9.11.16.

Example: let $\boxed{f(z) = \ln(1+z)}$, where we consider the branch which has value 0 when $z=0$.

(i) Expand f(z) in a Taylor's series about $z=0$

(ii) Determine the region of convergence for the series in (i)

(iii) Expand $\ln\left(\frac{1+z}{1-z}\right)$ in a Taylor's series about $z=0$.

Soln. (i) if $\boxed{|z|<1,}$ then

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

As power series can be integrated term-by-term, we get

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} a_n z^n \quad \text{for } |z|<1.$$

$\hookrightarrow$ (ii).

(ii)

region of convergence $a_n = \frac{(-1)^{n+1}}{n}$

$|z-0| < R$ when

$$1/R = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \qquad R = 1.$$

(iii)

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$\therefore \cancel{\frac{\ln(1+z)}{l}} \quad \ln\left(\frac{1+z}{1-z}\right) = 2z + \frac{2z^3}{3} + \frac{2z^5}{5} + \cdots$$

Exercise.

a) Taylor's series expansion of $f(z) = \sin z$ about $z = \pi/4$

b) Region of convergence of the series.

$$u = z - \pi/4 \Rightarrow z = u + \pi/4$$

$$f(z) = \sin z = \sin(u + \pi/4)$$

$$= \frac{1}{\sqrt{2}}\left[\sin u + \cos u\right]$$

$$= \frac{1}{\sqrt{2}}\left[\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right) + \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots\right)\right].$$

$$= \frac{1}{\sqrt{2}}\left[1 + u - \frac{u^2}{2!} - \frac{u^3}{3!} + \cdots\right]$$

$z - \pi/4$

$=$

- Taylor's series expansion of a fn. $f(z)$ is always about a pt. $z = z_0$ when $f(z)$ is analytic.

- Laurent's series expansion of a fn. $f(z)$ is always about an isolated singular pt. $z = z_0$ of $f(z)$

$D: R_1 < |z - z_0| < R_2$

Example: Expand $\boxed{f(z) = \dfrac{1}{(z+1)(z+3)}}$

in laurent series valid for

$1 < |z| < 3.$

Taylor's series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

laurent's series

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

Soln.

$$f(z) = \frac{1}{(z+1)(z+3)}$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right].$$

$|z| > 1$ i.e. $\dfrac{1}{|z|} < 1.$

$$\frac{1}{z+1} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]$$

$|z| < 3$

$$\frac{1}{z+3} = \frac{1}{3\left(1+\frac{z}{3}\right)} = \frac{1}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$\therefore$ for $1 < |z| < 3$, the laurent's series expansion of $f(z)$ is

$$\underbrace{\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z}}_{\text{principal part.}} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

analytic part

Example: Expand $f(z) = \dfrac{1}{z(z-1)}$ in laurent series valid for

(a) $0 < |z| < 1$, (b) $1 < |z|$,

(c) $0 < |z-1| < 1$, (d) $1 < |z-1|$.

Sol^n. (a) $\underline{|z| < 1}$

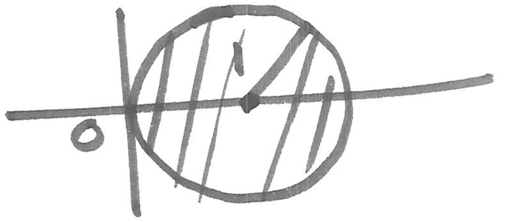
$$f(z) = -\frac{1}{z} \cdot \frac{1}{1-z}$$

$$= -\frac{1}{z}\left[1 + z + z^2 + z^3 + \cdots\right]$$

(b) $1 < |z|$ i.e. $\dfrac{1}{|z|} < 1$.

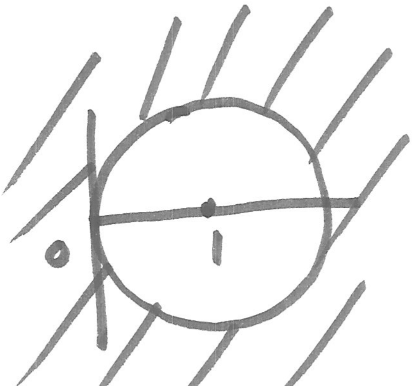
$$f(z) = \frac{1}{z^2} \frac{1}{\left(1 - \frac{1}{z}\right)} = \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$$

(c) $0 < |z-1| < 1$



$$f(z) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots$$

(d) $1 < |z-1|$



$$f(z) = \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \frac{1}{(z-1)^5} + \cdots$$

Classification of isolated singular pts.

$z = z_0$ (isolated singularity)	Laurent series.
1. Removable singularity	$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$
2. Pole of order n	$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

3. Simple pole

$$\frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

4. Essential singularity.

$$\cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

<u>Example:</u>

$$e^{\frac{3}{z}} = 1 + \frac{3}{z} + \frac{\left(\frac{3}{z}\right)^2}{2!} + \frac{\left(\frac{3}{z}\right)^3}{3!} + \cdots$$

$$= 1 + \frac{3}{z} + \frac{3^2}{2!}\cdot\frac{1}{z^2} + \frac{3^3}{3!}\cdot\frac{1}{z^3} + \cdots$$

$z = 0$ is an essential singularity of $e^{3/z}$.

<u>Example:</u> Find Laurent's series for the fn.

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

in

(a) $0 < |z-1| < 2$

(b) $0 < |z-3| < 2$.

Name the singularity

$\underline{\text{Sol}^n}$: (a) $\quad 0 < |z-1| < 2$

$$f(z) = \frac{1}{(z-1)^2\,(z-1-2)}$$

$$= \frac{1}{-2(z-1)^2 \cdot \left(1 - \frac{z-1}{2}\right)}$$

$$= \frac{1}{-2\,(z-1)^2}\left[1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \cdots\right]$$

$$= \underbrace{-\frac{1}{2\,(z-1)^2} - \frac{1}{4\,(z-1)}}_{\text{principal part.}} - \frac{1}{8} - \frac{1}{16}(z-1) + \cdots$$

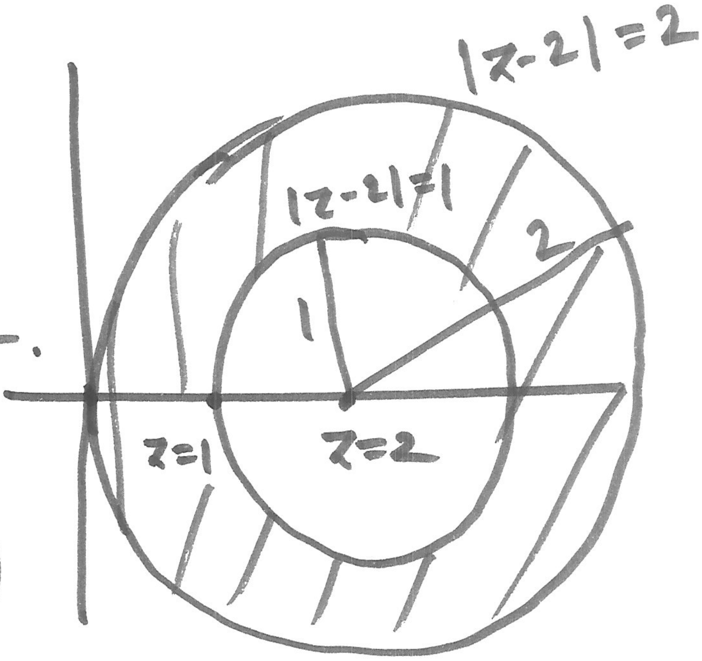
$\Downarrow$

$z = 1$ is a pole of order 2.

(b) Exercise.

Example: Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z-2| < 2$

Sol^n.



$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}.$$

$$1 < |z-2| \Rightarrow \frac{1}{|z-2|} < 1.$$

$$\frac{1}{z-1} = \frac{1}{z-2+1}$$

$$= \frac{1}{(z-2)\left(1 + \frac{1}{z-2}\right)}.$$

$= \cdots$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

$|z| < 1$

$|z-2| < 2$

$$\frac{1}{z} = \frac{1}{z-2+2} = \frac{1}{2\left(1 + \frac{z-2}{2}\right)}$$

$= \cdots$

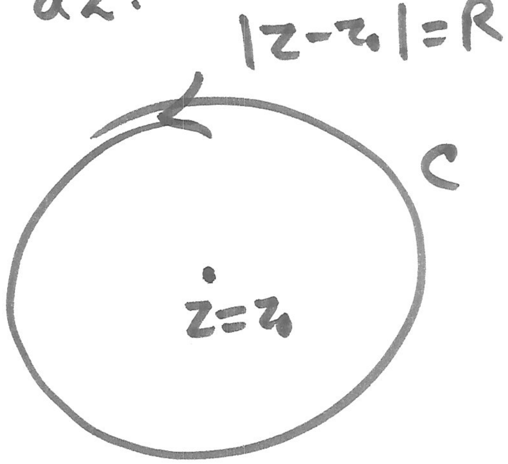
Residue & Residue Theorem.

- Lauret series expansion of $f(z)$ about an isolated singularity $z = z_0$, say,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0), \quad 0 < |z - z_0| <$$

region of convergence

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$



- $a_{-1} = \text{Res}\,(f(z), z_0).$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) \, dz.$$

$$\Rightarrow \oint_C f(z) dz = 2\pi i \, a_{-1}$$

find by Laurent's series expansion.

Example:

$$\oint_{C:|z|=2} e^{3/z}\,dz = 2\pi i \cdot \times 3$$
$$= 6\pi i$$

$$e^{3/z} = 1 + \frac{3}{z} + \frac{\left(\frac{3}{z}\right)^2}{2!} + \frac{\left(\frac{3}{z}\right)^3}{3!} + \cdots$$

$a_{-1} = 3.$