10.8.16 ✓

Example:

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

Prove that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Sol^n. $f_{yx}(0,0) = \dfrac{\partial}{\partial y}\left(f_x\right)$ at $(0,0)$.

$$= \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$$

Now

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{hk\dfrac{h^2-k^2}{h^2+k^2} - 0}{h}$$

$$= -k.$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$\therefore\ f_{yx}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1$

Now find $f_{xy}(0,0)$ in the same way $= 1$ (check)

- In general, $f_{xy} \neq f_{yx}$

sufficient conditions for $f_{xy} = f_{yx}$.

- Schwartz's Theorem.

- $f(x,y)$ defined in some region D of xy-plane.
- $(a,b) \in D$.
- f_x exists in some nbd. of (a,b)
- f_{xy} continuous at the (a,b).

$\Rightarrow$ f_{yx} at (a,b) exists & $f_{xy}(a,b) = f_{yx}(a,b)$

- Young's Theorem

If f_x, f_y exist in ~~some nbd of~~ (a,b) & if they are $\boxed{\text{differentiable}}$ at (a,b), then $f_{xy} = f_{yx}$ at (a,b)

The total differential: Concept of Differentiability.

- $z = f(x,y)$
- (x,y), $(x+\Delta x, y+\Delta y)$.
- $\Delta z \to$ consequent change in z.

e.g.
$x^2+y^2+z^2=1$.

$z = \pm\sqrt{1-x^2-y^2}$

$$z + \Delta z = f(x+\Delta x, y+\Delta y).$$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y).$$
$$= [f(x+\Delta x, \underline{y+\Delta y}) - f(x+\Delta x, \underline{y})]$$
$$+ [f(\underline{x+\Delta x}, y) - f(\underline{x}, y)].$$

$$= \Delta y f_y(x+\Delta x,\ y+\theta_2 \Delta y)$$

$$+ \Delta x\, f_x\left(x+\theta_1 \Delta x,\ y\right),$$

$0 < \theta_1, \theta_2 < 1$ (by MVTh.).

Let $\underline{f_x(x+\theta_1\Delta x,\ y)} - f_x(x,y) = \varepsilon_1$

$\underline{f_y(x+\Delta x,\ y+\theta_2\Delta y)} - f_y(x,y) = \varepsilon_2$

Then
$$\Delta z = \Delta x\left[\varepsilon_1 + f_x(x,y)\right] + \Delta y\left[\varepsilon_2 + f_y(x,y)\right].$$

$$= \boxed{\Delta x\, f_x(x,y) + \Delta y\, f_y(x,y)} + \Delta x . \varepsilon_1 + \Delta y . \varepsilon_2$$

$$dz = \Delta x . f_x(x,y) + \Delta y . f_y(x,y)$$

$$= \Delta x . \frac{\partial z}{\partial x} + \Delta y \frac{\partial z}{\partial y}.$$

- put $\underline{z = x}$ $\Rightarrow$ $dz = \Delta x = dx$

$\frac{\partial z}{\partial x} = 1,\ \frac{\partial z}{\partial y} = 0$

- put $z = y$ $\Rightarrow$ $dz = \Delta y = dy$

$$\Delta z = dz + (\varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y)$$

where

$$\boxed{\begin{aligned} dz &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \\ &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy. \end{aligned}}$$

Total differential

principal part of Δz.

i.e. $\underline{dx = \Delta x}$, $\underline{dy = \Delta y}$

$dz \neq \Delta z$.

- f is said to be <u>differentiable</u> at a point (x, y) if

$$\boxed{\begin{aligned} \Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= A \cdot \Delta x + B \cdot \Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y \end{aligned}}$$

where A, B are independent of $\Delta x, \Delta y$ & $\varepsilon_1, \varepsilon_2$ are fn^{s}. of $\Delta x, \Delta y$ s.t. $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$ in any manner.

Equivalently,

$$\Delta z = \overbrace{A.\Delta x + B.\Delta y}^{dz} + \varepsilon\rho$$

when $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ &

ε is a fn. of $\Delta x, \Delta y$ s.t.

$\varepsilon \to 0$ as $\rho \to 0$.

$\Delta x = \rho\cos\theta$

$\Delta y = \rho\sin\theta$.

i.e.

$$\boxed{\frac{\Delta z - dz}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \to 0 \text{ as } (\Delta x, \Delta y) \to (0,0)}$$

or

$$\frac{\Delta z - dz}{\rho} \to 0 \text{ as } \rho \to 0.$$

<u>Example:</u> $f(x,y) = xy$.

check the differentiability of f at $(0,0)$.

Sol^n.

$$\Delta f = f(x+\Delta x,\ y+\Delta y) - f(x,y)$$
$$= (x+\Delta x)(y+\Delta y) - xy$$
$$= x\Delta y + y\Delta x + \Delta x\,\Delta y$$

$$df = \Delta x f_x + \Delta y f_y = y\Delta x + x\Delta y$$

$$\varepsilon = \frac{\Delta f - df}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = \frac{\Delta x\,\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$\Delta x \le \sqrt{(\Delta x)^2+(\Delta y)^2}$$
$$\Delta y \le \sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$\le \sqrt{(\Delta x)^2+(\Delta y)^2}$$
$$\to 0 \text{ as } (\Delta x, \Delta y) \to (0,0)$$

$\therefore f(x,y) = xy$ is differentiable at (x,y)

- $h = \Delta x$, $k = \Delta y$.

$$\boxed{\varepsilon = \frac{\Delta f - df}{\sqrt{h^2+k^2}},\quad \Delta f = f(x+h, y+k) \ldots,\quad \boxed{df = f_x\,dx + f_y\,dy}}$$

Example: $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

(i) f continuum at $(0,0)$

(ii) f_x, f_y exist at $(0,0)$

(iii) not differentiable at $(0,0)$.

$$\varepsilon = \left[\frac{\Delta f - df}{\sqrt{h^2+k^2}}\right]_{(0,0)} =$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$df = \underbrace{f_x(0,0)}_{=0} h + \underbrace{f_y(0,0)}_{=0} k.$$

$$= 0$$

$$\Delta f = f(0+h, 0+k) - f(0,0)$$

$$= \frac{hk}{\sqrt{h^2+k^2}} - 0.$$

put
$h = \rho\cos\theta$
$k = \rho\sin\theta$

$$\therefore \varepsilon = \frac{hk}{h^2+k^2}$$

$$= \rho^2 \frac{\sin\theta\cos\theta}{\rho^2} \qquad (h,k)\to(0,0) \Rightarrow \rho\to 0.$$

$$= \sin\theta\cos\theta \not\to 0 \text{ as } \rho\to 0$$

$\Rightarrow$ The fn. f is not differentiable at $(0,0)$.

<u>Example:</u> $f(x,y) = |x|(1+y) = \begin{cases} x(1+y), & x \geq \\ -x(1+y), & x < \end{cases}$

check <u>differentiability</u> at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$\cancel{f_y(0,0)} = \lim_{h\to 0} \frac{|h|}{h} \text{ does not exist}$$

$\Rightarrow$ f is not differentiable at $(0,0)$.

Example: Check the differentiability of the following fun. at (0,0).

$$f(x,y) = \begin{cases} \dfrac{x^6 - 2y^4}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & (x,y) = (0,0) \end{cases}$$

Soln.

$f_x(0,0) = 0 = f_y(0,0)$. (Check)

$$df = h \overset{A}{f_x} + k \overset{B}{f_y} \Big] = 0$$ at (0,0).

$$\Delta f = f(0+h, 0+k) - f(0,0).$$

$$= \frac{h^6 - 2k^4}{h^2+k^2}$$

$$= df + h.\overbrace{\frac{h^5}{h^2+k^2}}^{\varepsilon_1} + k.\overbrace{\frac{(-2k^3)}{h^2+k^2}}^{\varepsilon_2}$$

Ans. So the fun. is diff. at (0,0)

$$= df + h.\varepsilon_1 + k.\varepsilon_2$$

Where $\varepsilon_1 = \dfrac{h^5}{h^2+k^2} \to 0$ as $(h,k) \to (0,0)$

and $\varepsilon_2 = \dfrac{-2k^3}{h^2+k^2} \to 0$ as $(h,k) \to (0,0)$