Mathematics-I

ROLLE'S THEOREM

Dr. Ratna Dutta
Department of Mathematics
Indian Institute of Technology
Kharagpur 721302

Rolle's Theorem.

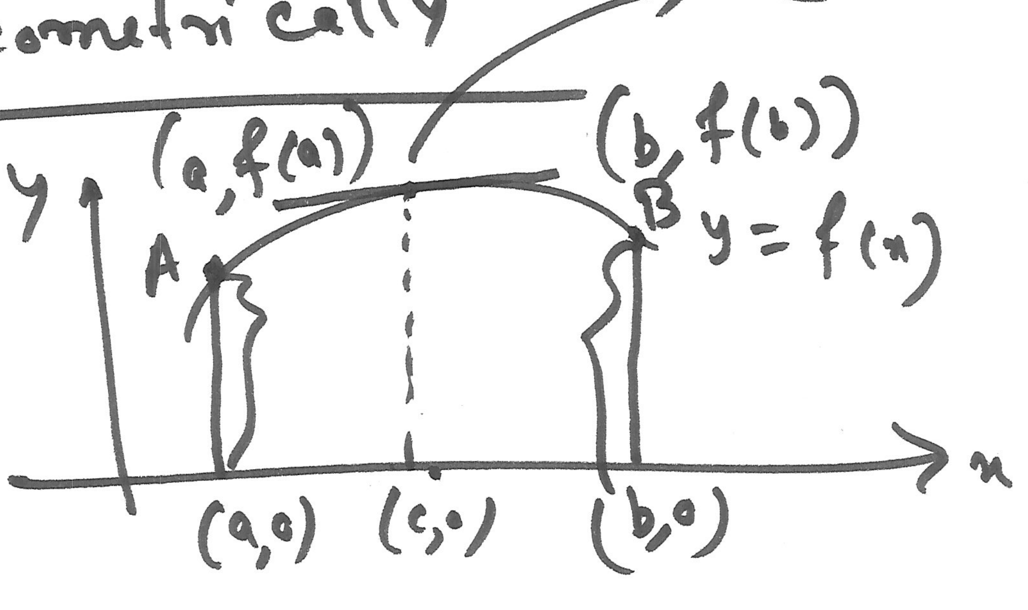
- Let f be a fn. defined on a closed interval $[a,b]$ satisfying:

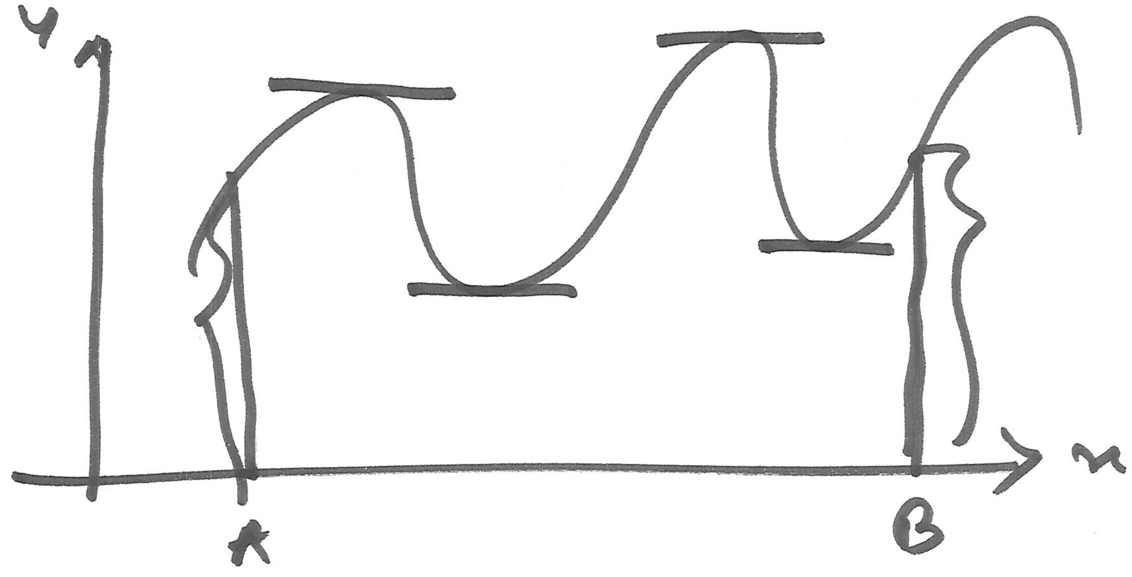
i) f is continuous in $[a,b]$;
ii) f is derivable in (a,b); and
iii) $f(a)=f(b)$.

Then Rolle's Th. states that $\exists$ **at least** one value c, $a<c<b$ st

$$f'(c)=0.$$

Geometrically





Example:

$$f(x) = x\sqrt{a^2-x^2} \quad , \quad [0, a]$$

- $f(x) \longrightarrow$ continuous in $[0, a]$.
- $f'(x) = \sqrt{a^2-x^2} + \dfrac{x\,(-2x)}{2\sqrt{a^2-x^2}}$

$$= \frac{a^2-2x^2}{\sqrt{a^2-x^2}} \text{ exists in } (0,a)$$

- $f(0) = 0 = f(a)$

All the conditions of Rolle's Th. are satisfied.

$\exists$ at least one c, $0<c<a$ s.t.

$f'(c) = 0.$

$c = ?$ **Ans.** $\frac{a}{\sqrt{2}}.$

Example:

$f(x) = \sin x \quad \left[\frac{\pi}{2}, \frac{100\pi}{2}\right]$

Example:

$f(x) = \frac{1}{x} + \frac{1}{1-x}$ in $[0, 1]$

↳ Roll's Th. ~~not~~

Conditions not satisfied.

as $f(0)$, $f(1)$ undefined.

Conclusion of

↳ Roll's Th. may/may not be true.

$c = \frac{1}{2}$

Corollary

If a, b are two roots of equ^n.

$f(x) = 0$,

then the equ^n. $f'(x) = 0$ must have at least one root between a, b, provided

i) $f(x) \rightarrow$ continuous in $[a, b]$

ii) $f'(x) \rightarrow$ exists in (a, b).

Obs 1. Conditions of Rolle's Th.

$\rightarrow$ a set of sufficient conditions.

$\rightarrow$ by no means, these conditions are necessary.

Example: Discuss the applicability of Rolle's Th. to the fn.

$$f(x) = \begin{cases} x^2+1, & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$$

Soln.

At $x = 1$

Check the continuity of $f(x)$.

$$f(1) = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3-x) = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+1) = 2$$

- Continuous in $[0, 2]$

At $x=1$ (Differentiability)

$$\lim_{h\to 0+} \frac{f(1+h)-f(1)}{h} = -1$$

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{\{(1+h)^2+1\} - 2}{h}$$

$$= 2$$

. not differentiable at $x=1$

$\therefore$ Rolle's Th. is not applicable.

Example: Prove if $a_0, a_1, \dots, a_n$ are real numbers s.t.

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \dots + \frac{a_{n-1}}{2} + a_n = 0,$$

then $\exists$ at least one number x,

$0 < x < 1$ s.t.

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$$

Sol^n.

$$f(x) = \frac{a_0}{n+1} x^{n+1} + a_1 \frac{x^n}{n} + \cdots + a_n x$$

→ continuous in $[0, 1]$

→ derivable in $(0, 1)$

→ $f(0) = 0$

$f(1) = 0$ by the given condition.

$\therefore$ By Rolle's Th., $\exists$ at least one c, $0 < c < 1$ s.t.

$f'(c) = 0$

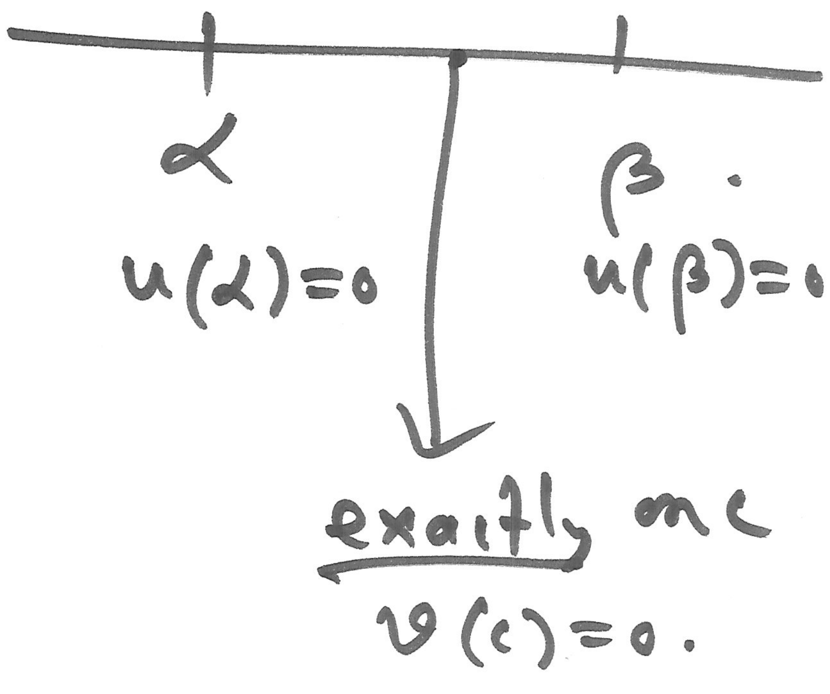
i.e. $a_0 c^n + a_1 c^{n-1} + \cdots + a_n = 0$.

Example: In any interval in which the fn^{ns}. $u(x)$, $v(x)$, $u'(x)$, $v'(x)$ & $\boxed{uv' - u'v \neq 0}$, the ~~roots~~ zer[os] of $u(x)$ & $v(x)$ seperate each other.

e.g. $u(x) = \cos x$
$v(x) = \sin x$.

Sol^n.

$\alpha, \beta \rightarrow$ two roots of $u(x) = 0$.



Sol^n: let $\alpha, \beta \rightarrow$ two consecutive roots of $u(x)=0$. $(\alpha < \beta)$.

Consider the f^n.

$$\boxed{f(x) = \frac{u(x)}{v(x)}}$$

$$f'(x) = \frac{u'v - uv'}{v^2}$$ exists if $v(x) \neq 0$ in ~~$[\alpha, \beta]$~~ (α, β).

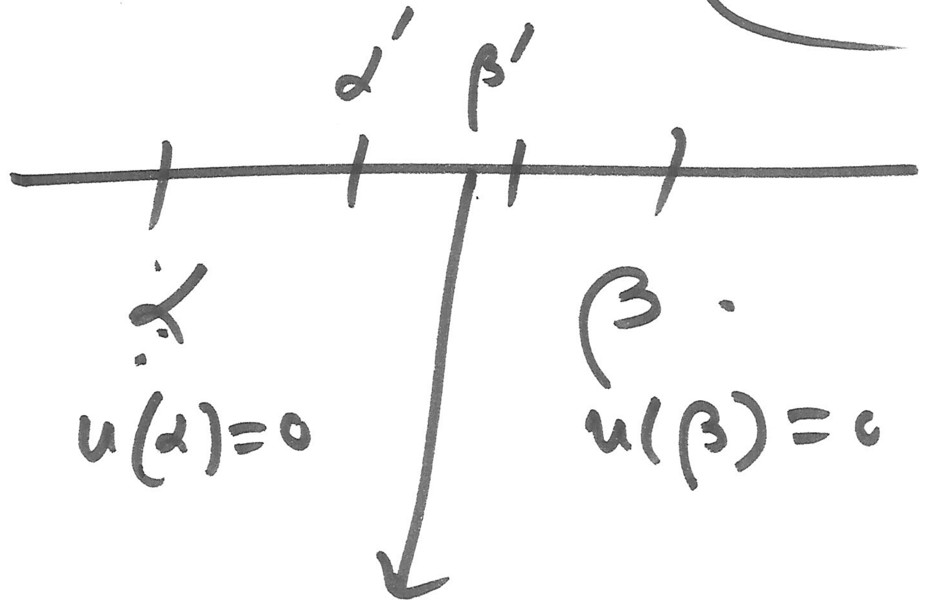
Claim $v(x)=0$ has exactly one root between (α, β).

. if not, then $f(x)$ → continuous in $[\alpha, \beta]$
→ derivable in (α, β).
→ $f(\alpha) = f(\beta) = 0$.

$\therefore$ By Rolle's Th. $\exists$ $\boxed{\text{at least}}$ one $c \in \cancel{(a, b)}\ (\alpha, \beta)$ s.t.

$$f'(c) = 0 \Rightarrow u'(c)\,v(c) - u(c)\,v'(c) = 0.$$

$(\rightarrow\leftarrow)$



at least one root of $v(x)=0$.