03.08.16

Example: Show that $\lim_{h\to 0} \theta = \frac{1}{n+1}$, when

θ is given by

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

1st. 2nd. 3rd.

Remainder after n-th term.

$$\boxed{R_n = \frac{h^n}{n!} f^n(a+\theta h)}$$

/ $0 < \theta < 1$.

provided $f^{n+1}(x)$ is continuous at a, $f^{n+1}(a) \neq 0$.

Sol^n: Apply Taylor's Th. with remainder after n terms & n+1 term.

$$\cancel{f(a+h)} = \cancel{f(a)} + \cancel{hf'(a)} + \cdots + \cancel{\frac{h^{n-1}}{(n-1)!} f^{n-1}(a)} + \frac{h^n}{n!} \times f^n(a+\theta h)$$

$$\cancel{f(a+h)} = \cancel{f(a)} + \cancel{hf'(a)} + \cdots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} \times f^{n+1}(a+\theta' h)$$

$$0 < \theta, \theta' < 1$$

$$\frac{h^n}{n!}\left[f^n(a+\theta h) - f^n(a)\right] = \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta' h)$$

or $$f^n(a+\theta h) = f^n(a) + \frac{h}{n+1} f^{n+1}(a+\theta' h)$$

$\downarrow$ Apply lagrange's MVTh.

$$\cancel{f^n(a)} + \theta h f^{n+1}(a+\theta\theta'' h) = \cancel{f^n(a)} + \frac{h}{n+1} f^{n+1}(a+\theta' h)$$

$$0 < \theta'' < 1$$

let $h \to 0$

$$\lim_{h \to 0} \left[\theta f^{n+1}(a+\theta\theta'' h)\right] = \lim_{h \to 0} \frac{1}{n+1} f^{n+1}(a+\theta' h)$$

$$\Rightarrow \lim_{h \to 0} \theta \cancel{f^{n+1}(a)} = \frac{1}{n+1} \cancel{f^{n+1}(a)}$$

as $f^{n+1}(x)$ continuous at $x=a$

$$\boxed{\lim_{h \to 0} \theta = \frac{1}{n+1}.}$$

Exercise

$$f(x) = y = \frac{1}{\sqrt{1+2x}}.$$

Expand y by Maclaurin's Theorem with remainder after n terms.

$$R_n = \frac{h^n}{n!} f^n(a+\theta h)$$

Taylor's Infinite Series

If (i) f is defined in $[a-h, a+h]$

(ii) for each +ve int. n, $f^{n}(x)$ exists for $x \in [a-h, a+h]$

iii) $\boxed{\lim_{n \to \infty} R_n = 0}$ for each $x \in [a-h, a+h]$,

where R_n is the remainder after n-terms in the Taylor's Theorem,

we have for each $x \in [a-h, a+h]$,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \quad (1)$$

Maclaurin's Infinite series

put $a=0$ in (1).

Example: Expand $\sin x$ in a finite series in power of x

Sol^n:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x), \quad 0<\theta<1.$$

R_n

Claim $\boxed{R_n \to 0 \text{ as } n \to \infty}$ R_n.

$f(x) = \sin x.$

$$\boxed{f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)}.$$

$f'(x) = \cos x$
$= \sin\left(\frac{\pi}{2}+x\right)$

$f''(x) = -\sin x$
$= \sin\left(\frac{2\pi}{2}+x\right)$

$f^n(0) = \sin \frac{n\pi}{2}.$

$$f(x) = \sin x = 0 + x + 0 \cdot \frac{x^2}{2!} + \frac{x^3}{3!}(-1) + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{n-1}}{(n-1)!} \sin (n-1)\frac{\pi}{2} + R_n$$

$$R_n = \frac{x^n}{n!} \sin\left(\frac{n\pi}{2} + \theta x\right), \quad 0 < \theta <$$

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0}$$ why?

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{x^n}{n!} \sin\left(\frac{n\pi}{2} + \theta x\right)$$

bdd. fun

$$= 0.$$

ratio test for $\sum_{n=1} a_n$

let $a_n = \dfrac{x^n}{n!}$

$$\boxed{\begin{array}{l}\textbf{Theorem:} \quad \text{if } \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l, \quad 0 \le l < 1 \\ \text{then } \displaystyle\lim_{n\to\infty} a_n = 0.\end{array}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right|$$

$\to 0$ as $n \to \infty$

$\forall$ real x.

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

i.e. $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$.

<u>Series</u> $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots\cdots$

$S_n = a_1 + a_2 + \cdots + a_n$

<u>Series converges if</u> $S_n \to S$ (finite) as $n\to\infty$.

<u>Ratio test</u>

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l,\ 0 \le l < 1$$

then <u>series converges.</u>

$$\Rightarrow S_n \to S \text{ as } n\to\infty.$$

$$\lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} a_n$$

$$0 = S - S$$

$\sum_{n=1}^{\infty} a_n$ Converge. $\Rightarrow$ $\lim_{n\to\infty} a_n = 0$.

$$\not\Leftarrow$$

e.g. Harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$$

$\frac{1}{n} \to 0$ as $n\to\infty$, but the series does not converg

<u>Example:</u>

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

$\forall$ real x

$\cancel{\forall x \in \mathbb{R}}$.

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$\forall$ real x

$\cancel{\forall x \in \mathbb{R}}$.

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

$\cancel{\forall x \in \mathbb{R}}$

$\forall$ real x

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

$$-1 < x \leq 1$$

$$(1+x)^{m} = \begin{cases} 1+\binom{m}{1}x+\binom{m}{2}x^2+\dots+\binom{m}{m}x^m & \forall x \in \mathbb{R} \text{ if } m \text{ is a +ve integer} \\ 1+mx+\frac{m(m-1)}{2}x^2+\dots\dots & -1 < x < 1 \text{ \& } m \text{ is not a +ve integer.} \end{cases}$$

Power Series expansion

- $\displaystyle\sum_{n=0}^{\infty} a_n x^n = \underbrace{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n}_{\to S_n(x)} + \dots$

$\to x$ real variable

$a_0, a_1, \dots$ real const.

- Converges if $\boxed{\lim_{n\to\infty} S_n(x) = S(x)}$

When $S_n(x) = a_0 + a_1 x + \dots + a_{n-1}x^{n-1}$.

otherwise, diverges.

- $x = 0 \to$ it always converges.
- $\forall x$ in $]-R, R[$, the power series

converges

→ R is called the radius of convergence.

$]-R, R[$ is called interval of convergence.

- $R = \infty$ if the power series converges $\forall$ real x.

How to find R?

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n$$

$\downarrow$ Apply. ~~converges by~~ Ratio test

if $$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{a_{n+1}\, x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x|.$$

$\Rightarrow l$, $0 \le l < 1$
as $n \to \infty$

$\Rightarrow \sum_{n=0}^{\infty} b_n$ converges by Ratio test.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = \frac{1}{R}|x| < 1$

$\left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{R}$.

$\Rightarrow |x| < R$.

R → radius of convergence

Exercise. Cauchy fn.

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Try to get the Maclaurin's series expansion.

convergent to 0, not $f(x)$

- $R_n \to 0$ as $n \to \infty$ NASC for the equality of fn with its Maclaurin's series

- $R_n \to 0$ $n \to \infty$ only sufficient condition for convergence of Maclaurin's series.

Exercise.

Prove that the Maclaurin's series expansion of $\sin^{-1}x$ upto x^7 term is

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots$$

Sol^n:

$$y = \sin^{-1}x.$$

$$y' = \frac{1}{\sqrt{1-x^2}}$$

$$(y')^2(1-x^2) - 1 = 0.$$

diff. w.r.to x

$$y''(1-x^2) - xy' = 0.$$

↙ diff. n-times & using <u>Leibnitz's rule.</u>

$$y_{n+2}(1-x^2) + \binom{n}{1} y_{n+1}(-2x) + \binom{n}{2} y_n(-2) - x y_{n+1} - \binom{n}{1} y_n = 0.$$

<u>Leibnitz's rule.</u>

$$\frac{d^n}{dx^n}\{u(x)v(x)\} = \underline{u_n} + \binom{n}{1}\underline{u_{n-1}}v + \binom{n}{2}u_{n-2}v_2 + \cdots + \binom{n}{n}v_n$$

$$\boxed{y_{n+2} = \frac{(2n+1)\, x\, y_{n+1} + n^2 y_n}{1-x^2}}$$

at $x=0$, $y_{n+2}(0) = n^2 y_n(0)$

<u>Exercise.</u> $\tan^{-1}x$??