Mathematics-I

FUNCTIONS OF SEVERAL VARIABLES-III

(a) Chain Rule.

(b) Leibnitz's Rule for Differentiation under the Sign of Integration.

(c) Harmonic Function and Euler's Theorem.

(d) Taylor's Expansion of Functions of two Variables.

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· Composite functions

Chain rules for fu^{n}: of 2 variables

1. $z = f(x,y)$, $x = \phi(t)$, $y = \psi(t)$

$f, \phi, \psi \to$ ~~all~~ all differentiable fu^{ns}!

$\Rightarrow$ z is differentiable fu^{n}: of t &

$$\underline{\underline{\frac{dz}{dt}}} = \frac{\partial z}{\partial x} \underline{\underline{\frac{dx}{dt}}} + \frac{\partial z}{\partial y} \underline{\underline{\frac{dy}{dt}}} \quad \text{(chain rule)}.$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad \text{(total differential)}.$$

2. $z = f(x,y)$, $x = \phi(u,v)$, $y = \psi(u,v)$.

$f, \phi, \psi \to$ all diff. fu^{ns}.

$\Rightarrow$ z is a diff fu^{n}: of u, v &

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{aligned}\right\} \text{(chain rule)}$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad \text{(total differential)}.$$

Chain rules for fu^{n} of 3 variables

- $f(x, y, z)$, $x = \phi(t)$, $y = \psi(t)$, $z = \theta(t)$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

- $f(x, y, z)$, $x = \phi(u, v, w)$, $y = \psi(u, v, w)$, $z = \theta(u, v, w)$.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}.$$

$$\frac{\partial f}{\partial v} =$$

$$\frac{\partial f}{\partial w} =$$

Example: $u = \phi(\overbrace{x-y}^{g}, \overbrace{y-z}^{h}, \overbrace{z-x}^{t})$.

Prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

Sol^{n}. $u = \phi(g, h, t)$, $g = x - y$, $h = y - z$, $t = z - x$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial g}\cdot\frac{\partial g}{\partial x} + \frac{\partial u}{\partial h}\frac{\partial h}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}.$$

$$= \frac{\partial u}{\partial g} - \frac{\partial u}{\partial t}$$

$$u_y = \frac{\partial u}{\partial y} = -\cancel{\frac{\partial u}{\partial g}} + \cancel{\frac{\partial u}{\partial h}}$$

$$u_z = \frac{\partial u}{\partial z} = -\cancel{\frac{\partial u}{\partial h}} + \cancel{\frac{\partial u}{\partial t}}.$$

$$u_x + u_y + u_z = 0.$$

<u>Example:</u> $f \to$ homogeneous fn. of x, y, z of deg. n.

Prove Euler's Theorem for f.

i.e. $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf.$$

<u>Soln.</u>

$$f(x,y,z) = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right).$$

$$= x^n \phi(u,v), \quad u = \frac{y}{x},\ v = \frac{z}{x}$$

$$\frac{\partial f}{\partial x} = n x^{n-1}\phi(u,v) + x^n \underbrace{\left\{\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}\right\}}_{\frac{\partial \phi}{\partial x}}$$

$$= n x^{n-1}\phi(u,v) + x^n \left\{\frac{\partial \phi}{\partial u}\left(-\frac{y}{x^2}\right) + \frac{\partial \phi}{\partial v}\cdot\left(-\frac{z}{x^2}\right)\right\}$$

$$= n x^{n-1}\phi(u,v) - x^{n-2}\left\{y\frac{\partial \phi}{\partial u} + z\frac{\partial \phi}{\partial v}\right\}.$$

$$\frac{\partial f}{\partial y} = x^n \frac{\partial \phi}{\partial y} = x^n \left\{ \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} \right\}$$

$$= x^n \left\{ \frac{1}{x}\frac{\partial \phi}{\partial u} + 0 \right\}$$

$$= x^{n-1}\frac{\partial \phi}{\partial u}.$$

$$\frac{\partial f}{\partial z} = x^n \frac{\partial \phi}{\partial z} = x^n \left\{ \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z} \right\}$$

$$= x^n \left\{ \frac{\partial \phi}{\partial u}\cdot 0 + \frac{\partial \phi}{\partial v}\cdot \frac{1}{x} \right\}$$

$$= x^{n-1}\frac{\partial \phi}{\partial v}.$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = n\underbrace{x^n \phi(u,v)}_{=f}$$

$$- x^{n-1}\left\{ \cancel{y\frac{\partial \phi}{\partial u}} + \cancel{z\frac{\partial \phi}{\partial v}} \right\}.$$

$$+ \cancel{x^{n-1} y \frac{\partial \phi}{\partial u}} + \cancel{z x^{n-1}\frac{\partial \phi}{\partial v}}.$$

$$= nf.$$

Jacobian

$x_1, x_2 \rightarrow$ two diff. fn^{s} of r, θ.

- Jacobian of x_1, x_2 w.r.to r, θ, denoted by $\dfrac{\partial(x_1, x_2)}{\partial(r, \theta)}$, is defined as:

$$\frac{\partial(x_1, x_2)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{vmatrix}.$$

- $x_1, x_2, \ldots, x_n \longrightarrow y_1, y_2, \ldots, y_n$.

$$\frac{\partial(x_1, x_2, \ldots x_n)}{\partial(y_1, y_2, \ldots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ & \cdots & & \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}.$$

<u>Example:</u> Polar co-ordinate.

$$x = r\cos\theta, \quad y = r\sin\theta.$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r.$$

<u>Example:</u>

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \begin{aligned} &0 < \phi < 2\pi \\ &-\pi < \theta < \pi. \end{aligned}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix} = r^2 \sin\theta$$

- $$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{1}{\dfrac{\partial(r,\theta)}{\partial(x,y)}}$$

- <u>leibnitz's rule for differentiation under the sign of integration</u>

- $$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = f(h(x))\,h'(x) - f(g(x))\,g'(x).$$

- $$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(x,y)\,dy\right]$$

$$= \int_{g(x)}^{h(x)} \frac{\partial f}{\partial x}\, dy + f(x, h(x)) \cdot h'(x) - f(x, g(x)) \cdot g'(x)$$

<u>Example:</u>

$$\lim_{x \to \frac{\pi}{4}} \frac{\int_{2}^{\sec^2 x} f(t)\, dt}{x^2 - \frac{\pi^2}{16}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{f(\sec^2 x) \cdot 2\sec x \cdot \sec x \tan x - f(2) \cdot 0}{2x}$$

$$= \frac{f(2) \cdot 2\sqrt{2} \cdot \sqrt{2} \cdot 1}{\cancel{2} \cdot \frac{\pi}{\cancel{4}\, 2}} = \frac{8}{\pi} f(2)$$

<u>Example:</u> If $y(x) = \int_{\frac{\pi^2}{16}}^{x^2} \underbrace{\frac{\cos x \cdot \cos\sqrt{\theta}}{1 + \sin^2\sqrt{\theta}}}_{f(x,\theta)}\, d\theta$

then $\left.\frac{dy}{dx}\right]_{x=\pi} = ?$

Sol^n: By Leibnitz's rule,

$$\frac{dy}{dx} = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\partial f}{\partial x}\, d\theta + f\left(x, \theta = x^2\right) \cdot \frac{d}{dx}(x^2) - f\left(x, \theta = \frac{\pi^2}{16}\right) \cdot \underbrace{\frac{d}{dx}\left(\frac{\pi^2}{16}\right)}_{=0}.$$

$$= -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cancel{\sin x}\, \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta + 2x\, \frac{\cos x \cdot \cos\sqrt{x^2}}{1+\sin^2\sqrt{x^2}} - 0$$

$$\left.\frac{dy}{dx}\right]_{x=\pi} = 2\pi \cdot \frac{(-1)(-1)}{1+0}$$

$$= 2\pi.$$

· Example: $f(x,y) = \sqrt{|xy|}$, $x = u$, $y = u$.

Not differentiable at $(0,0)$.

$$\times \quad \frac{df}{du} = \frac{\partial f}{\partial x} \overbrace{\frac{dx}{du}}^{=1} + \frac{\partial f}{\partial y} \overbrace{\frac{dy}{du}}^{=1}$$

$$= f_x + f_y .$$

$$\left.\frac{df}{du}\right]_{(0,0)} = f_x(0,0) + f_y(0,0)$$

$$= 0 + 0 = 0.$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0.$$

$$f_y(0,0) = 0$$

• $f(x,y) = \sqrt{|u^2|} = |u|$.

$$\frac{df}{du} = \begin{cases} 1 & \text{if } u > 0 \\ -1 & \text{if } u < 0. \end{cases}$$

does not exist at $u = 0$.

__Example:__ If $u = \log(x^3+y^3+z^3-3xyz)$,

then prove that

i) $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z}.$$

__Sol^n:__ $$x^3+y^3+z^3-3xyz = (x+y+z)\,\cancel{(x+\omega y+}$$
$$(x+\omega y+\omega^2 z)$$
$$(x+\omega^2 y+\omega z).$$

$$\therefore u = \log(x+y+z)+\log(x+\omega y+\omega^2 z)$$
$$+\log(x+\omega^2 y+\omega z).$$

$$\boxed{\begin{matrix}\omega^3=1\\ 1+\omega+\omega^2=0\end{matrix}}$$

$$\frac{\partial u}{\partial x} = \frac{1}{x+y+z}+\frac{1}{x+\omega y+\omega^2 z}+\frac{1}{x+\omega y+\omega^2 z}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y+z}+\frac{\omega}{x+\omega y+\omega^2 z}+\frac{\omega^2}{x+\omega^2 y+\omega z}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x+y+z}+\frac{\omega^2}{x+\omega y+\omega^2 z}+\frac{\omega}{x+\omega^2 y+\omega z}.$$

$$u_x+u_y+u_z = \frac{3}{x+y+z}.$$

ii) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

$\underline{\text{LHS}}$ $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$= \text{RHS}.$

(iii) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u$$

$$= -\frac{3}{(x+y+z)^2}.$$

$\nabla^2 u$

$$\nabla^2 \equiv \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{1}{x+y+z}+\frac{1}{x+\omega y+\omega^2 z}+\frac{1}{x+\omega^2 y+\omega z}\right]$$

$$\frac{\partial^2 u}{\partial y^2} =$$

$$\frac{\partial^2 u}{\partial z^2} =$$

$$\nabla^2 u = -\frac{3}{(x+y+z)^2}$$

<u>Laplacian</u> (f^n of 3 variables x, y, z)

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

- f^{ns} of 2 variables x, y

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

<u>Harmonic f^n.</u> $\nabla^2 f = 0.$

<u>Example:</u> $f(x, y) = \log \sqrt{x^2+y^2}$

Prove that f is a harmonic f^n.

i.e. to prove that $f_{xx} + f_{yy} = 0$.

Example: Show that $\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ is a harmonic fu^n.

Euler's Theorem.

For fu^{ns}. of 2 variables.

. $f(x,y) \rightarrow$ hom. of deg n.

$$\boxed{x f_x + y f_y = n f}$$

i) $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$ (Euler's Th.) — ①.

ii) $x^2\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2 f}{\partial x \partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = n(n-1)f$

(more general result).

proof -

Differentiating ① partially w.r.to x

$$x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial x \partial y} = n\frac{\partial f}{\partial x}. \quad \text{—②}.$$

Differentiating (1) partially w.r.to y

$$x\frac{\partial^2 f}{\partial y\,\partial x} + y\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} = n\frac{\partial f}{\partial y} \quad \text{—— (3).}$$

(2) $\times\, x$ + (3) $\times\, y$

$$x^2\frac{\partial^2 f}{\partial x^2} + x\frac{\partial f}{\partial x} + xy\frac{\partial^2 f}{\partial x\,\partial y} + xy\frac{\partial^2 f}{\partial y\,\partial x} + y^2\frac{\partial^2 f}{\partial y^2} + y\frac{\partial f}{\partial y} = nx\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y}$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2}$$

$$= (n-1)\underbrace{\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial x}\right)}_{nf \text{ by (1).}}$$

$$= n(n-1)f.$$

Example: $\boxed{u = \sin^{-1} \dfrac{x^2+y^2}{x+y}.}$

w.r.to x

i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$

w.r.to y

ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \tan^3 u.$

Example: Verify Euler's Theorem for

$$f(x,y,z) = 3x^2yz + 5xy^2z + 5z^4.$$

$$f(tx, ty, tz) = t^4 f(x,y,z)$$

$\Rightarrow f$ is a hom. fn. in x, y, z of deg. 4.

Claim

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 4f.$$

LHS =

Example: Given $u = z e^{ax+by}$,

$z \rightarrow$ hom. in x, y of deg n.

prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = (ax+by+n)u$

Sol^n:

$$\text{LHS} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial}{\partial x}\left(z e^{ax+by}\right) \cancel{+ y\frac{\partial}{\partial y}\left(z e^{ax+by}\right)}$$

$$+ y\frac{\partial}{\partial y}\left(z e^{ax+by}\right)$$

$$= x\left[\frac{\partial z}{\partial x} e^{ax+by} + z \cdot a\, e^{ax+by}\right]$$

$$+ y\left[\frac{\partial z}{\partial y} e^{ax+by} + z \cdot b\, e^{ax+by}\right].$$

$$= \underbrace{\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)}_{\text{using Euler's Th.}} e^{ax+by} + z(ax+by) \times e^{ax+by}$$

$$= n z e^{ax+by} + z(ax+by)e^{ax+by}.$$

$$= (ax+by+n)\underbrace{z e^{ax+by}}_{u} = (ax+by+n)u$$

$$= \text{RHS}.$$

Summary

- Repeated limits exist & unequal

$\Longrightarrow$ double limit $\boxed{\text{does not}}$ exist

$\not\Longleftarrow$

- Repeated limits exist & equal

$\Rightarrow$ double limit may not exist.

- Repeated limit do not exist.

$\Rightarrow$ double limit may exist.

Continuity of $f(x, y)$ at (a, b).

- $f(a, b)$ defined
- $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists.
- $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

- Partial derivatives $\rightarrow$ $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}, f_{xxx}, \ldots$

- $f_{xy} \neq f_{yx}$.

Sufficient condition for $f_{xy} = f_{yx}$

(Young's Th) f_x, f_y both exist & both are differentiable.

$\Rightarrow$ $f_{xy} = f_{yx}$ at that pt.

Implicit functions

- if $F(x,y)=0$ defines y as a fn. of x, say $y=f(x)$ so that
$$F(x, f(x))=0,$$
we say that $y=f(x)$ is implicitely defined by $F(x,y)=0$

- $F(x,y,z)=0 \longrightarrow z=f(x,y)$ implicitly defined by $F(x,y,z)=0$.

Example:

1. $x^2+y^2+1=0 \to$ not satisfied by any pair of real values (x,y).

2. $x^2+y^2=0 \to$ only at $(0,0)$.

3. $x^2+y^2+z^2-1=0$. implicitely defines
$$z=\pm\sqrt{1-x^2-y^2}$$

- Derivatives of implicit fn^{ns}.

- $F(x,y)=0 \Rightarrow dF = F_x\,dx + F_y\,dy = 0$

curve $\quad y=f(x) \quad \dfrac{dy}{dx} \Rightarrow \boxed{\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}}$.

- $F(x,y,z)=0 \Rightarrow dF = 0$

surface

i.e. $F_x\,dx + F_y\,dy + F_z\,dz = 0$

$\boxed{z = f(x,y)}$

$dz = z_x\,dx + z_y\,dy$

$z_x,\ z_y$

$dz = -\dfrac{F_x}{F_z}\,dx - \dfrac{F_y}{F_z}\,dy$

$$\boxed{z_x = -\frac{F_x}{F_z}, \quad z_y = -\frac{F_y}{F_z}}$$

$y' = -\dfrac{F_x}{F_y}$

$y'' =$

$\boxed{\begin{array}{c} y=f(x) \\ \text{or} \\ F(x,y)=0 \end{array}}$

radius of curvature

$= \dfrac{(1+y'^2)^{3/2}}{|y''|}$

$$y'' = -\frac{\left\{ F_y \frac{d}{dx}(F_x) - F_x \frac{d}{dx}(F_y) \right\}}{F_y^2}.$$

$$= -\frac{\left\{ \begin{array}{l} F_y\left[\frac{\partial}{\partial x}(F_x)\frac{dx}{dx} + \frac{\partial}{\partial y}(F_x)\frac{dy}{dx} \right] \\ - F_x\left[\frac{\partial}{\partial x}(F_y)\frac{dx}{dx} + \frac{\partial}{\partial y}(F_y)\frac{dy}{dx} \right] \end{array} \right\}}{F_y^2}.$$

$$= -\frac{F_y\left(F_{xx} + F_{yx}\, y'\right) - F_x\left[F_{xy} + F_{yy}\, y'\right]}{F_y^2}$$

$$\leftarrow \left[y' = -\frac{F_x}{F_y} \right]$$

$$= -\frac{\left(F_y^2 F_{xx} - 2F_xF_y F_{xy} + F_{yy}F_x^2\right)}{F_y^3}.$$

<u>radius of curvature</u> $\longrightarrow$ For $F(x,y)=0$

$$\rho = \frac{(1+y'^2)^{3/2}}{y''}$$

$$= \frac{-\left(F_x^2 + F_y^2\right)^{3/2}}{F_y^2F_{xx} - 2F_xF_yF_{xy} + F_x^2F_{yy}}$$

Example: Folium of Descartes:

$$x^3 + y^3 - 3axy = 0.$$

$$\downarrow$$

$$F(x,y) = 0.$$

Find y', y''.

Sol^n:

$$y' = -\frac{F_x}{F_y} = -\frac{x^2 - ay}{y^2 - ax}.$$

$$y'' = -\frac{2a^3xy}{(y^2-ax)^3}. \text{ (check).}$$

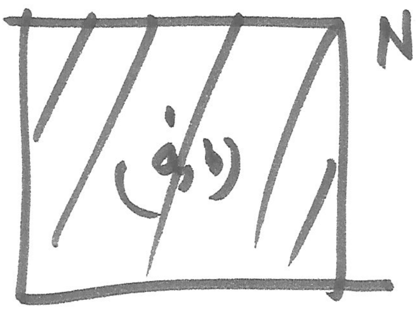
find the curvature

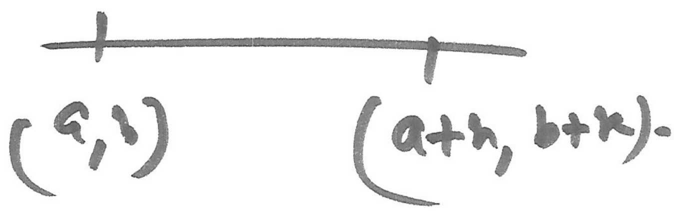
$$\text{Curvature} = \left| \frac{y''}{(1+y'^2)^{3/2}} \right|$$

Taylor's Expansion of fun. of two variables

MVTh. $f = f(x, y)$, f_x, f_y continuous in some nbd. N of (a, b).

$$f(a+h, b+k) - f(a, b)$$
$$= h f_x(a + \theta h, b + \theta k)$$
$$+ k f_y(a + \theta h, b + \theta k),$$
$$0 < \theta < 1.$$





<u>Taylor's Theorem.</u>

- $f = f(x,y)$, defined over a certain domain D having <u>continuo</u> partial ~~der~~ derivatives of order n.

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b)$$

$$+ \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 \frac{f(a,b)}{\lfloor 2} + \cdots$$

$$\cdots + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} \frac{f(a,b)}{\lfloor (n-1)} + \underline{\underline{R_n}}$$

where R_n = remainder after n terms (Lagrange's ~~f~~ form)

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n} \frac{\cancel{f(a,b)}\, f(a+\theta h,\ b+\theta k)}{\lfloor n},$$

$$0 < \theta < 1$$

- <u>Note.</u> • $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$

- $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f = h^n \frac{\partial^n f}{\partial x^n} + \binom{n}{1} h^{n-1} k \frac{\partial^n f}{\partial x^{n-1}\partial y}$

$$+\binom{n}{2} h^{n-2} k^2 \frac{\partial^n f}{\partial x^{n-2}\,\partial y^2} + \cdots$$

$$\cdots + \binom{n}{n} \cancel{h}\, k^n \frac{\partial^n f}{\partial y^n}$$

- $$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f = h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x\,\partial y} + k^2 \frac{\partial^2 f}{\partial y^2}$$

- put $a = 0,\ b = 0,\ h = x,\ k = y \Rightarrow$ Maclaurin's Theorem.

$\cancel{f(a,b)}$:

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\underbrace{\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0)}_{x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)} + \cdots$$

- put $a + h = x,\ b + k = y$ so that

$h = x - a,\ \cancel{y}\ k = y - b.$

$\Rightarrow$ Taylor's expansion of $f(x,y)$ about (a,b).

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b)$$
$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)$$
$$+ \frac{1}{3!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^3 f(a,b)$$
$$+ \cdots\cdots$$

Example: Expand $f(x,y) = x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$.

or

~~Ex~~ Expand $f(x,y)$ about the pt. $(1,-2)$.

Sol^n.
$$f(x,y) = f(1,-2) \cancel{+ \left[(x-1)\frac{\partial}{\partial x} + (y+2)\right.}$$
$$+ \left[(x-1)\frac{\partial}{\partial x} + (y+2)\frac{\partial}{\partial y}\right] f(1,-2)$$
$$+ \frac{1}{2!}\left[(x-1)^2 \frac{\partial^2 f}{\partial x^2}(1,-2) + 2(x-1)(y+2) f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2)\right]$$

$+ \cdots \quad \cdots$

$f(1,-2) = -10$	$f_{xxx}(\cancel{0}1,-2) = 0$
$f_x(1,-2) = -4$	$f_{yxx}(1,-2) = 0$
$f_y(1,-2) = 4$	$f_{yyy}(1,-2) = 0$
$f_{xx}(1,-2) = -4$	$f_{xxy}(1,-2) = 2$
$f_{xy}(1,-2) = 2$	$f_{yxy}(1,-2) = 0$
$f_{yy}(1,-2) = 0$	$f_{xyy}(1,-2) = 0.$

$$\begin{aligned} f(x,y) = & -10 - 4(x-1) + 4(y+2) \\ & - 2(x-1)^2 + 2(x-1)(y+2) \\ & + (x-1)^2(y+2) . \end{aligned}$$

<u>Exercise:</u> Using Taylor's Theorem, prove that

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy + xe^{x} - y}{x\cos y + \sin xy} = 2$$

(Without using L'Hospital rule) $(y = -x)$