1.8.16

- Cauchy's MVTh.

$g(x)$, $f(x)$ over $[a,b]$.

- both continuous in $[a,b]$
- both derivable in (a,b).
- $g'(x) \neq 0$ anywhere in (a,b).

Then $\exists$ at least one c, $a<c<b$ s.t.

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

proof.

let

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}\left(g(x)-g(a)\right)$$

$\Downarrow$

satisfies all the conditions of the Rolle's Th.

$h(a) = 0 = h(b)$

$\Rightarrow$ $\exists$ at least one $c \in (a,b)$ s.t.

$h'(c) = 0$.

$\Downarrow$

result follows.

- h-θ form of Cauchy's MVTh.

$[a,b] \rightarrow [a, a+h]$.

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, \quad 0<\theta<1$$

- if $f(a)=0=g(a)$, then Cauchy's MVTh. reduces to

$$\frac{f(b)}{g(b)} = \frac{f'(c)}{g'(c)}, \quad a<c<b.$$

Example:

i) $f(x)=e^x, \quad g(x)=e^{-x}, \quad [a,b]$

$\downarrow$

$c = \dfrac{a+b}{2} \rightarrow \boxed{a.m.}$

ii) $f(x)=\sqrt{x}, \quad g(x)=\dfrac{1}{\sqrt{x}}, \quad [a,b]$

$\downarrow$

$c=\sqrt{ab} \rightarrow \boxed{g.m.}$

iii) $f(x)=\dfrac{1}{x^2}, \quad g(x)=\dfrac{1}{x}, \quad [a,b]$

$\downarrow$

$c = \dfrac{2}{\frac{1}{a}+\frac{1}{b}} \rightarrow \boxed{h.m.}$

Example: Use Cauchy's MVTh. to evaluate

$$\lim_{x \to 1} \left[\frac{\cos \frac{\pi x}{2}}{\log\left(\frac{1}{x}\right)} \right].$$

Sol^n: let $f(x) = \cos \frac{\pi x}{2}$

$g(x) = \log\left(\frac{1}{x}\right)$, $[x, 1]$.

Apply Cauchy's MVTh.

$$\frac{f(1) - f(x)}{g(1) - g(x)} = \frac{f'(c)}{g'(c)}, \quad \boxed{x < c < 1}$$

$$\frac{\cos \frac{\pi}{2} - \cos \frac{\pi x}{2}}{\log 1 - \log\left(\frac{1}{x}\right)} = \frac{-\frac{\pi}{2} \sin \frac{\pi c}{2}}{c \cdot \left(-\frac{1}{c^2}\right)}.$$

$$\frac{\cos \frac{\pi x}{2}}{\log\left(\frac{1}{x}\right)} = \frac{\pi}{2} c \sin \frac{\pi c}{2}$$

$$\lim_{x \to 1} \frac{\cos \frac{\pi x}{2}}{\log\left(\frac{1}{x}\right)} = \lim_{c \to 1} \frac{\pi}{2} c \sin \frac{\pi c}{2} = \frac{\pi}{2}$$

Example: $f(x)$, $\phi(x)$, $\psi(x)$ satisfy the 1st. two conditions of Rolle's Th. in $[a,b]$, then $\exists$ a value $c \in (a,b)$

s.t. $$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0.$$

Sol^n: let

$$F(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f(x) & \phi(x) & \psi(x) \end{vmatrix}.$$

$F(a) = 0 = F(b)$.

$F(x)$ $\Rightarrow$ cont. in $[a,b]$
$\Rightarrow$ derivable in (a,b)
$\Rightarrow$ $F(a) = F(b)$.

$\therefore$ By Rolle's Th., $F'(c) = 0$ for some c, $a < c < b$.

$\Downarrow$

i.e. result.

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0, \quad a < c < b.$$

i) $\boxed{\phi(x) = \text{const.} = K \text{ (say)}}$

$$\begin{vmatrix} f(a) & K & \psi(a) \\ f(b) & K & \psi(b) \\ f'(c) & 0 & \psi'(c) \end{vmatrix} = 0.$$

Simplify.

$$\frac{f(b) - f(a)}{\psi(b) - \psi(a)} = \frac{f'(c)}{\psi'(c)}$$

Cauchy's MVTh.

ii) $\boxed{\psi(x) = x.}$ → Lagrange's MVTh.

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

iii) put $f(b) = f(a)$

$\exists\, f'(c) = 0.$ → Rolle's Th.

Proof of Rolle's Th.

- $f(x)$ continuous in $[a,b]$.

$\Rightarrow$ has max M & min m.

$$M = \max_{x \in [a,b]} f(x)$$

$$m = \min_{x \in [a,b]} f(x).$$

max^m, min^m attained.

$\Rightarrow \exists$ some $\alpha \in [a,b]$ s.t.

$$\boxed{M = f(\alpha)}$$

Case I. $m = M$ (trivial case).

$f(x) = M \ \forall \ x \in [a,b]$.

$\Rightarrow f'(x) = 0 \ \forall \ x \in [a,b]$.

Case II $M \neq m$. $\qquad f(a) = f(b)$

$\cancel{M = f(\alpha)}$ $\qquad$ let $M \neq f(a)$.

A) $f(a) = f(b)$, & $M \neq m$,

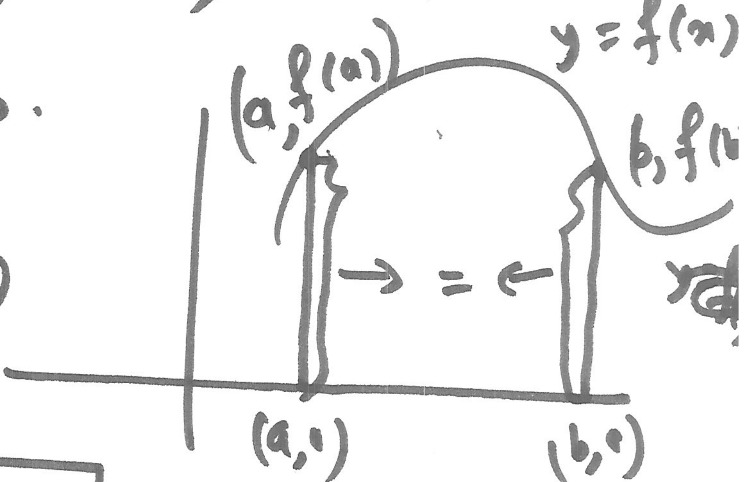
one of M or $m \neq f(a)$

let $M \neq f(a)$

$f(\alpha) \neq f(a) \Rightarrow \alpha \neq a$.

Also $f(\alpha) \neq f(b) \Rightarrow \alpha \neq b$.

$a < \alpha < b$.



By hypothesis, $f'(\alpha)$ exists.

Claim $\boxed{f'(\alpha) = 0}$

proof

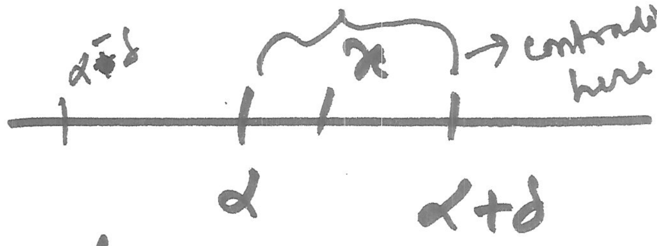
if not, either i) $f'(\alpha)$ is a finite +ve no. or +d

$\Rightarrow f'(\alpha) > 0$.

or ii) $f'(\alpha) < 0$.

Case (i) if $f'(\alpha) > 0$.

$$f'(\alpha) = \lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{\underbrace{(x - \alpha)}_{>0}} > 0$$



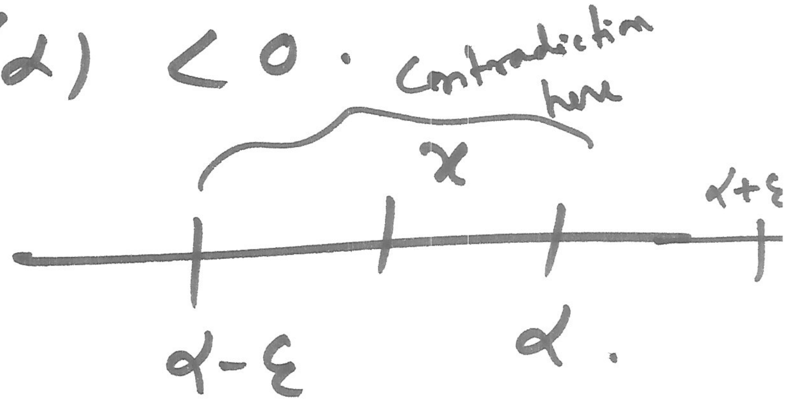
$\Rightarrow \exists$ an open interval $\delta > 0$

$(\alpha, \alpha+\delta)$ at every pt. of which.

$$f(x) > f(\alpha) = M.$$

$(\rightarrow\leftarrow)$.

Case (ii) if $f'(\alpha) < 0$.



$\varepsilon > 0$

$(\alpha-\varepsilon, \alpha)$.

Indeterminate forms : L'Hospital's Rule.

A> $\left(\frac{0}{0}\right)$ form.

If f, g two fu^{ns}. s.t.

i) $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$

ii) $f'(x)$, $g'(x)$ exist & $g'(x) \neq 0$ $\forall x \in (a-\delta, a+\delta)$, $\delta > 0$, except possibly at a, and

iii) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

<u>Example:</u> $\lim_{x \to 0} \frac{\sin x}{x}$ $\left(\frac{0}{0}\right)$.

$= \lim_{x \to 0} \frac{\cos x}{1} = 1$.

Example: Determine the values of a, b, c so that

$$\frac{a e^{x} - b\cos x + c e^{-x}}{x \sin x} \to 2 \text{ as } x \to 0$$

Sol^n:

$$\lim_{x \to 0} \frac{a e^{x} - b\cos x + c e^{-x}}{x \sin x} \quad \left(\frac{0}{0}\right).$$

$$\boxed{a - b + c = 0.}$$

$$= \lim_{x \to 0} \frac{a e^{x} + b\sin x - c e^{-x}}{\sin x + x\cos x.} \quad \left(\frac{0}{0}\right)$$

$$\boxed{a - c = 0.}$$

$$a = c.$$

$$= \lim_{x \to 0}$$

$$2a = b.$$

$$\boxed{\begin{aligned} a &= 1 \\ b &= 2 \\ c &= 1 \end{aligned}} \quad \text{Ans.}$$

$$\boxed{a + b + c = 4}$$

B) $\left(\frac{\infty}{\infty}\right)$ form.

<u>Example:</u> Find $\lim_{x\to 0} \log_{\tan^2 x} \tan^2 2x$.

<u>Soln:</u>

$$\lim_{x\to 0} \frac{\log_e \tan^2 2x}{\log_e \tan^2 x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} 2\,\frac{\sec^2 2x}{\sec^2 x} \cdot \frac{\tan x}{\tan 2x}$$

$= 1.$

$$\boxed{\lim_{x\to 0} \frac{\tan x}{x} = 1}$$

$$\lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x}$$

$$= \lim_{x\to 0} \frac{\cos x}{1} \cdot \times 1 \quad \left(\frac{0}{0}\right)$$

$$= 1$$

C. Other ~~forms~~ Indeterminate form.

$0 \times \infty$, $\infty - \infty$, $\underline{0^0, \infty^0, 1^{\pm\infty}}$

$0 \times \infty \rightarrow \frac{0}{0}$ or $\frac{\infty}{\infty}$.

$\infty - \infty \rightarrow \quad f(x) - g(x)$. $[\infty - \infty]$

$$= \underbrace{f(x)\,g(x)}_{\infty} \underbrace{\left[\frac{1}{g(x)} - \frac{1}{f(x)} \right]}_{\times \ 0}.$$

Example: $\lim_{x \to 0+} x^m (\log x)^n \rightarrow 0 \times \infty$,

m, n +ve.

Sol^n. $\lim_{x \to 0+} \frac{(\log x)^n}{\left(x^{-\frac{m}{n}}\right)^n} \quad \left(\frac{\infty}{\infty}\right)$.

$$= \lim_{x \to 0+} \left(\frac{\log x}{x^{-\frac{m}{n}}} \right)^n .$$

Now $\lim\limits_{x \to 0+} \dfrac{\log x}{x^{-\frac{m}{n}}}$ $\left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x \to 1+} \frac{\frac{1}{x}}{-\frac{m}{n}\, x^{-\frac{m}{n}-1}}$$

$$= \lim_{x \to 0+} -\frac{n}{m} \cdot x^{\frac{m}{n}}$$

$= 0$ as m, n both +ve.

~~Example~~: Exercise $\lim\limits_{x \to \alpha} \left[x - \sqrt[n]{(x-a_1)(x-a_2)\cdots(x-a_n)} \right]$

$(\alpha - \alpha)$.

Example: let $L = \lim\limits_{x \to 0} \left[\dfrac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4} \right]$

$a > 0$. If L is finite, then prove that $L = \dfrac{1}{64}$.

Sol^n. $$L = \lim_{x \to 0} \left[\frac{a - \sqrt{a^2 - x^2}}{x^4} - \frac{1}{4x^2} \right]$$

$(\infty - \infty)$

$$= \lim_{x \to 0} \left[\frac{\cancel{a^2} - (\cancel{a^2} - x^2)}{x^4 \left(a + \sqrt{a^2 - x^2}\right)} - \frac{1}{4x^2} \right]$$

$$= \lim_{x \to 0} \left[\frac{4 - a - \sqrt{a^2 - x^2}}{4x^2 \left(a + \sqrt{a^2 - x^2}\right)} \right] \left(\frac{0}{0}\right)$$

Complete it.

$$4 - a - a = 0$$
$$\Downarrow$$
$$a = 2$$

Exercise: If $\lim_{x \to 0} \left[1 + x \ln(1 + b^2)\right]^{\frac{1}{x}} = 2b \sin^2\theta, b > 0.$

and $\theta \in [-\pi, \pi]$, then

the value of θ is $\pm \frac{\pi}{2}$.

Example: Prove that

$$\lim_{x\to\infty}\left(\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+\cdots+a_n^{\frac{1}{x}}}{n}\right)^{nx}=a_1a_2\cdots a_n$$

$\to 1^{\infty}$

Sol^n:

Let $$y=\left(\frac{\sum_{i=1}^{n}a_i^{\frac{1}{x}}}{n}\right)^{nx}.$$

$$\therefore \log y = nx\left(\log\sum_{i=1}^{n}a_i^{\frac{1}{x}}-\log n\right).$$

$$\lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{\log\left(\sum_{i=1}^{n}a_i^{\frac{1}{x}}\right)-\log n}{\frac{1}{nx}}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{1}{\sum_{i=1}^{n}a_i^{\frac{1}{x}}}\left(-\frac{1}{x^2}\right)\sum_{i=1}^{n}a_i^{\frac{1}{x}}\log a_i}{\frac{1}{n}\cdot\left(-\frac{1}{x^2}\right)}.$$

$$= \lim_{x \to \alpha} \frac{\frac{1}{\sum_{i=1}^{n} a_i^{\frac{1}{x}}} \sum_{i=1}^{n} a_i^{\frac{1}{x}} \log a_i}{\frac{1}{n}}$$

$$= \sum_{i=1}^{n} \log a_i$$

$$= \log a_1 + \log a_2 + \dots + \log a_n$$

$$\log \lim_{x \to \alpha} y = \log (a_1 a_2 \dots a_n)$$

$$\Rightarrow \lim_{x \to \alpha} y = a_1 a_2 \dots a_n$$

(proved)

• L'Hospital's Rule $\left(\frac{0}{0}\right)$

i) $f, g \to 0$ as $x \to a$.

ii) f', g' both exist in a δ-nbd. of a. $g' \neq 0$.

iii) $\frac{f'}{g'} \to l$ (finite) as $x \to a$.

$\Rightarrow \frac{f}{g} \to l$ as $x \to a$.

<u>proof.</u>

• f', g' exists

$\Rightarrow f, g$ continuous in $[a, x]$.

• f, g derivable in (a, x).

• $f(a) = g(a) = 0$ as f, g continuous at $x = a$. & $f, g \to 0$ as $x \to a$

iii) $g' \neq 0$ $\forall x$ in (a, x).

By Cauchy's MVTh. for $f(x)$, $g(x)$ in $[a, x]$.

$$\frac{f(x) - \overset{=0}{f(a)}}{g(x) - \underset{\overset{\shortparallel}{0}}{g(a)}} = \frac{f'(c)}{g'(c)}, \quad c \in (a, x)$$

$a < c < x$

$x \to a \Rightarrow c \to a$

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{\substack{\cancel{x \to a} \\ c \to a}} \frac{f'(c)}{g'(c)} = l}$$

<u>Note:</u> Conditions of L'Hospital's rule are only sufficient, by no means necessary.

<u>Example:</u>

$$\lim_{x \to \infty} \frac{e^{-2x}(\cos x + 2\sin x)}{e^{-x}(\cos x + \sin x)} \cdot \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \infty} \frac{e^{-2x}(-5\sin x)}{e^{-x}(-2\sin x)}$$

does not exist.

$$\begin{cases} f(x) = e^{-2x}(\cos x + 2\sin x) \\ g(x) = e^{-x}(\cos x + \sin x) \end{cases}$$

(i), (ii) satisfied.

but (iii) violated. $\lim_{x\to\alpha} \frac{f'}{g'}$ does not exist.

$$\lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = \begin{cases} 0 & \text{if } \sin x \neq 0 \\ \text{does not exist} & \text{o.w.} \end{cases}$$

Conclusion of L'Hospital's rule may or may not true

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \lim_{x\to\alpha} e^{-x}\left(\frac{1+2\tan x}{1+\tan x}\right)$$

does not exist.

Conclusion of L'Hospital's rule not true for this example. (Show it).

Example: $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x} + \sin\frac{1}{x}$

$x \to 0$.

Condition (iii) violated

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} \Rightarrow ✓$$

$$= \lim_{x \to 0} \frac{1}{1 + \cos\frac{1}{x}}.$$

does not exist.

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{\sin x}{x}}.$$

$$= \lim_{x \to 0} \frac{1}{1 + x\frac{\sin x}{x}}. = 1.$$

Conclusion of L'Hospital's rule holds although condition (iii) does not hold.

Generalized MVTh. — Taylor's Th.

- If a f^n. f is such that
  i) the $(n-1)$th. derivative f^{n-1} continuous in $[a, a+h]$
  ii) the nth. derivative f^n exists in $(a, a+h)$

then

$$\boxed{f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n}$$

where $\boxed{R_n = \frac{h^n}{n!} f^n(a+\theta h), \quad 0<\theta<1.}$

Lagrange's.

Remainder after n-terms.

- $n=1$ → Lagrange's MVTh.

Maclaurin's Theorem.

$[a, a+h] \to [0, x]$. putting $a=0$, $a+h=x$.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

when $R_n = \dfrac{x^n}{n!} f^n(\theta x)$, $0 < \theta < 1$.

Example: $f(x) = e^x$.

By Maclaurin's Theorem with remainder after n-term,

$$f(x) = \underline{f(0)} + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x),$$

$0 < \theta < 1$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\theta x}$$

$R_n = \frac{x^n}{n!} e^{\theta x}$, $0 < \theta < 1$.

error term

$x = 1$

$$e = \underbrace{1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!}}_{} + R_n$$

Estimate error in e

$$R_n = \frac{e^\theta}{n!} < \frac{e}{n!} < \frac{3}{n!}$$

(as $2 < e < 3$)

find n to calculate e with an error at most 10^{-5}

i.e. $R_n < \frac{3}{n!} < 10^{-5}$

$\Rightarrow n = 10$.

$\therefore$ value of e correct to six decimal places is

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{9!}$$

$= 2.718281$.

e is irrational. ✓

if not, let $e = \frac{p}{q}$, p, q two integers

expand e by Maclaurin's Th.

Choose $\boxed{n-1 > q.}$

$$(n-1)!\,e^{=\frac{p}{q}} = (n-1)!\Big[\underbrace{1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{(n-1)!}}_{\text{integer}} + \frac{e^{\theta}}{n!}$$

LHS → integer.

$0 < \theta < 1$

$e^{\theta} < e < 3.$

$$\frac{e^{\theta}}{n} < \frac{e}{n} < \frac{3}{n} < 1$$

for sufficiently large n.

RHS fraction.