \boxed{dz = -\frac{p}{r}dx - \frac{q}{r}dy}$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{p}{r}, \quad \frac{\partial z}{\partial y} = -\frac{q}{r}.$$

$$\phi_x = 0 = 2(x-a) + 2(z-c)\frac{\partial z}{\partial x}.$$

$$\phi_y = 0 = 2(y-b) + 2(z-c)\frac{\partial z}{\partial y}.$$

$$\Rightarrow \left.\begin{aligned} (x-a) + (z-c)\left(-\frac{p}{r}\right) &= 0. \\ (y-b) + (z-c)\left(-\frac{q}{r}\right) &= 0. \end{aligned}\right\}$$

$$\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r} \cancel{= \frac{1}{px+qy+rz}}$$

$$= \frac{p(x-a) + q(y-b) + r(z-c)}{p^2+q^2+r^2}.$$

$$= \frac{s - pa - qb - rc}{p^2+q^2+r^2} = \underset{(\text{say})}{R}$$

$$\left.\begin{aligned} x &= a + pR \\ y &= b + qR \\ z &= c + rR. \end{aligned}\right\} \rightarrow \text{critical pt.}$$

$$\phi_{xx} = 2 + 2\frac{p^2}{r^2} \qquad \phi_x = 2(x-a) + 2(z-c) \times \left(-\frac{p}{r}\right)$$

$$\phi_{yy} = 2 + 2\frac{q^2}{r^2}$$

$$\phi_{xy} = \frac{2pq}{r^2}$$

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = 4\left(1 + \frac{p^2}{r^2} + \frac{q^2}{r^2}\right).$$

$$> 0.$$

$\Rightarrow$ ϕ has min. at the critical pt. $x = a + pR$, $y = b + qR$, $z = c + rR$.

when $R = \dfrac{s - pa - qb - rc}{p^2 + q^2 + r^2}$.

$$\phi_{min} = p^2R^2 + q^2R^2 + r^2R^2$$

$$= \frac{(s - pa - qb - rc)^2}{p^2 + q^2 + r^2}$$

General Rule. To find stationary pt. of

$$f(x_1, \dots, x_n, u_1, \dots, u_m) = 0$$

s.t. $\boxed{g_1 = 0,\ g_2 = 0, \dots, g_m = 0.}$

where $g_i = g_i(x_1, \dots, x_n, u_1, \dots, u_m)$, $1 \le i \le m$

- $df = 0$ at stationary pts. of f.
- $\boxed{dg_i = 0}$ for $i = 1, 2, \dots, m$.

$\Downarrow$

$$\frac{\partial g_i}{\partial x_1} dx_1 + \frac{\partial g_i}{\partial x_2} dx_2 + \dots + \frac{\partial g_i}{\partial x_n} dx_n$$

$$+ \frac{\partial g_i}{\partial u_1} du_1 + \frac{\partial g_i}{\partial u_2} du_2 + \dots + \frac{\partial g_i}{\partial u_m} du_m = 0.$$

$i = 1, 2, \dots, m$.

$\Downarrow$

find $du_1, \dots, du_m$ in terms of $dx_1, \dots, dx_n$

& Substitute in $df = 0$.

$$df = f_{x_1} dx_1 + \dots + f_{x_n} dx_n + \boxed{f_{u_1} du_1 + \dots + f_{u_m} du_m}$$

equate coeff. of $dx_i = 0$,
$i = 1, 2, \cdots, n$.

n eqns.
together with m eqns.
$g_1 = 0 = g_2 = \cdots = g_m$.

Solve for $n+m$ variables.

Lagrange's multiplier method.

Example: Use Lagrange's method to find the shortest distance from the pt. $(0, b)$ to the parabola
$$x^2 = 4y$$

Sol^n.

min.

$$f(x,y) = (x-0)^2 + (y-b)^2$$

s.t. $x^2 = 4y$. $\quad \underbrace{x^2 - 4y}_{g} = 0$.

let $L(x,y) = x^2 + (y-b)^2 + \lambda(x^2 - 4y)$.

$\rightarrow$ variables x, y, λ.

$$\left.\boxed{\begin{array}{c} L_x = 0 \\ L_y = 0 \\ x^2 - 4y = 0 \end{array}}\right\}$$

$\rightarrow$ 3 eqn^{ns}.

$\rightarrow$ solve for x, y, λ.

$L_x = 0 \Rightarrow 2x + 2\lambda x = 0 \Rightarrow x = 0$ or $\lambda = -1$

$L_y = 0 \Rightarrow 2(y-b) - 4\lambda = 0 \Rightarrow y = b + 2\lambda$.

Case I $x = 0$.

Then $x^2 - 4y = 0 \Rightarrow y = 0$.

$\rightarrow (0,0)$, a critical pt.

$\rightarrow f(0,0) = b^2$.

$\lambda = \frac{y-b}{2} = -\frac{b}{2}$, $d^2L = (2-b)(dx)^2 + 2(dy)^2 > 0$

$\Rightarrow (1,0) \rightarrow$ f has min if $2-b>0$.

& S.D $= b$.

<u>Case 2</u>

$\lambda = -1$.

Then $y = b-2$

$\Rightarrow x = \pm 2\sqrt{y} = \pm 2\sqrt{b-2}$

$$d^2L = (2+2\lambda)(dx)^2 + 2(dy)^2 > 0.$$

if $b \geq 2$, $(\pm 2\sqrt{b-2},\ b-2)$ are critical pts. & S.D $= 2\sqrt{b-1}$

$\downarrow$

min.

Lagrange method of undetermined multipliers

max / min

$$f(x_1, \dots, x_n, u_1, \dots, u_m)$$

s.t. $\left.\begin{array}{l} g_1 = 0 \\ g_2 = 0 \\ \vdots \\ g_m = 0 \end{array}\right\}$ — ① where $g_i = g_i(x_1, \dots, x_n, u_1, \dots, u_m)$

- $L = f + \lambda_1 g_1 + \lambda_2 g_2 + \dots + \lambda_m g_m$.

  $\rightarrow$ fn. of $\underbrace{x_1, \dots, x_n}_{n}, \underbrace{u_1, \dots, u_m}_{m}, \underbrace{\lambda_1, \dots, \lambda_m}_{m}$.

- $dL = 0 \rightarrow \boxed{L_{x_1} = 0 = L_{x_2} = \dots = L_{x_n} = L_{u_1} = \dots = L_{u_m}}$ — ②

  $\hookrightarrow$ $n+m$ eqns.

- $d^2L > 0 \rightarrow$ min

  $< 0 \rightarrow$ max.

How it comes? $\quad L = f + \lambda_1 g_1 + \cdots + \lambda_m g_m$

$$
\begin{aligned}
1 \times \; & df = 0 \\
\lambda_1 \times \; & dg_1 = 0 \\
\lambda_2 \times \; & dg_2 = 0 \\
\vdots \quad & \;\; \vdots \\
\lambda_m \times \; & dg_m = 0
\end{aligned}
$$

$$dL = 0 = (\;)\,dx_1 + (\;)\,dx_2 + \cdots (\;)\,dx_n + (\;)\,du_1 + \cdots + (\;)\,du_m.$$

- choose $\lambda_1, \ldots, \lambda_m$ s.t. coeff. of $du_i = 0$, $i = 1, \ldots, m$. $\longrightarrow$ m eqn's.
- $x_1, \ldots, x_n \longrightarrow$ ind. variables. co-eff. of $dx_i = 0$, $i = 1, \ldots, n$ $\longrightarrow$ n eqn's.
- given m constraints $\rightarrow g_1 = 0 = \cdots = g_m$ $\rightarrow$ m eqn's.

- $n + 2m$ many eqn's.
- variables $\underbrace{x_1, \ldots, x_n, \; u_1, \ldots, u_m, \; \lambda_1, \ldots, \lambda_m}_{n+2m}$

Example: find the max^m. value of $f(x,y,z) = x^2y^2z^2$

s.t. $x^2+y^2+z^2 = a^2$ (x, y, z are +ve)

$a \neq 0$.

Sol^n. Construct a Lagrangian fu^n.

$$L(x,y,z) = x^2y^2z^2 + \lambda(x^2+y^2+z^2-a^2),$$

λ is the Lagrangian undetermined multiplier.

- Stationary pts. $\longrightarrow$ $dL = 0$.

$$L_x = 2xy^2z^2 + 2\lambda x = 0 \Rightarrow x(y^2z^2+\lambda) = 0$$

$$L_y = 2yx^2z^2 + 2\lambda y = 0 \Rightarrow y(x^2z^2+\lambda) = 0$$

$$L_z = 2x^2y^2z + 2\lambda z = 0. \Rightarrow z(x^2y^2+\lambda) = 0.$$

- $x=0, y=0, z=0$ $\longrightarrow$ ignored as $x^2+y^2+z^2 = a^2 \neq 0$
- $\lambda = \underbrace{-y^2z^2 = -x^2z^2} = -x^2y^2$

$$\Rightarrow x^2 = y^2 = z^2 = \frac{x^2+y^2+z^2}{3} = \frac{a^2}{3}.$$

$x = \pm\frac{a}{\sqrt{3}} = y = z$. $\longrightarrow$ gives stationary pt

checking max^m.

$$d^2L = L_{xx}(dx)^2 + L_{yy}(dy)^2 + L_{zz}(dz)^2 + 2L_{xy}\,dx\,dy + 2L_{xz}\,dx\,dz + 2L_{yz}\,dy\,dz.$$

$L_{xx} = 0 = L_{yy} = L_{zz}$ at $\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}\right)$

$L_{xx} = 2y^2z^2 + 2\lambda$

$L_{xy} = 4xyz^2$

$$L_{xy} = \frac{4a^4}{9} = L_{xz} = L_{yz}.$$

$$d^2L = 2 \cdot \frac{4a^4}{9}\left[dx\,dy + dy\,dz + dx\,dz\right]$$

$$= \frac{8a^4}{9}\left[dx\,dy + (dx+dy)\,dz\right].$$

$$= \frac{8a^4}{9}\left[dx\,dy - (dx+dy)^2\right]$$

$$= \frac{8a^4}{9}\left[-(dx)^2 - (dy)^2 - dx\,dy\right]$$

$$< 0$$

$x^2 + y^2 + z^2 = a^2$

$2x\,dx + 2y\,dy + 2z\,dz = 0$

$\Rightarrow dz = -\frac{x\,dx + y\,dy}{z}$

$$dz\Big]_{\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}\right)} = -(dx+dy).$$

<u>Exercise:</u>

(i) Maximise $x^a y^b z^c$, where a, b, c are +ve constants, s.t. the condition $x^k + y^k + z^k = 1$, where x, y, z are ~~non-zero~~ non-negative variables & $k > 0$.

(ii) Hence prove that for any six +ve real numbers u, v, w, a, b, c,

$$\left(\frac{u}{a}\right)^a \left(\frac{v}{b}\right)^b \left(\frac{w}{c}\right)^c \leq \left(\frac{u+v+w}{a+b+c}\right)^{a+b+c}$$