24.8.16

Example: If $u = \log(x^3+y^3+z^3-3xyz)$,

then prove that

(i) $u_x + u_y + u_z = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z}$

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{9}{(x+y+z)^2}$

(iii) $\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}}_{\nabla^2 u} = -\frac{3}{(x+y+z)^2}$

Sol^n.

$$x^3+y^3+z^3-3xyz$$
$$= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z), \quad \omega^3=1$$
$$1+\omega+\omega^2=0$$

$$u = \log(x+y+z) + \log(x+\omega y+\omega^2 z) + \log(x+\omega^2 y+\omega z).$$

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{1}{x+\omega y+\omega^2 z} + \frac{1}{x+\omega^2 y+\omega}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{1}{x+y+z} + \frac{\omega}{x+\omega y+\omega^2 z} + \frac{\omega^2}{x+\omega^2 y+}$$

$$u_z = \frac{\partial u}{\partial z} = \frac{1}{x+y+z} + \frac{\omega^2}{x+\omega y+\omega^2 z} + \frac{\omega}{x+\omega^2 y+}$$

$$u_x + u_y + u_z = \frac{3}{x+y+z}$$ proving (i)

(ii) $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+}\right)$$

Complete it.

<u>Example:</u> If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n for which

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

Sol^n:

$$\theta = t^n e^{-\frac{r^2}{4t}}.$$

Find n s.t.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}.$$

$$\theta = \theta(r,t) = t^n e^{-\frac{r^2}{4t}}$$

$$\frac{\partial \theta}{\partial t} = \underline{n t^{n-1} e^{-\frac{r^2}{4t}}} + t^n . e^{-\frac{r^2}{4t}} \left(\frac{r^2}{4t^2}\right)$$

$$= \frac{n}{t}\theta + \frac{\theta r^2}{4t^2} \quad —1.$$

$$\frac{\partial \theta}{\partial r} = t^n e^{-\frac{r^2}{4t}} \left(-\frac{\cancel{2}r}{\cancel{4}_2 t}\right).$$

$$= -\frac{r\theta}{2t}$$

$$r^2 \frac{\partial \theta}{\partial r} = -\frac{r^3\theta}{2t}.$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) =$$

Ans.

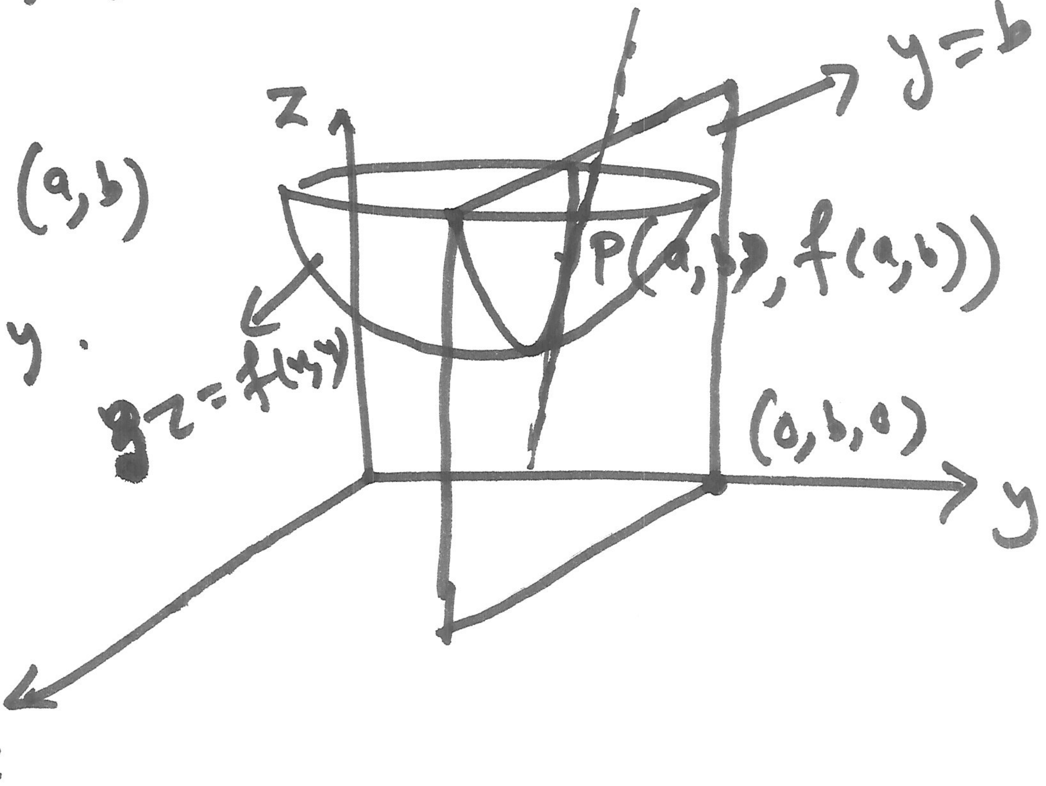
$$\boxed{n = -3/2}$$

1st. order partial derivatives interpreted geometrically

- $z = f(x, y) \rightarrow$ surface in space.
- $\left.\dfrac{\partial z}{\partial x}\right]_{(a,b)}$

  $\Downarrow$

  fixing y.



$\left.\dfrac{\partial z}{\partial x}\right]_{(a,b)} \rightarrow \tan\psi$.

$\rightarrow$ slope of the curve at $P(a, b, f(a,b))$

$\psi \rightarrow$ angle between $\underline{x\text{-axis}}$ and the tangent at P on the curve of intersection of $z = f(x,y)$ with the plane $y = b$.

$$\left.\frac{\partial z}{\partial y}\right]_{(a,b)} \to \tan\phi .$$

$\downarrow$ $\to$ ϕ angle bet. y-axis & the tangent at P on the curve of intersection of $z = f(x,y)$ with the plane $x = a$.

fixing x.

$\downarrow$

$x = a$

Example: Find the slope of the curve of intersection of the ellipsoid

$$\frac{x^2}{24} + \frac{y^2}{12} + \frac{z^2}{6} = 1$$

made by the plane $y = 1$ at the pt.

$$\left(4, 1, \sqrt{\frac{3}{2}}\right)$$

Sol^n.

$$\left.\frac{\partial z}{\partial x}\right]_{(4,1)} = ?$$

Implicit fnⁿ

- $F(x,y) = 0 \xrightarrow{\text{defines}} y = f(x)$ implicitly

  if

  $F(x, f(x)) = 0.$

e.g. · $3x + 2y - 6 = 0$.

defines implicit fnⁿ: $y = \frac{1}{2}(6 - 3x)$

- $F(x,y,z) = 0 \xrightarrow{\text{defines}} z = f(x,y)$ implicitly

  if

  $F(x,y, f(x,y)) = 0.$

e.g. $x^2+y^2+z^2-1 = 0$.

defines two implicit fnˢ.

$z = \pm\sqrt{1-x^2-y^2}$

for $x^2+y^2 \le 1$.

Example:

1. $x^2+y^2+1 = 0.$

   $y = \pm\sqrt{-1-x^2}$

2. $x^2+y^2 = 0.$

do not define any implicit fnⁿ.

3. $x + y + \sin y = 0$.

$x = f(y) = -y - \sin y$.

$x + \sin x + y + \sin y = 0$

$y = f(x)$

or

$x = f(y)$.

$F(x, y) = 0$

finding derivatives of implicit fn's.

- $F(x, y) = 0$ defines $y = f(x)$ implicitely.

$$\frac{dy}{dx} ?$$

- $F(x, y, z) = 0$ defines implicitely the fn. $z = f(x, y)$

z_x, z_y?

$\bullet\ F(x,y)=0 \Rightarrow dF=0=F_x\,dx+F_y\,dy$

$\downarrow$

$y=f(x)$

$\frac{dy}{dx}=y'=?$

$$\boxed{\frac{dy}{dx}=-\frac{F_x}{F_y}}$$

$\bullet\ F(x,y,z)=0 \Rightarrow dF=0=F_x\,dx+F_y\,dy+F_z\,dz$

$\downarrow$

$z=f(x,y) \Rightarrow dz=z_x\,dx+z_y\,dy.$

$$dz=-\frac{F_x}{F_z}dx-\frac{F_y}{F_z}dy.$$

$$\boxed{z_x=-\frac{F_x}{F_z},\quad z_y=-\frac{F_y}{F_z}}$$

Example: ~~Botti~~ Folium of Descartes:

$$x^3 + y^3 - 3axy = 0.$$

find y' & y''

Soln. $F(x,y) = x^3 + y^3 - 3axy = 0.$

$$\boxed{y' = -\frac{F_x}{F_y}} = -\frac{3x^2 - 3ay}{3y^2 - 3ax}$$

$$= -\frac{x^2 - ay}{y^2 - ax}.$$

$$\boxed{y'' = ?}$$

$$y'' = -\frac{\left[F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2\right]}{(F_y)^3}.$$

How? ?

$$= -\frac{2a^3xy}{(y^2 - ax)^3} \text{ (check).}$$

$$F(x,y)=0 \longrightarrow y=f(x).$$

$$y' = -\frac{F_x}{F_y}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(-\frac{F_x}{F_y}\right).$$

$$= -\frac{F_y \frac{d}{dx}\left(\overset{x,y}{F_x}\right) - F_x \frac{d}{dx}\left(\overset{x,y}{F_y}\right)}{(F_y)^2}.$$

$$= -\frac{F_y\left[\frac{\partial}{\partial x}(F_x)\cdot\frac{dx}{dx} + \frac{\partial}{\partial y}(F_x)\cdot\frac{dy}{dx}\right] - F_x\left[\frac{\partial}{\partial x}(F_y)\frac{dx}{dx} + \frac{\partial}{\partial y}(F_y)\frac{dy}{dx}\right]}{(F_y)^2}.$$

$$= -\frac{\left[F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2\right]}{(F_y)^3}.$$