22.8.16

Functions of three variables: Notion of Differentiability

- $f(x,y,z)$ over $D \subseteq \mathbb{R}^3$.
- $x \uparrow \Delta x$, $y \uparrow \Delta y$, $z \uparrow \Delta z$
- $\Delta f = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$

- <u>f is diff. at (x,y,z)</u> if

$$\Delta f = (A.\Delta x + B.\Delta y + C.\Delta z) + (\varepsilon_1.\Delta x + \varepsilon_2.\Delta y + \varepsilon_3.\Delta z)$$

where A, B, C are ind. of $\Delta x, \Delta y, \Delta z$ &

$\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0,0,0)$

- equivalently if

$$\boxed{\Delta f = df + \varepsilon\rho} \quad \text{where } \varepsilon \to 0 \text{ as } \rho \to 0$$

$$\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

df: total differential or principle part of Δf

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

Spherical polar co-ordinates

$\Delta x = \rho \sin\theta \cos\phi \qquad 0 < \phi < 2\pi$

$\Delta y = \rho \sin\theta \sin\phi \qquad -\pi < \theta < \pi.$

$\Delta z = \rho \cos\theta$

$\varepsilon = \varepsilon_1 \sin\theta \cos\phi + \varepsilon_2 \sin\theta \sin\phi + \varepsilon_3 \cos\theta$

$\rightarrow 0$ as $\rho \rightarrow 0$

Apply in the 1st. def^n. of diff.

<u>Homogeneous f^n:</u> : <u>Euler's Theorem</u>

• $f(x_1, x_2, .., x_n) \rightarrow$ homogeneous f^n. of deg. m in variables $x_1, x_2, .., x_n$

<u>if</u>

$f(tx_1, tx_2, .., tx_n) = t^m f(x_1, x_2, .., x_n)$

for every +ve value of t.

- $f(x,y) \rightarrow$ hom. in x, y of deg. $\underline{\underline{m}}$

<u>if</u>

$$f(x,y) = x^m \phi\left(\frac{y}{x}\right)$$

or

$$f(x,y) = y^m \psi\left(\frac{x}{y}\right)$$

<u>Example</u>:

$$f(x,y) = \frac{x+y}{\sqrt{x}+\sqrt{y}} \rightarrow \text{hom. in } x,y \text{ of deg. } \frac{1}{2}$$

$$f(x,y) = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)} = \sqrt{x}\left(\frac{1+\frac{y}{x}}{1+\sqrt{\frac{y}{x}}}\right)$$

- $f(x,y,z) \rightarrow$ hom. in x, y, z of deg m

<u>if</u>

$$f(x,y,z) = x^m \phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

Euler's Theorem

- $u = f(x,y) \to$ hom. of deg n in x, y

Then $$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

proof $$u = f(x,y) = x^n \phi\left(\frac{y}{x}\right).$$

(1) — $u_x = n x^{n-1} \phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$

(2) — $u_y = x^n \phi'\left(\frac{y}{x}\right) \frac{1}{x}$.

(1) $\times x$ + (2) $\times y$ $\Rightarrow$

$$x u_x + y u_y = n \underbrace{x^n \phi\left(\frac{y}{x}\right)}_{=u} - \cancel{x^{n-2+1} y \phi'\left(\frac{y}{x}\right)} + \cancel{x^{n-1} y \phi'\left(\frac{y}{x}\right)}$$

$= nu$

A more general result

- $u = f(x, y) \rightarrow$ hom. in x, y of deg. n

Then
$$\boxed{x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u}$$

provided u has continuous 1st. order & 2nd. order partial derivatives.

proof. By Euler's Th.,

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu} \quad \text{--- (1)}.$$

(1) $\rightarrow$ w.r.to x differentiate partially, get eqn. (2)

(1) $\rightarrow$ w.r.to y differentiate partially, get eqn. (3).

(2) $\times x$ + (3) $\times y$ $\rightarrow$ get the given result

Example: Let $u = \sin^{-1} \dfrac{x^2+y^2}{x+y}$.

Prove that

(i) $x u_x + y u_y = \tan u$

(ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \tan^3 u$.

Soln.

(i) $u = \sin^{-1} \dfrac{x^2+y^2}{x+y}$

$\sin u = \dfrac{x^2+y^2}{x+y}$ → hom. fn. in x, y of deg. 1

$= \dfrac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)}$

$= x\,\phi\left(\frac{y}{x}\right)$

$x \dfrac{\partial}{\partial x}(\sin u) + y \dfrac{\partial}{\partial y}(\sin u) = 1 \cdot \sin u$.

$x \cos u\, u_x + y \cos u\, u_y = \sin u$

$\Rightarrow x u_x + y u_y = \tan u$.

(ii) ~~Exercise~~. Exercise.

Harmonic fn.

- $f(x,y)$.
- Laplacian $\quad \nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

$$\nabla^2 f = f_{xx} + f_{yy}.$$

- $f(x,y,z)$
- Laplacian $\quad \nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}.$
- $\nabla^2 f = f_{xx} + f_{yy} + f_{zz}.$

Harmonic fn. $\quad \nabla^2 f = 0.$

Example: Show that $\dfrac{1}{\sqrt{x^2+y^2+z^2}}$ is a harmonic fn.

Soln.

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

do it

Example: $f(x,y) = \log\sqrt{x^2+y^2}$

Harmonic fn.

Example: $u = \log\left\{\dfrac{x^4+y^4}{x+y}\right\}$

$x u_x + y u_y = ?$

Solⁿ.

$$e^u = \frac{x^4+y^4}{x+y} \rightarrow \text{hom. deg } 3$$

Ans. $\rightarrow$ 3.

Example: If $\boxed{u = z e^{ax+by}}$, where z is a hom. funⁿ. in x, y of deg. n,

prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \cancel{nz}$.
$(ax+by+n)z$

Solⁿ. As z is hom. in x, y of deg. n,

we have $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nz$ —— (1)

(by Euler's Th.)

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial}{\partial x}\left(z e^{ax+by}\right) + y\frac{\partial}{\partial y}\left(z e^{ax+by}\right)$$

$$= x\left(\frac{\partial z}{\partial x} e^{ax+by} + z a e^{ax+by}\right)$$

$$+ y\left(\frac{\partial z}{\partial y} e^{ax+by} + z b e^{ax+by}\right)$$

$$= e^{ax+by}\left[nz + zax + zby\right]$$

$$= u\left(n + ax + by\right) \longrightarrow \text{(proved)}.$$

- $f(x_1, x_2, \dots, x_n) \rightarrow$ hom. f^n in $x_1, x_2, \dots, x_n$ of deg. m.

Euler's Theorem true for this f?

i.e. $$\boxed{x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_n \frac{\partial f}{\partial x_n} = m f(x_1, \dots, x_n)}$$

$\downarrow$

true or not?

Composite functions.

Chain Rules for f^{n}s of 2 variables

1. $z = f(x, y)$, $x = \phi(t)$, $y = \psi(t)$,

   f, ϕ, ψ all diff. f^{n}s.

   $\Rightarrow$ z is also diff. f^{n} of t &

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$ (chain rule)

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$ (total differential)

2. $z = f(x, y)$, $x = \phi(u, v)$, $y = \psi(u, v)$,

   $f, \phi, \psi \rightarrow$ all diff. f^{n}s.

   $\Rightarrow$ z is a diff. f^{n} of u, v and.

$$\boxed{\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \end{aligned}}$$ (chain rule)

& $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$ (total differential)

Chain rules for fn^{ns}. of 3 variables

- $f(x,y,z)$, $x=\phi(t)$, $y=\psi(t)$, $z=\theta(t)$

f, ϕ, ψ, θ all diff. fn^{ns}.

$\Rightarrow f$ is diff. fn^{n}. of t &

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}$$

(Chain rule)

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

- $f(x,y,z)$, $x=\phi(u,v,w)$, $y=\psi(u,v,w)$,
$z=\theta(u,v,w)$

$\rightarrow f$ is diff. fn^{n}. of u,v,w.

chain rule

$$\begin{cases} f_u = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \\ f_v = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}. \\ f_w = \quad \dots \end{cases}$$

$df = ?$

Example: If $u = \phi(x-y,\ y-z,\ z-x)$, find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$

Sol^n: $g = x-y,\quad h = y-z,\quad t = z-x$

$\therefore\ u = \phi(g, h, t)$, g, t, h are themselves fu^{n}. of x, y, z.

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial g}\frac{\partial g}{\partial x} + \frac{\partial u}{\partial h}\frac{\partial h}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial g}\cdot 1 + \frac{\partial u}{\partial h}\cdot 0 + \frac{\partial u}{\partial t}\cdot(-1)$$

$$= \frac{\partial u}{\partial g} - \frac{\partial u}{\partial t}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial g}\overset{=-1}{\frac{\partial g}{\partial y}} + \frac{\partial u}{\partial h}\overset{=1}{\frac{\partial h}{\partial y}} + \frac{\partial u}{\partial t}\overset{=0}{\frac{\partial t}{\partial y}}$$

$$= -\frac{\partial u}{\partial g} + \frac{\partial u}{\partial h}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial g}\overset{=0}{\frac{\partial g}{\partial z}} + \frac{\partial u}{\partial h}\overset{=-1}{\frac{\partial h}{\partial z}} + \frac{\partial u}{\partial t}\overset{=1}{\frac{\partial t}{\partial z}}$$

$$= -\frac{\partial u}{\partial h} + \frac{\partial u}{\partial t}.$$

Ans. 0

Exercise:

Let $u = \dfrac{y^2 - x^2}{x^2 y^2}$, $v = \dfrac{z^2 - y^2}{y^2 z^2}$,

$$x \neq 0, \ y \neq 0, \ z \neq 0$$

and $\omega = f(u, v)$ over R^2, with continuous partial derivatives

Prove that $x^3 \dfrac{\partial \omega}{\partial x} + y^3 \dfrac{\partial \omega}{\partial y} + z^3 \dfrac{\partial \omega}{\partial z} = 0$

at $(1, 2, 3)$.

Example: $f \to$ hom. fn. in x, y, z of deg n.

Prove Euler's Th. for f.

i.e. $$\boxed{x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = n f}.$$

soln: $f = x^n \phi\left(\underbrace{\tfrac{y}{x}}_{u}, \underbrace{\tfrac{z}{x}}_{v}\right)$.

$\therefore \boxed{f = x^n \phi(u, v),}$ $u = \dfrac{y}{x}$, $v = \dfrac{z}{x}$.

① — $\dfrac{\partial f}{\partial x} = n x^{n-1} \phi(u,v) + x^n \left\{ \dfrac{\partial \phi}{\partial u} \dfrac{\partial u}{\partial x} + \dfrac{\partial \phi}{\partial v} \dfrac{\partial v}{\partial x} \right\}$

$$= n x^{n-1} \phi(u,v) - x^{n-2} \left\{ y \frac{\partial \phi}{\partial u} + z \frac{\partial \phi}{\partial v} \right\}$$

② — $\dfrac{\partial f}{\partial y} = x^n \left\{ \dfrac{\partial \phi}{\partial u} \dfrac{\partial u}{\partial y} + \dfrac{\partial \phi}{\partial v} \dfrac{\partial v}{\partial y} \right\}$

$$= x^n \left\{ \frac{\partial \phi}{\partial u} \cdot \frac{1}{x} + \frac{\partial \phi}{\partial v} \cdot 0 \right\}$$

③ — $\dfrac{\partial f}{\partial z} = x^n \left\{ \dfrac{\partial \phi}{\partial u} \dfrac{\partial u}{\partial z} + \dfrac{\partial \phi}{\partial v} \dfrac{\partial v}{\partial z} \right\}$

$$= x^n \left\{ \frac{\partial \phi}{\partial u} \cdot 0 + \frac{\partial \phi}{\partial v} \cdot \frac{1}{x} \right\}.$$

① $\times x$ + ② $\times y$ + ③ $\times z$ $\Rightarrow$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = n \underbrace{x^n \phi(u,v)}_{f} - x^{n-1} \cancel{\left(y \frac{\partial \phi}{\partial u} + z \frac{\partial \phi}{\partial v} \right)}$$

$$+ x^{n-1} \cancel{y} \frac{\partial \phi}{\partial y}$$

$$+ x^{n-1} \cancel{z} \frac{\partial \phi}{\partial v}.$$

$$= nf.$$

Leibnitz's rule for differentiation under integral sign.

- $$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

- $$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} \underline{f(x,y)}\,dy\right] = \int_{g(x)}^{h(x)} \frac{\partial f}{\partial x}\,dy + f(x,h(x))\,h'(x) - f(x,g(x))\,g'(x).$$

Exercise.
Use chain rule to prove this.

Example:

$$\lim_{x \to \frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\, dt}{x^2 - \frac{\pi^2}{16}} . \quad \left(\frac{0}{0}\right)$$

$$= \frac{8}{\pi} f(2).$$

Example: If $y(x) = \displaystyle\int_{\frac{\pi^2}{16}}^{x^2} \underbrace{\frac{\cos x . \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}}_{f(x,\theta).} \, d\theta$

then $\left[\dfrac{dy}{dx}\right]_{x=\pi} = ?$

Sol^n.

Using Leibnitz's rule for diff. under the sign of integral, we get

$$\frac{dy}{dx} = \int_{\frac{\pi^2}{16}}^{x^2} \frac{\partial f}{\partial x} d\theta + f\left(x, \theta = x^2\right) \cdot \frac{d}{dx}(x^2)$$
$$- f\left(x, \theta = \frac{\pi^2}{16}\right) \cdot \underbrace{\frac{d}{dx}\left(\frac{\pi^2}{16}\right)}_{= 0}$$

$$= 2x \cdot \frac{\cos x \cdot \cos\sqrt{x^2}}{1+\sin^2\sqrt{x^2}}$$

$$f(x,\theta) = \frac{\cos x \cdot \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}.$$

$$+ \int_{\frac{\pi^2}{16}}^{x^2} \frac{-\sin x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta .$$

$$\therefore \left.\frac{dy}{dx}\right]_{x=\pi} = \frac{2\pi \cdot (\cos\pi)^2}{1+\sin^2\pi}.$$

$$= 2\pi .$$

Example: $f(x,y) = \sqrt{|xy|}$, $x=u$, $y=u$

Using chain rule

$$x = u$$
$$y = u.$$

$$\frac{df}{du} = \frac{\partial f}{\partial x}\underbrace{\frac{dx}{du}}_{=1} + \frac{\partial f}{\partial y}\underbrace{\frac{dy}{du}}_{=1} = f_x + f_y .$$

$$\left.\frac{df}{du}\right]_{(0,0)} = 0 \qquad f_x(0,0) = 0 = f_y(0,0)$$

false result

$f(u) = |u|$.

which does not have derivative at $u=0$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0.$$

. Chain rule cannot be applied to the fn.

$$\boxed{f(x,y) = \sqrt{xy}}, \quad x=u,\ y=v$$

as this f is not a diff. fn. of x,y.

(Prove it).

Jacobians.

- $x_1, x_2 \rightarrow$ two differentiable fun^{ns}. of r, θ.
- Jacobian of x_1, x_2 w.r.to r, θ is defined as

$$\frac{\partial(x_1, x_2)}{\partial(r, \theta)} = \begin{array}{cl} \begin{vmatrix} \dfrac{\partial x_1}{\partial r} & \dfrac{\partial x_1}{\partial \theta} \\[2ex] \dfrac{\partial x_2}{\partial r} & \dfrac{\partial x_2}{\partial \theta} \end{vmatrix} & \begin{array}{l} \rightarrow x_1 \\[4ex] \rightarrow x_2 \end{array} \end{array}.$$

Example: $x = r\cos\theta, \quad y = r\sin\theta$.

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = r.$$

Example:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

$$0 < \phi < \pi$$
$$-\pi < \theta < \pi.$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= r^2 \sin\theta \quad \text{(check)}.$$

Jacobian of x, y, z
w.r.to r, θ, ϕ.