16.11.16

Residues.

- $f(z)$ is a fn. of complex variable z.
- $z_1 \Rightarrow$ an isolated singularity of $f(z)$.

Then Laurent's series expansion of $f(z)$

in $$R_1 < |z - z_0| < R_2$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz.$$

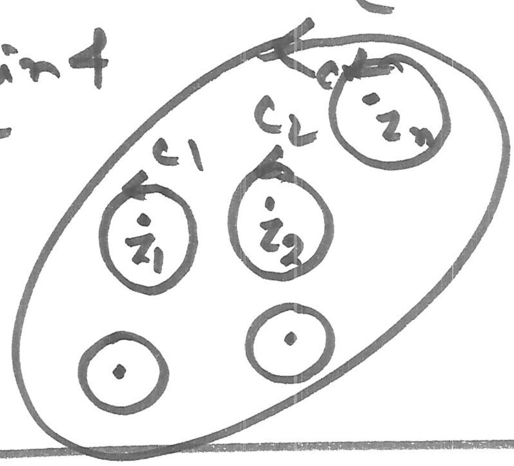
- a_{-1} = Residue of $f(z)$ at z_0.

$$= \text{Res}\,(f(z),\ z_0)$$

$$= \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

Cauchy's Residue Theorem.

- $D \to$ simply connected domain.
- $C \to$ a closed contour lying entirely within D.
- $f(z) \to$ analytic within & on C, except at a finite no. of pts. $z_1, z_2, \dots, z_n$



Then

$$\boxed{\oint_C f(z)\,dz = 2\pi i \left[\mathrm{Res}(f(z), z_1) + \mathrm{Res}(f(z), z_2) + \cdots + \mathrm{Res}(f(z), z_n) \right]}$$

Cauchy's Residue Theorem.

proof. radius r_k of the circle C_k with center at z_k is chosen so small s.t. $C_1, C_2, \dots, C_n$ are non-overlapping.

$$\therefore \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_r} f(z)\,dz$$

(using decomposition of line integrals)

$$= 2\pi i \operatorname{Res}\left(f(z), z_1\right) + 2\pi i \operatorname{Res}\left(f(z), z_2\right) + \cdots + 2\pi i \operatorname{Res}\left(f(z), z_n\right)$$

$$= 2\pi i \sum_{k=1}^{n} \underline{\underline{\operatorname{Res}\left(f(z), z_k\right)}}$$

Residues at simple poles.

- f has a <u>simple at $z = z_0$</u>.

Then $\operatorname{Res}\left(f(z), z_0\right) = \lim_{z \to z_0} (z - z_0) f(z)$.

__proof.__ Laurent's series of $f(z)$ about $z = z_0 \Rightarrow$

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$(z-z_0)f(z) = a_1 + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 + \cdots$$

$$\boxed{\lim_{z\to z_0}(z-z_0)f(z)} = a_{-1} = \operatorname{Res}(f(z), z_0) .$$

__Residues at a pole of order n__

- f has a pole of order n at $z = z_0$.

Then $\operatorname{Res}(f(z), z_0) = \cancel{\frac{1}{(n-1)!}\lim}$

$$= \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right].$$

__proof:__

Laurent's series of $f(z)$ about $z=z_0 \rightarrow$

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} + a_0(z-z_0)^n + a_1(z-z_0)^{n+1} + \cdots$$

$$\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right] = a_{-1}(n-1)! + a_0\, n!\,(z-z_0) + a_1 (n+1)!\,(z-z_0)^2 + \cdots$$

$$\lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right] = a_{-1}(n-1)!$$

$a_{-1} \rightarrow \operatorname{Res}(f(z), z_0)$

$$\Rightarrow \operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]$$

Computing Residues

Example: $f(z) = \dfrac{1}{(z-1)^2 (z-3)}$.

$\to$ $z=1$ is a pole of order 2

$\to$ $z=3$ is a simple pole.

$$\operatorname{Res}\left[f(z), 3\right] = \lim_{z\to 3} (z-3) f(z) = \lim_{z\to 3} \frac{1}{(z-1)^2}$$
$$= \frac{1}{4}.$$

$$\operatorname{Res}\left[f(z), 1\right] = \frac{1}{1!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right]$$

$$= -1/4$$

line integral evaluation by the Residue Th.

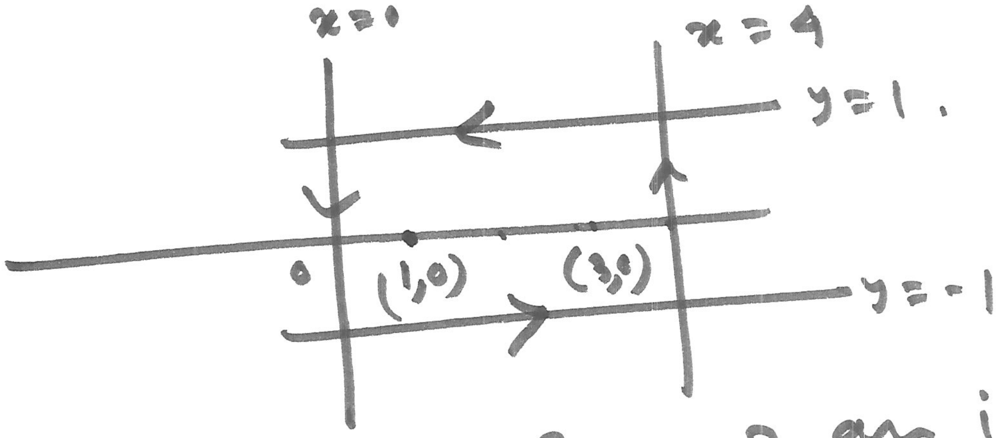
Example: Evaluate $\oint_C \dfrac{1}{(z-1)^2(z-3)}\,dz$ by the Residue Th. when

(i) C : rectangle defined by $x=0, x=4, y=-1,$
$y=1$

(ii) C : $|z|=2$.

Soln.

(i)

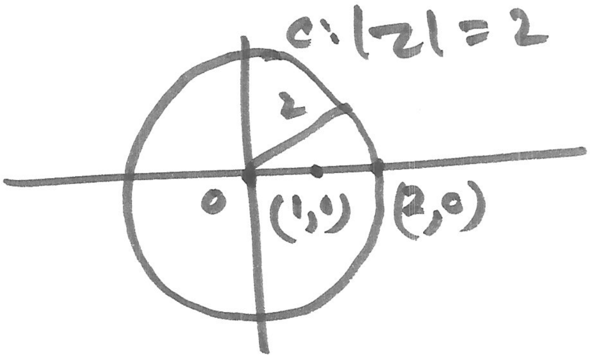


both $z=1$ & $z=3$ are inside C

$$\oint_C f(z)\,dz = 2\pi i\left[\operatorname{Res}(f(z),1) + \operatorname{Res}(f(z),3)\right]$$

$$= 2\pi i\left[\frac{1}{4} - \frac{1}{4}\right] = 0.$$

(ii)



only $z=1$ in inside c

$z=3$ outside c.

$$\therefore \oint_C f(z)\,dz = 2\pi i \operatorname{Res}(f(z),1)$$

$$= 2\pi i \times\left(\frac{1}{4}\right).$$

Example: Evaluate $\oint_C \underbrace{\frac{e^z}{z^4+5z^3}}_{f(z)} dz$

$C : |z| = 2$

using Cauchy's Residue Th.

Soln: ✓ $\text{Res}\left(f(z), \underset{\text{pole of order 3.}}{0}\right) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}\left[(z-0)^3 f(z)\right]$

✗ $\text{Res}\left(f(z), -5\right) =$

simple pole outside $|z|=2$

$$\boxed{\oint_C \frac{e^z}{z^4+5z^3} dz = \frac{17\pi i}{125}.}$$

Note:

- $f(z) = \dfrac{g(z)}{h(z)}$, $g(z_0) \neq 0$. &

  $h(z)$ has a zero of order 1 at $z = z_0$.

Then

$$\boxed{\operatorname{Res}\left(f(z), z_0\right) = \frac{g(z_0)}{h'(z_0)}}.$$

why?

$$\operatorname{Res}\left(f(z), z_0\right) = \lim_{z \to z_0} (z - z_0) f(z).$$

$$= \lim_{z \to z_0} \frac{g(z)}{\dfrac{h(z) - h(z_0)}{z - z_0}}$$

$$= \frac{g(z_0)}{h'(z_0)}$$

Example:

$$f(z) = \frac{1}{z^4+1} = \frac{g(z)}{h(z)}, \quad g(z)=1$$

has a simple pole at

z_1, z_2, z_3, z_4

$$z^4 = -1 = e^{\pi i} = e^{2k\pi i + \pi i}$$

$$z = e^{(2k+1)\frac{\pi i}{4}}, \quad k=0,1,2,3$$

$$\operatorname{Res}(f(z), z_1) = \frac{g(z_1)}{h'(z_1)} = \frac{1}{4z_1^3} = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$\operatorname{Res}(f(z), z_2) = \quad \cdots$$

$$\operatorname{Res}(f(z), z_3) = \quad \cdots$$

$$\operatorname{Res}(f(z), z_4) = \quad \cdots$$

$$\int_0^{\pi} \cos^{2n}\theta \, d\theta$$

Evaluation of Real Integrals.

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

Example:

$$\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\, d\theta .$$

$$= \oint_C \frac{1}{\left(2+\frac{z+\frac{1}{z}}{2}\right)^2} \frac{dz}{iz}$$

$C: |z|=1$

$$= \frac{4}{i} \oint_C \frac{z\, dz}{(z^2+4z+1)^2}$$

$z = e^{i\theta}$.

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$$

$dz = e^{i\theta} i\, d\theta$.

$$d\theta = \frac{dz}{zi}$$

θ from 0 to 2π

$\Rightarrow |z|=1$.

$$f(z) = \frac{z}{(z^2+4z+1)^2}$$

$$z^2+4z+1 = 0.$$

$$z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= -2 \pm \sqrt{3}$$

$z_0 = -2+\sqrt{3}$ is inside $c: |z|=1$.

$\longrightarrow$ ~~simple~~ pole of order 2 of $f(z)$

$$\text{Res}\left(f(z), z_0\right) = \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right].$$

$$= \frac{1}{6\sqrt{3}} \quad \text{(check)}.$$

$$\therefore \int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\, d\theta = \frac{4}{i}\, 2\pi i \, \text{Res}\left(f(z), z_0\right)$$

$$= \frac{4}{i} \cdot 2\pi i \cdot \frac{1}{6\sqrt{3}}.$$