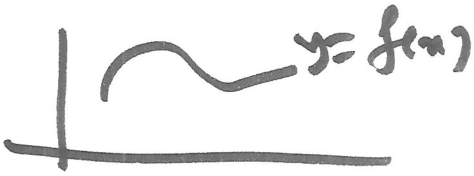
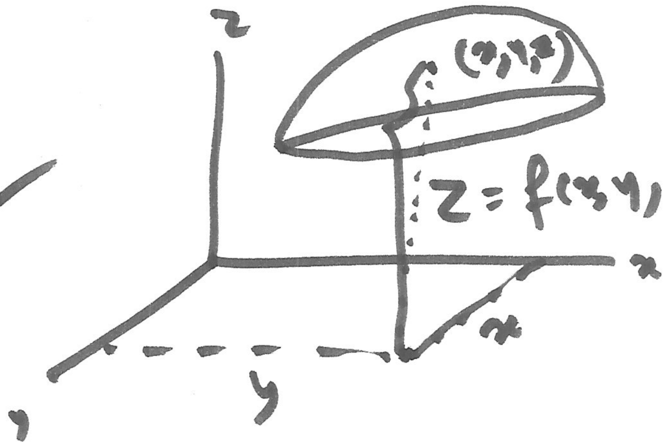
Functions of Several variables

9.8.16

- $y = f(x)$



- $z = f(x,y)$ ✓
- $h = f(x,y,z)$ ✓

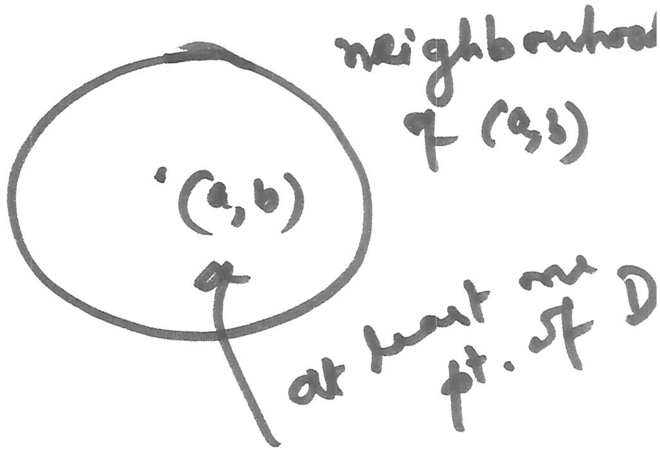


- $g = f(x_1, x_2, \dots, x_n)$

Limit of $f(x,y)$

- $f(x,y)$ defined over a certain domain D.
- (a,b) is a cluster point / limit pt. of D

Double limit



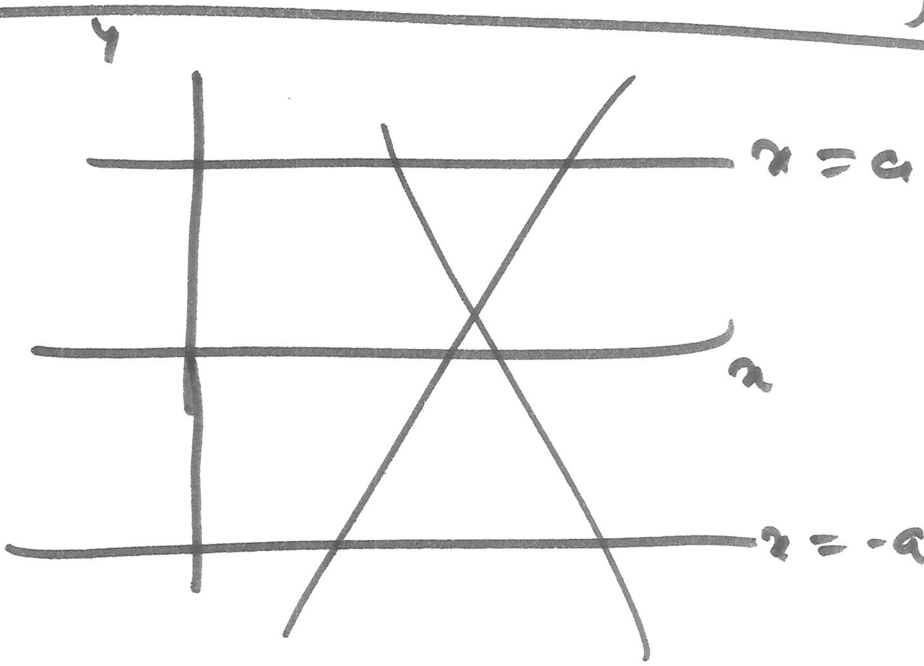
$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

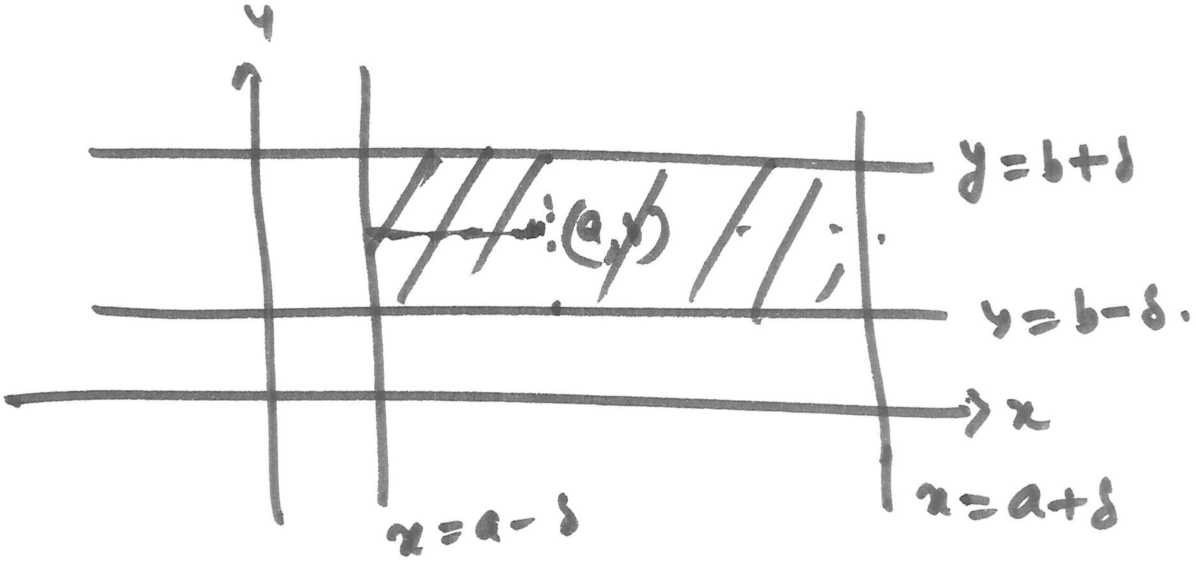
or

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = l$$

$\forall\ \varepsilon > 0$ (arb. small), $\exists$ a $\delta > 0$ (depending on ε) s.t.

$$\boxed{|f(x,y) - l| < \varepsilon \text{ whenever } |x-a| < \delta,\ |y-b| < \delta.}$$





$$\boxed{|f(x,y) - l| < \varepsilon \text{ whenever } (x-a)^2 + (y-b)^2 < \delta^2}$$

Note. The limit l must be unique, along whatever path we may approach to (a,b).

Example: $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

$$= \lim_{\substack{x\to 0\\ y\to 0\\ y=mx}} \frac{x^2 - m^2x^2}{x^2+m^2x^2} = \frac{1-m^2}{1+m^2}$$

Repeated limit / Iterated limit.

$$\lim_{y\to b}\lim_{x\to a} f(x,y) \quad , \quad \lim_{x\to a}\lim_{y\to b} f(x,y)$$

Example: $f(x,y) = \dfrac{x^2 y}{x^4+y^2}$, $(a,b)=(0,0)$

Double limit exists or not ?

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=mx^2}} \frac{x^2\cdot mx^2}{x^4+m^2x^4} = \frac{m}{1+m^2}$$

$\boxed{y = mx^2}$ ✓

Example: $f(x,y) = \dfrac{xy}{x^2+y^2}$

- $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist
- $\boxed{\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0}$
- $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0$

Note: In case double limit exists, the two repeated limit, if exist, are necessarily equal.

(Converse not true)

$\Rightarrow$ if two repeated limits exists & are unequal, the double limit cannot exist.

Example: $f(x,y) = \dfrac{x+y}{x-y}$.

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = 1 \checkmark$$

$$\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = -1 \checkmark$$

So $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ cannot exist.

$y = mx$ $\qquad \displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \frac{1+m}{1-m}.$

Example: Show that

$$\lim_{(x,y)\to(0,0)} xy\,\frac{x^2-y^2}{x^2+y^2} = 0 .$$

To prove that $\forall \varepsilon > 0$ (arb. small), $\exists\ \delta > 0$ s.t.

$$\left| xy\,\frac{x^2-y^2}{x^2+y^2} - 0 \right| < \varepsilon$$

whenever $|x-0| < \delta$, $|y-0| < \delta$.

Soln: let $\boxed{x = r\cos\theta,\ y = r\sin\theta}$.

$$\left| xy\,\frac{x^2-y^2}{x^2+y^2} \right| = \left| r^2 \cos\theta \sin\theta \cos 2\theta \right|$$

$$= \left| \frac{r^2}{4} \sin 4\theta \right|$$

$(x,y) \to (0,0)$
$\Rightarrow r \to 0.$

$$\leq \frac{r^2}{4} = \frac{x^2+y^2}{4} < \varepsilon$$

whenever $\frac{x^2}{4} < \frac{\varepsilon}{2}$, $\frac{y^2}{4} < \frac{\varepsilon}{2}$

i.e. $x^2 < 2\varepsilon$, $y^2 < 2\varepsilon$.

$|x| < \sqrt{2\varepsilon}$, $|y| < \sqrt{2\varepsilon}$

choose $\delta = \sqrt{2\varepsilon}$.

Example: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{5}{2}$ ✓

Sol^n: Let $|x-0|<\delta$, $|y-0|<\delta$, $\delta>0$

$$\left|\frac{3x^2-y^2+5}{x^2+y^2+2} - \frac{5}{2}\right| = \left|\frac{x^2-7y^2}{2(x^2+y^2+2)}\right|$$

$x^2+y^2+2 \geq 2$

$$< \frac{1}{4}\left(|x|^2+7|y|^2\right)$$

whenever $|x|<\delta$, $|y|<\delta$.

$$= \frac{8\delta^2}{4} = 2\delta^2 = \varepsilon.$$

$$\varepsilon = 2\delta^2 \Rightarrow \delta = \sqrt{\frac{\varepsilon}{2}}.$$

$\Rightarrow$ the result.

Exercise. Prove that $\displaystyle\lim_{(x,y)\to(3,1)} (x^2+2x-y^2) = 7$

using $\varepsilon-\delta$ approach.

Example:

$$f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0 & , xy = 0 \end{cases}$$

- Repeated limits? $\rightarrow$ None exists
- Double limit? $\rightarrow$

Claim $\lim_{(x,y)\to(0,0)} f(x,y) = 0.$

To prove that

$|f(x,y) - 0| < \varepsilon$ whenever $|x| < \delta, |y| < \delta$

$(\forall \varepsilon > 0, \exists \delta > 0)$

proof.

$$\left| x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right) \right|$$

$x = r\cos\theta$
$y = r\sin\theta$.

$$\leq |x| + |y|.$$

$$\leq 2r < \varepsilon$$

whenever $r < \frac{\varepsilon}{2}$

$$\sqrt{x^2+y^2} < \frac{\varepsilon}{2}$$

$|x| < \frac{\varepsilon}{2\sqrt{2}}$, $|y| < \frac{\varepsilon}{2\sqrt{2}}$

$\frac{\varepsilon}{2\sqrt{2}} \rightarrow \delta$

$x^2+y^2 < \frac{\varepsilon^2}{4}$ i.e. $x^2 < \frac{\varepsilon^2}{8}$, $y^2 < \frac{\varepsilon^2}{8}$

Example:

does not exist.

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

when

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

along the path

$$x = my^3 + y.$$

Example:

$$f(x,y) = \begin{cases} \dfrac{2(x^3+y^3)}{x^2+2y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Path 1: $y = mx \to \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

Path 1: $y = -\dfrac{x^2}{2}e^x \to \lim\limits_{(x,y)\to(0,0)} f(x,y) = -2$

$\Rightarrow$ limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$$

$x = my^3 + y$

$$= \lim_{y\to 0} \frac{(my^3+y)^3 + y^3}{my^3 + \cancel{y} - \cancel{y}}$$

$$= \lim_{y\to 0} \frac{(my^2+1)^3 + 1}{m} = \frac{2}{m}$$

$\Rightarrow$ Double limit does not exist.

Exercise

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2-x}$$

Continuity.

- $f(x,y)$ is <u>continuous</u> at (a,b)

if

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).}$$

i.e. $\forall\, \underline{\varepsilon > 0}$, $\exists$ a $\delta > 0$ s.t.
(arb. small)

$|f(x,y) - f(a,b)| < \varepsilon$ whenever

$$\sqrt{(x-a)^2+(y-b)^2} < \delta$$

or

whenever $|x-a| < \delta$, $|y-b| < \delta$

Example:

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Example:

$$f(x,y) = \begin{cases} x^2+2y, & (x,y) \neq (1,2) \\ \cancel{0} \to 5, & (x,y) = (1,2) \end{cases}$$

not continuous $\qquad$ continuous

Example:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Sol^n: Do it by yourselves.

Algebra of limits

$f(x,y)$, $g(x,y)$ $\to$ defined on ~~some~~ some nbd. of (a,b).

$\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = l$, $\quad \displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = m$.

Then (i) $\displaystyle\lim_{(x,y)\to(a,b)} (f \pm g) = \lim_{(x,y)\to(a,b)} f \pm \lim_{(x,y)\to(a,b)} g$.

$= l \pm m.$

ii) $\lim (fg) = \lim f \cdot \lim g = l$

iii) $\lim \dfrac{f}{g} = \dfrac{\lim f}{\lim g} = \dfrac{l}{m}$, provided $m \neq 0$.

<u>Example:</u> find $\displaystyle \lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}$.

algebraically.

<u>Sol^n.</u>

$x = r\cos\theta,\ y = r\sin\theta.$

$= 0$ (prove it).

Partial Derivatives of 1st. order

- $z = f(x, y)$.
- $\frac{\partial f}{\partial x}$ or f_x or Z_x
- $\frac{\partial f}{\partial y}$ or f_y or Z_y

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Example: $f(x, y) = \frac{x+y-1}{x+y+1}$

$f_x(2, 1) = ?$

$$f_x(2, 1) = \lim_{h \to 0} \frac{f(2+h, 1) - f(2, 1)}{h}$$

$$= \frac{\frac{2+h+\cancel{1}-\cancel{1}}{2+h+1+1} - \frac{2+1-1}{2+1+1}}{}$$

$= 1/8$

$$f_x(x,y) = \frac{(\cancel{x}+\cancel{y}+1)\cdot 1 - (\cancel{x}+\cancel{y}-1)\cdot 1}{(x+y+1)^2}$$

$$= \frac{2}{(x+y+1)^2}$$

$$f_x(x,y)\Big]_{(2,1)} = \frac{2}{4^2} = \frac{2}{16} = 1/8$$

<u>Example</u>:

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$$

- not continuous at $(0,0)$ as double limit does not exist
- $f_x(0,0)$, $f_y(0,0)$?

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$f_y(0,0) = 0$ (check)

Partial derivatives of higher order

- $f(x,y) \longrightarrow$ fn. of x, y

1st. order

$\underbrace{f_x(x,y), f_y(x,y)}_{}\longrightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

themselves fn^{ns}. of (x,y).

2nd. order

$f_{xx}(x,y) = \frac{\partial}{\partial x}\left\{\frac{\partial f}{\partial x}(x,y)\right\}$

$\frac{\partial^2 f}{\partial x \partial x}$

$f_{yy}(x,y) =$

$f_{xy}(x,y) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y} f(x,y)\right\}$

$\frac{\partial^2 f}{\partial x \partial y}$

$f_{yx}(x,y) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x} f(x,y)\right\}$.

3rd. order

Example: $f(x,y) = \begin{cases} +xy & \text{when } |x| \geq |y| \\ -xy & \text{when } |x| < |y| \end{cases}$

Show that $f_{xy}(0,0) = 1$, $\boxed{f_{yx}(0,0) = -1}$

Soln.

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$$

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial f}{\partial y}\right)\right]_{(0,0)} = \lim_{h \to 0} \frac{h - 0}{h} = 1.$$

f_y

$$f_y(0+h,0) = f_y(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k \to 0} \frac{+hk - 0}{k}$$

$$= h$$

& $$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0-0}{k} = 0.$$

$k \to 0$ means k is sufficiently small, but h is fixed.